International Journal of Pure and Applied Mathematics
Volume 104 No. 4 2015, 517-521
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v104i4.3

ijpam.eu

THE COMPLETE SET OF SOLUTIONS OF THE DIOPHANTINE EQUATION $p^x + q^y = z^2$ FOR TWIN PRIMES p AND q

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Abstract: The main purpose of this paper is to correct the result of A. Suvarnamani that was published in this journal. In particular, A. Suvarnamani showed in [6] that $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ is the "unique solution" to the Diophantine equation

$$p^x + q^y = z^2 \tag{1}$$

where p is an odd prime, $q-p = 2$ and x, y and z are non-negative integers. The author, however, did not realize that $(p, q, x, y, z) \in \{(17, 19, 1, 1, 6), (71, 73, 1, 1, 12)\}$ also satisfies equation (1) (cf. [4]).

In the present paper, we give more solutions to (1). That is, we show that if the well-known Twin Prime Conjecture is true, then the Diophantine equation given by (1), where p and q are twin primes, has infinitely many solutions (p, q, x, y, z) in positive integers. Furthermore, we show that if the sum of p and q is a square, then (1) has the unique solution $(x, y, z) = (1, 1, \sqrt{p+q})$ in non-negative integers.

AMS Subject Classification: 11D61
Key Words: Diophantine equation, twin primes, integer solutions

Received: May 17, 2015

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1. Introduction

In 1844, Charles Catalan conjectured that the only solution to the Diophantine equation $a^x - b^y = 1$ with $\min\{a, b, x, y\} > 1$ is the 4-tuple $(a, b, x, y) = (3, 2, 2, 3)$ (cf. [1]). This was finally proven in 2002 by Preda Miháilescu (cf. [3]). Corollary to this result is that the Diophantine equation $1 + p^x = z^2$, where p is a prime and $\min\{p, x, z\} > 1$, has the unique solution $(p, x, z) = (2, 3, 3)$.

Recently, a solution to the Diophantine equation (1) where p is an odd prime was given in [6]. Most precisely, A. Suvarnamani showed that the 5-tuple $(p, q, x, y, z) = (3, 5, 1, 0, 2)$ is the "unique solution" to the equation (1). However, we found out that there's an error in Suvarnami's work. A counterexample would show that $(17, 19, 1, 1, 6)$ and $(71, 73, 1, 1, 12)$ are also solutions to (1). In fact, the case where $(p, q) = (17, 19), (71, 73)$ in (1) had already been studied by the J. F. T. Rabago in [4].

The flaw in the Suvarnami's proof for his main result is as follows: "... If we suppose that $p^x + q^y = z^2$ to have a solution (x, y) in positive integers, then z is odd." This is absurd, since, if p is assumed to be prime (i.e., $p \equiv 1, 3 \pmod 4$ and $q = p + 2$), this means that q is also odd. Hence, z must be an even (positive) integer. So the problem on determining the set of all solutions to (1), where p is a prime and $q = p + 2$, remains open. This gives us a motivation to further investigate the Diophantine equation (1).

So, in this paper, we shall answer the following questions:

(Q1) Does the Diophantine equation of the form given by (1) have infinitely many solutions (p, q, x, y, z) in $\mathbb{N}_0$ when p and q are twin primes?

(Q2) Given twin primes p and q, does a solution (x, y, z) to the Diophantine equation (1) always exist in positive integers? Is the solution unique?

Throughout the paper we denote $\mathbb{N}_0$ and $\mathbb{N}$ as the set of all non-negative and positive integers, respectively. Furthermore, we use the notation $a \equiv_m b$ to denote $a \equiv b \pmod m$.

2. Main Result

We begin with the following lemmata.

Lemma 1. *Let $q > p > 3$ be twin primes. Then, $12 | (p + q)$.*

Proof. Let p and q be twin primes such that $p > 3$. Then, $p = 6l \pm 1$ and $q = 6l \pm 1 + 2$ for some $l \in \mathbb{N}$. Since $p < q$ then p must be of the form $6l - 1$ and $q = 6l + 1$. It follows that $p + q = 12l$, proving the lemma. □

Lemma 2. *Let p and q be twin primes. Then,* (1) *has infinitely many solutions in $\mathbb{N}$ of the form $(p, q, x, y, z) = (p, q, 1, 1, 6l)$ where $l \in \mathbb{N}$.*

Proof. The proof relies on the infinitude of twin primes (cf. [2]). We first assume that the conjecture is true. We also note that we can express $36l^2$ as $(6(3l^2) - 1) + (6(3l^2) + 1) = p + q$. Thus, if $p = 6(3l^2) - 1$ and $q = 6(3l^2) + 1$ are twin primes, then $(p, q, x, y, z) = (6(3l^2) - 1, 6(3l^2) + 1, 1, 1, 6l)$ is a solution to $p^x + q^y = z^2$. So, if there exists an infinite pair of primes (p, q) of distance two, then there also exists an infinite number of solutions (p, q, x, y, z) to (1) in $\mathbb{N}$. □

Remark 1. We note that even if $q = p + 2$ is not prime, the equation (1) still has infinitely many solutions (p, q, x, y, z) in $\mathbb{N}$ of the form $(6(3l^2) - 1, 6(3l^2) + 1, 1, 1, 6l)$ where $l \in \mathbb{N}$.

We now proceed to state our main result.

Theorem 1. *Let p and q be fixed twin primes and their sum be a perfect square. Then, the equation* (1) *has the unique solution $(x, y, z) = (1, 1, \sqrt{p + q})$.*

Before we prove Theorem 1, we first prove the following lemmas.

Lemma 3. *The Diophantine equation $1 + p^x = z^2$, where p is an odd prime has the exactly two solutions solutions (p, x, z) in $\mathbb{N}_0$; namely,* $(3, 1, 2)$ *and* $(2, 3, 3)$.

Proof. It suffices to assume that $x > 0$ since $z^2 = 2$ is impossible. If $\min\{p, x, z\} > 1$, then it immediately follows that $(2, 3, 3)$ is a unique solution to $1+p^x = z^2$. If no restriction is imposed, then it follows that $(z+1)(z-1) = p^x$. Hence, $2 = p^\beta - p^\alpha = p^\alpha(p^{\beta-\alpha} - 1)$ where $\beta > \alpha$ and $\alpha + \beta = x$. Since $p \neq 2$, then $\alpha = 0$ and $p^x - 1 = 2$. Thus, $p = 3$ and $x = 1$, which yield the solution $(p, y, z) = (3, 1, 2)$ of $1 + p^x = z^2$. This concludes the lemma. □

Remark 2. It is stated in [6, Lemma 2.2] that "If q is an odd prime number and y, z are non-negative integers, then the Diophantine equation $1 + q^y = z^2$ has no solution." We remark that this statement is only true provided q is assumed to be a prime number greater than 3. We also remark, by using Lemma (3), that $(1, 0, 2)$ is a solution to any Diophantine equations of the form $3^x + b^y = z^2$.

In the succeeding discussions, we will assume WLOG that x, y, and z are in $\mathbb{N}$ since the least possible pair of twin primes that sums up to a perfect square is 17 and 19, and $17^x + 1 = z^2$ (resp. $1 + 19^y = z^2$) has no solution in $\mathbb{N}_0$ by Miháilescu Theorem (cf. [3]).

Lemma 4. *Let p and q be fixed twin primes. Then, equation* (1) *is never possible for* $\min\{x, y\} > 1$.

Proof. Let p and q be twin primes. If $p \equiv_4 1, -1$, then $q \equiv_4 -1, 1$, respectively. First, suppose that $p \equiv_4 1$ and $q \equiv_4 -1$. Then, x, y must be of different parity and z is even. So we consider two possibilities: (i) x is even and y is odd; and, (ii) x is odd and y is even.

If $x = 2k$ for some $k \in \mathbb{N}$, then $q^y = (z + p^k)(z - p^k)$. It follows that $2p^k = (z + p^k) - (z - p^k) = q^\alpha(q^{\beta-\alpha} - 1)$ where $\alpha < \beta$ and $\alpha + \beta = y$. Since $p \neq 2, q$, then $\alpha = 0$ and $2p^k = q^y - 1$. Note that if $p \equiv_4 1$, then $p \equiv_6 -1$ and $2p^k \equiv_3 1, 2$. Furthermore, since $q \equiv_6 1$ and $q^y - 1 \equiv_3 0$ whenever $q \equiv_4 -1$, then $2p^k \not\equiv_3 q^y + 1$.

If $y = 2l$ for some $l \in \mathbb{N}$, then $p^x = (z+q^l)(z-q^l)$. So $2q^l = (z+q^l)-(z-q^l) = p^\alpha(p^{\beta-\alpha} - 1)$ where $\alpha < \beta$ and $\alpha + \beta = x$. Again, because $p \neq 2, q$, then $\alpha = 0$ and $2q^l = p^x - 1$. Note that if $q \equiv_4 -1$, then $2q^l \equiv_4 2$. Also, since $p^x \equiv_4 1$ implies that $p^x - 1 \equiv_4 0$, then $2q^l \not\equiv_4 p^x - 1$.

Now, suppose that $p \equiv_4 -1$ and $q \equiv_4 1$. Then, again, x, y are of different parity and z is even. If $x = 2k$, then we obtain $2p^k = q^y - 1$. So $2p^k \equiv_4 2$ and $q^y - 1 \equiv_4 0$ which implies that $2p^k \not\equiv_4 q^y - 1$. Similarly, if $y = 2l$, then we get $2q^l = p^x - 1$. Since $2q^l \equiv_6 2$ and $p^x - 1 \equiv_6 4$, $2q^l \not\equiv_6 p^x - 1$.

We have shown that if p and q are twin primes, then the equation (1) has no solution (x, y, z) in $\mathbb{N}$. This proves the lemma. □

We now prove our main result.

Proof of Theorem 1. Let p, q be twin primes. Without loss of generality, we assume $p < q$. By assumption, the sum of p and q is a perfect square. Clearly, $p \geq 17$ and $q \geq 19$, and $(x, y, z) = (1, 1, \sqrt{p+q})$ is a solution to $p^x + q^y = z^2$. Now, to show that the solution (x, y, z) of $p^x + q^y = z^2$ is unique, it suffices to assume that $\min(x, y) > 1$. From Lemma 2, we see that p and q are of different residue classes modulo 4; that is, if $p \equiv_4 1, -1$, then $q \equiv_4 -1, 1$. Using Lemma 3 and Lemma 4, we obtain no other solution (x, y, z) to $p^x + q^y = z^2$ in $\mathbb{N}_0$ except $(1, 1, \sqrt{p+q})$. This proves the main result.

A consequence of our main result is given in the next corollary.

Corollary 2. *Let p and q be fixed such that p, q are twin primes. Then, the Diophantine equation $p^x + q^y = z^2$ has at most one solution in $\mathbb{N}_0$.*

We have the following table for some particular values of p and q.

$p^x + q^y = z^2$	unique solution in $\mathbb{N}_0$
$881^x + 883^y = z^2$	$(1, 1, 42)$
$1151^x + 1153^y = z^2$	$(1, 1, 48)$
$2591^x + 2593^y = z^2$	$(1, 1, 72)$
$3527^x + 3529^y = z^2$	$(1, 1, 84)$
$4049^x + 4051^y = z^2$	$(1, 1, 90)$

3. Conclusion

In this paper, we have shown that the answer to (Q1) is affirmative by Lemma 2, while the answer to (Q2) is no by using Theorem 1. It seems that by Theorem 1, the equation $p^x + q^y = z^2$ will have the unique solution $(x, y, z) = (1, 1\sqrt{p+q})$ only when the sum of p and q is a perfect square. The answer to (Q2) can easily be supported by a result of Sroysang (cf. [5]).

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