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PROPERTIES OF b-θ-COMPACT SPACES

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Abstract: In this paper, we present and study the notion of firm b-θ-continuity to investigate b-θ-compactness. We also present some properties of b-θ-compactness in terms of nets and ultranets.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide, Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure

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of A and the interior of A, respectively. A set A is called b-open [1] ($= \gamma$-open [2]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of b-open set is called b-closed. The intersection of b-closed sets of X containing A is called b-closure [1] of A and is denoted by $b\,\mathrm{Cl}(A)$. A set A is b-closed if and only if $A = b\,\mathrm{Cl}(A)$. The b-θ-closure [3], denoted by $b\,\mathrm{Cl}_\theta(A)$, is the set of all $x \in X$ such that $b\,\mathrm{Cl}(U) \cap A \neq \varnothing$ for every b-open set U containing x. A subset A is called b-θ-closed [3] if $A = \mathrm{Cl}_\theta(A)$. By [3], it is proved that, for a subset A, $b\,\mathrm{Cl}_\theta(A)$ is the intersection of all b-θ-closed sets containing A. The complement of a b-θ-closed set is called b-θ-open. The family of all b-θ-open (resp. b-θ-closed) sets of (X, τ) is denoted by $B\theta O(X, \tau)$ (resp. B$\theta C(X, \tau)$). In this paper, we present and study the notion of firm b-θ-continuity to investigate b-θ-compactness. We also present some properties of b-θ-compactness in terms of nets and ultranets. Moreover, we introduce and investigate some basic properties of b-θ-(m, n)-compact spaces.

2. Characterization of b-θ-Compact Spaces

Definition 2.1. A subset K of a nonempty set X is said to be b-θ-compact relative to (X, τ) if every cover of K by b-θ-open sets of X has a finite subcover. We say that (X, τ) is b-θ-compact if X is b-θ-compact.

Definition 2.2. A function $f : X \to Y$ is said to have property $\mathcal{P}$ if for every b-θ-open cover ∇ of Y there exists a finite cover (the members of which need not be necessarily b-θ-open) $\{A_1, A_2, ...A_n\}$ of X such that for each $i \in \{1, 2, ..., n\}$, there exists $U_i \in \nabla$ such that $f(A_i) \subset U_i$.
Recall that a function $f : X \to Y$ is said to be quasi b-θ-continuous if $f^{-1}(V)$ is b-θ-open in X for every b-θ-open set V of Y.

Theorem 2.3. *A topological space X is b-θ-compact if and only if for every topological space Y and every quasi b-θ-continuous function $f : X \to Y$, f has the property $\mathcal{P}$.*

Proof. Let the topological space X be b-θ-compact and the function $f : X \to Y$ be quasi b-θ-continuous. Suppose that Ω be a b-θ-open cover of Y. The set $f(X)$ is b-θ-compact relative to Y. This means that there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ of Ω which cover $f(X)$. Then the sets $A_1 = f^{-1}(U_1)$, $A_2 =f^{-1}(U_2)$,....,$A_n = f^{-1}(U_n)$ from a cover of X such that $f(A_i) \subset U_i$ for each $i \in \{1, 2, ..., n\}$. Conversely, suppose that X is a topological space such that for every topological space Y and every quasi b-θ-continuous function $f : X \to Y$, f has property $\mathcal{P}$. It follows that the identity function $id_X : X \to X$ has the property $\mathcal{P}$. Hence, for every b-θ-open cover Ω of X, there exists a finite cover $A_1, A_2, ...A_n$ of X such that for each $i \in \{1, 2, ..., n\}$ there exists a set $U_i \in \Omega$ such that $A_i = id_X(A_i) \subset U_i$. Then $U_1, U_2, ...U_n$ are finite b-θ-subcover of Ω. Since Ω was an arbitrary b-θ-open cover of X, the space X is b-θ-compact. □

Theorem 2.4. *A function $f : X \to Y$ is called firmly b-θ-continuous if for every b-θ-open cover ∇ of Y there exists a finite b-θ-open cover Ω of X such that for every $U \in \Theta$, there exists a set $G \in \Omega$ such that $f(U) \subset G$.*

Remark 2.5. It should be noted that if the topological space, then every quasi b-θ-continuous function $f : X \to Y$ is firmly b-θ-continuous.

Lemma 2.6. *Let X,Y,Z and W be topological spaces. Let $g : X \to Y$ and $h : Z \to W$ be quasi b-θ-continuous functions and let $f : Y \to Z$ be firmly b-θ-continuous. Then the functions $f \circ g : X \to Z$ and $h \circ f : Y \to W$ are firmly b-θ-continuous.*

Lemma 2.7. *Let X and Y be topological spaces. Suppose that $f : X \to Y$ is a quasi b-θ-continuous function which has the property $\mathcal{P}$. Then f is firmly b-θ-continuous.*

Theorem 2.8. *The following statements are equivalent for a topological space (X, τ):*

(i) X is b-θ-compact.

(ii) The identity function $id_X : X \to X$ is firmly b-θ-continuous;

(iii) Every quasi b-θ-continuous function from X to X is firmly b-θ-continuous;

(iv) Every quasi b-θ-continuous function from X to a topological space Y is firmly b-θ-continuous;

(v) Every quasi b-θ-continuous function from X to a topological space Y has the property $\mathcal{P}$;

(vi) For each topological space Y and each quasi b-θ-continuous function $f : Y \to X$, f is firmly b-θ-continuous.

Proof. (i)⇒(ii): Let X be a b-θ-compact space. The identity function $id_X : X \to X$ is quasi b-θ-continuous and by Remark 2.5 id_X is firmly b-θ-continuous.

(ii)⇒(iii): Let $f : X \to X$ is any quasi b-θ-continuous function. By (ii), the identity function $id_X : X \to X$ is firmly b-θ-continuous Therefore by Lemma 2.6 $f = id_X : X \to X$ is firmly b-θ-continuous.

(iii)⇒(iv): Suppose that $f : X \to Y$ is any quasi b-θ-continuous function. The identity function $id_X : X \to X$ is firmly b-θ-continuous and by (iii) id_X is firmly b-θ-continuous. As a consequence of Lemma 2.6, We have that $f = f \circ id_X : X \to Y$ is firmly b-θ-continuous.

(iv)⇒(v): Obvious.

(v)⇒(i): This is an immediate consequence of Lemma 2.3.

(vi)⇒(ii): Suppose that $id_X : X \to X$ is the identity function. Then id_X is quasi b-θ-continuous and by (vi) id_X is firmly b-θ-continuous.

(i)⇒(vi): Suppose that ∇ is b-θ-open cover of X. Since X is b-θ-compact, then there is a finite b-θ-subcover $U_1, U_2, ..., U_n$ of ∇. Let $A_i = f^{-1}(U_i)$ for $i \in \{1, 2, ..., n\}$. We have that $f(A_i) \subset U_i$ for every $i \in \{1, 2, ..., n\}$. Therefore, f is firmly b-θ-continuous. □

Definition 2.9. A topological space (X, τ) is said to be b-θ-T_1 if for each pair of distinct points x and y of X, there exists b-θ-open sets U and V of X such that $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$.

Theorem 2.10. *If $f : X \to Y$ is a firmly b-θ-continuous function, where X is a topological space and Y is b-θ-T_1 space, then f is quasi b-θ-continuous.*

Proof. Let x be an arbitrary point of X and V be a b-θ-open set of Y containing $f(x)$. We define a b-θ-open cover Ω of Y such that $\Omega = \{V, Y \backslash f(x)\}$. Since f is firmly b-θ-continuous, it follows that there exists a finite b-θ-open cover $\{P_1, P_2, ..., P_n\}$ of X such that $f(P_i) \subset V$ or $f(P_i) \subset Y \backslash f(x)$ for every $i \in \{1, 2, ..., n\}$. Let $x \in P_j$ for some index j. Since $f(P_j)$ contains $f(x)$, so it follows that $f(P_j) \subset V$. This shows that f is quasi b-θ-continuous. □

3. Properties of b-θ-Compact Spaces in Terms of Nets and Ultranets

Definition 3.1. Let (X, τ) be a topological space, $x \in X$ and $\{x_\ell, \ell \in L\}$ be a net of X. We say that a net $\{x_\ell, \ell \in L\}$ b-θ-converges to x if for each b-θ-open set U containing x, there exists an element $\ell_0 \in L$ such that $\ell \geq \ell_0$ implies $x_\ell \in U$.

Definition 3.2. Let (X, τ) be a topological space, $G = \{F_i : i \in I\}$ be a filterbase of X and $x \in X$. A filterbase G is said to be b-θ-converges to x if there exists a member $F_i \in G$ such that $F_i \subseteq U$ for each b-θ-open set containing x.

Proposition 3.3. *If $x \in U$ and $U \in B\theta C(X,\tau)$, then there exists a net $\{x_i\}_{i\in I}$ that b-θ-converges to x and $x_i \in U$ for each $i \in I$.*

Proof. Suppose that $x \in U$ and $U \in B\theta C(X,\tau)$ which means $U = b\,\mathrm{Cl}_\theta(U)$. This means that if $x \in N$ and $N \in B\theta O(X,\tau)$ then $N \cap U \neq \emptyset$. It follows that there exists an element $x_N \in N \cap U$. This implies that $\{x_N\}_{N\in I}$. b-θ-converges to x. □

Proposition 3.4. *Let $\{x_i\}_{i\in I}$ be a net in (X,τ) and $U \in B\theta C(X,\tau)$, where $x_i \in U$ for each $i \in I$. If $\{x_i\}_{i\in I}$ b-θ-converges to x, then $x \in U$.*

Proof. Assume that $\{x_i\}_{i\in I}$ b-θ-converges to x, then $x \notin U$. Then there exists a b-θ-open set N such that $x \in N$ and $N \cap U = \emptyset$. This means that there exists $i_0 \in I$ such that $x_i \in N$ for each $i \geq i_0$. Then x_i is not an element of U for each $i \geq i_0$. But this is a contradiction and hence the result. □

Definition 3.5. A point y is a b-θ-cluster point of $\{x_i\}_{i\in I}$ if for each $i_0 \in I$ and $U \in B\theta O(X,\tau)$ such that $y \in U$, there exists an $i_1 \geq i_0$ such that $x_{i_1} \in U$.

Proposition 3.6. *Let $(\ell_i)_{i\in I}$ be an ultranet and y be a b-θ-cluster point of the net. Then the ultranet $(\ell_i)_{i\in I}$ b-θ-converges to y.*

Proof. Suppose that $(\ell_i)_{i\in I}$ is an ulteranet in a topological space (X,τ) and y be a b-θ-cluster point of the net $(\ell_i)_{i\in I}$. Suppose that, $(\ell_i)_{i\in I}$ does not b-θ-converges to y. This means that there exists $U \in B\theta O(X,\tau)$ such that $y \in U$ and ℓ_i is not an element of U for any $i \in I$. So y is not a b-θ-cluster point of $(\ell_i)_{i\in I}$. □

Proposition 3.7. *Let $(\ell_i)_{i\in I}$ be a net in a topological space (X,τ). Then $y \in X$ is a b-θ-cluster point of $(\ell_i)_{i\in I}$, if and only if a subnet of $(\ell_i)_{i\in I}$ b-θ-converges to y.*

Proof. Let $(\ell_i)_{i\in I}$ have a subnet $(\ell_{k_\mathrm{j}})_{j\in J}$ that b-θ-converges to y and J be a directed set. Now suppose that $y \in X$ is not a b-θ-cluster point of $(\ell_i)_{i\in I}$. This means that there exists $U \in B\theta O(X,\tau)$ and $i_0 \in I$ such that s_{i_1} is not an element of U for every $i_1 \geq i_0$. Then $(\ell_{k_\mathrm{j}})_{j\in J}$ does not b-θ-converges to y. Conversely, assume that y is a b-θ-cluster point of $(\ell_i)_{i\in I}$. $J = \{(i,U) : i \in I, y \in U, U \in B\theta O(X,\tau)$ and $\ell_i \in U\}$ is a partially ordered set where $(i,U) \leq (i_1,V)$, if $i \leq i_1$ and $V \subset U$. (i) $(i,U) \leq (i,U)$ for every $(i,U) \in J$. Because, $i \leq i$ and $U \subset U$ for every $i \in I$ and $U \in B\theta O(X,\tau)$. (ii) Let $(i,U) \leq (i_1,V)$ and $(i_1,V) \leq (i,U)$. Then, $i \leq i_1$, $V \subset U$ and $i_1 \leq i$, $U \subset V$. This follows that $i = i_1$, $V = U$. Then $(i_1,V) = (i,U)$. (iii) Let $(i,U),(i_1,V)$ and $(i_2,W) \in J$

such that $(i,U) \leq (i_1,V)$ and $(i_1,V) \leq (i_2,W)$. Since I is a directed set, $i \leq i_2$ where $i \leq i_1$ and $i_1 \leq i_2$. Also, we know that $W \subset U$ where $V \subset U$ and $W \subset V$. Then $(i,U) \leq (i_2,W)$ where $i \leq i_2$ and $W \subset U$. Consequently, J is a partially ordered set. Now let $(i,U),(i_1,V) \in J$. Then $U \cap V \in B\theta O(X,\tau)$. We know that $U \cap V \subset U$ and $U \cap V \subset V$ and $y \in U \cap V$. Since y is a b-θ-cluster point of $(\ell_i)_{i\in I}$, there exists $i_2 \in I$ such that $i \leq i_2$, $i_1 \leq i_2$ and $s_{i_2} \in U \cap V$. Then $(i_1,V) \leq (i_2,U\cap V)$ and $(i,U) \leq (i_2,U\cap V)$. This means that J is a directed set. Define $k : J \to I$ by $k(i,A) = i$. (a) $(i,U) \leq (i_1,V)$ means that $i \leq i_1$. Then $k(i,U) \leq k(i_1,V)$. (b). Let $i,i_1 \in I$ and $U \in B\theta O(X,\tau)$ which contains y. Then there exists $i_2 \in I$ such that $i \leq i_2, i_1 \leq i_2$ and $\ell_{i_2} \in U$. This means that $(i_2,U) \in J$, $i \leq k(i_2,U)$ and $i_1 \leq k(i_1,U)$. This follows that $\{\ell_{k(i,U)}\}_{i\in I}$. Consider the set $U \in B\theta O(X,\tau)$ which contains y. There exists $i_0 \in I$ such that $\ell_{i_0} \in U$. This $(i_0,U) \in J$. For every $(i,V) \in J$ that $(i_0,U) \leq (i,V), V \subset U$ and $\ell_i \in V$. This follows that $\ell_{k(i,V)} \in U$ for every $(i_0,U) \leq (i,V)$. So the subnet, $\{\ell_{k(i,U)}\}_{(i,U)\in J}$, b-θ-converges to y. □

Proposition 3.8. *Let (X,τ) be topological space. Then the following statements are equivalent:*

(i)(X,τ) is b-θ-compact.

(ii) For any family Ψ of b-θ-closed subsets of X such that $\cap_{k\in\Psi}K = \emptyset$, there exists a finite subfamily $\Phi \subset \Psi$ such that $\cap_{L\in\Phi}L = \emptyset$.

(iii)$\cap_{k\in\Psi}K \neq \emptyset$ for any family Ψ of b-θ-closed subsets of X such that $\cap_{L\in\Phi}L \neq \emptyset$ where $\Phi \subset \Psi$ is a finite subfamily.

Proof . $(i) \Rightarrow (ii)$: Let (X,τ) be b-θ-compact and Ψ be a family of b-θ-closed subsets such that $\cap_{k\in\Psi}K = \emptyset$. Then $[\cap_{k\in\Psi}K]^c = [\emptyset]^c$. This means $\cup_{k\in\Psi}K^c = X$. There exists a finite subfamily $\Phi \subset \Psi$ such that $\cap_{L\in\Phi}L = \emptyset$.

$(ii) \Rightarrow (iii)$: Let Ψ be a family of b-θ-closed subsets of X. From the assumption if $\cap_{k\in\Psi}K \neq \emptyset$, then there exists a finite subfamily $\Phi \subset \Psi$ such that $\cap_{L\in\Phi}L = \emptyset$. This means that if Ψ does not have any finite subfamily Φ such that $\cap_{L\in\Phi}L = \emptyset$, then $\cap_{k\in\Psi}K = \emptyset$.

$(iii) \Rightarrow (ii)$: Let Ψ be a family of b-θ-closed subsets of X. From the assumption if $\cap_{L\in\Phi}L \neq \emptyset$ for any subfamily $\Phi \subset \Psi$, then $\cap_{k\in\Psi}K \neq \emptyset$. This means that, if $\cap_{k\in\Psi}K = \emptyset$, then there exists at least one subfamily $\Phi \subset \Psi$ such that $\cap_{L\in\Phi}L = \emptyset$.

$(ii) \Rightarrow (i)$: Let $\{U_i\}_{i\in I}$ be a b-θ-open cover of X. Then, $\cup_{i\in I}U_i = X$. This means that $\cap_{i\in I}U_i^c = \emptyset$ and $U_i^c \in B\theta C(X,\tau)$ for each $i \in I$. It follows from the assumption that there exists a finite subfamily $J \subset I$ such that $\cap_{j\in J} U_j^c = \emptyset$. So $\cup_{j\in J} U_j = X$. Therefore (X,τ) is b-θ-compact. $\square$

Proposition 3.9. *A topological space (X,τ) is b-θ-compact if and only if every net has at least one b-θ-cluster point in the topological space.*

Proof. Let (X,τ) be b-θ-compact and $\{x_i\}_{i\in I}$ be any net in this space. Let as consider a family $b\,\mathrm{Cl}_\theta(B_j)$ of subsets, where $B_j = \{x_i : j \le i\}$. Then, $b\,\mathrm{Cl}_\theta(B_j) \in B\theta C(X,\tau)$ for any $j \in I$ and the intersection of finitely many of $b\,\mathrm{Cl}_\theta(B_j)$ is nonempty. It follows from proposition 3.8 that $\cap_{j\in J} b\,\mathrm{Cl}_\theta(B_j) \neq \emptyset$ for (X,τ) is b-θ-compact. Let $y \in \cap_{j\in J} b\,\mathrm{Cl}_\theta(B_j)$. Then $y \in b\,\mathrm{Cl}_\theta(B_j)$ for any $j \in I$. Consider $y \in U$, $U \in B\theta O(X,\tau)$ and $r \in I$. Then $U \cap B_r \neq \emptyset$. So $U \cap B_k \neq \emptyset$ for any $k \in I$ such that $k \ge r$. Consequently y is a b-θ-cluster point of $\{x_i\}_{i\in I}$. Now suppose that every net in (X,τ) has at least one b-θ-cluster point. Let $\{F_i\}_{i\in I}$ be a family of b-θ-closed sets where intersection of finitely many of F_i's is nonempty. Consider the set $J = \{\cap_{j=1}^n G_{i_j} : \{G_{i_j}\}_{j=1}^n \subset \{F_i\}_{i\in I}\}$ and the relation " $\le$ ", where $A \le B$ whenever $B \subset A$ and $A, B \in J$. (i)$A \subset A$ for every set $A \in J$. This means that $A \le A$ for every set $A \in J$. (ii) We know that if $A \supset B$ and $B \supset A$, then $A = B$. So $A \le B$ and $B \le A$ then $A = B$.(iii) We know that if $C \supset B$ and $B \supset A$, then $C \supset A$. So, if $C \le B$ and $B \le A$, then $C \le A$. This means that $(J,\le)$ is a directed set and partially ordered. Let us consider the function $\ell : J \to X$ such that $\ell(A) \in A$ for every $A \in J$. Then $\{\ell_A\}_{A\in J}$ is a net in X and by the assumption has a b-θ-cluster point. Let y be the b-θ-cluster point of $\{\ell_A\}_{A\in J}$. We know that if $A \in J$ and $F_k \le A$, then $A \subset F_k$, where $F_k \in \{F_i\}_{i\in I}$. So $\ell_B \in F_k$ whenever $A \le B$. Then, $\{\ell_A\}_{A\in J}$ is residually in F_k. By proposition 3.7, since y is a b-θ-cluster point of $\{\ell_A\}_{A\in J}$, a subset of $\{\ell_A\}_{A\in J}$ b-θ-converges to y. Since $\{\ell_A\}_{A\in J}$ is residually in F_k for each k, such a subnet would be residually in F_k for each k. By Proposition 3.4, $y \in F_k$ for each k. So $\cap_{i\in I} F_i \neq \emptyset$. By Proposition 3.8, (X,τ) is b-θ-compact. $\square$

Proposition 3.10. *A topological space (X,τ) is b-θ-compact if and only if every ultranet in it is b-θ-convergent.*

Proof. Suppose (X,τ) is b-θ-compact and $\{\ell_i\}_{i\in I}$ is an ultranet in (X,τ). By Proposition 3.9, $\{\ell_i\}_{i\in I}$ has at least one b-θ-cluster point. From Proposition 3.6, $\{\ell_i\}_{i\in I}$ b-θ-converges to its b-θ-cluster point. Hence, $\{\ell_i\}_{i\in I}$ is b-θ-convergent. Conversely, assume that every ultra net in (X,τ) b-θ-convergent. Let $\{\ell_i\}_{i\in I}$ be a net in (X,τ). Since every net has a subnet which is an ultranet, so there exists a subnet of $\{\ell_i\}_{i\in I}$ which is an ultranet. This ultranet b-θ-converges to a point which is b-θ-cluster point of $\{\ell_i\}_{i\in I}$. $\square$

4. b-θ-(m,n)-Compact Spaces

Definition 4.1. A space (X,τ) is said to be b-θ-(m,n)-compact if from every b-θ-open covering $\{U_i : i \in I\}$ of X whose cardinality I, denoted by card I, is a at most n, one can select a subcovering $\{U_j : j \in J\}$ of X whose card J is at most m.

Definition 4.2. A subset A of a space (X,τ) is said to be a b-θ-(m,n)-compact subspace if the subspace A is b-θ-(m,n)-compact.

Definition 4.3. A space (X,τ) is said to be a completely b-θ-(m,n)-compact if every subspace X is b-θ-(m,n)-compact.

Remark 4.4. It should be noted that a b-θ-(1,n)-compact space is a b-θ-n-compact space and b-θ-$(1,\infty)$-compact space is the usual b-θ-compact space. A b-θ-(ω,∞)-compact space is a b-θ-Lindeloff space.

Definition 4.5. A family $\{U_i : i \in I\}$ of subsets of a set X is said to have the m-intersection property if every subfamily of cardinality at most m has a non-void intersection.

Theorem 4.6. *A space (X,τ) is b-θ-(m,n)-compact if and only if every family $\{P_i\}$ of b-θ-closed sets $P_i \subseteq X$ having the m-intersection property also has the n-intersection property.*

Proof . The proof is a consequence of the following equivalent statements:(i) X is b-θ-(m,n)-compact. (ii) If $\{U_i : i \in I\}$ is a b-θ-open cover of X such that card $I \leq n$, then there is a subcover $\{U_{i_j} : j \in J\}$ of X such that card $J \leq m$. (iii) If $\{U_i : i \in I\}$ is a b-θ-open sets such that card $I \leq n$ and every subfamily $\{U_{i_j}\}$ of card $J \leq m$ has the property $X \backslash (U_i U_{i_j}) \neq \emptyset$, then $X \backslash (U_{i \in I} U_{i_j}) \neq \emptyset$. (iv) If $\{U_i : i \in I\}$ is a family of b-θ-open sets such that $X \backslash (U_{j \in J} U_{i_j}) \neq \emptyset$ whenever card $J \leq m$, then $X \backslash (U_{j \in J} U_{i_j}) \neq \emptyset$ whenever card $J \leq n$. (v) If $\{P_i : i \in I\}$ is a family of b-θ-closed sets having the m-intersection property, then $\{P_i\}$ has also the n-intersection property. □

Theorem 4.7. *If a space X is b-θ-(m,n)-compact and if Y is a b-θ-closed subset of X, then Y is a b-θ-(m,n)-compact subspace of X.*

Proof . Suppose that $\{U_i : i \in I\}$ is a b-θ-open cover of Y such that card $I \leq n$. By adding $U_j = X \backslash Y$, we obtain a b-θ-open cover of X with cardinality at most n. By eliminating U_j, we have a subcover of $\{U_j\}$ whose cardinality is at most m. □

Theorem 4.8. *If X is a space such that every b-θ-open subset of X is a b-θ-(m, n)-compact subspace of X, then X is completely b-θ-(m, n)-compact.*

Proof. Let $Y \subset X$ and $\{U_i : i \in I\}$ be b-θ-open cover of Y such that card $I \leq n$. Then the family $\{U_i : i \in I\}$ is a b-θ-open cover of the b-θ-open set $\cup_i U_i$. Then there is a subfamily $\{U_{i_j} : j \in J\}$ of card $J \leq m$ which covers $\cup_i U_i$. This subfamily also covers the set Y and therefore Y is b-θ-(m, n)-compact. □

Theorem 4.9. *Let X be a topological space and $\{Y_k : k \in K\}$ be a family of subsets of X. If every Y_k is b-θ-(m, n)-compact for some $m \geq cardK$, then $U_{k\in K} Y_k$ is a b-θ-(m, n)-compact subspace of X.*

Proof. If $\{U_i : i \in I\}$ is a b-θ-open cover of $Y = \cup_K Y_k$, then it is a b-θ-open cover of Y_k for every $k \in K$. If card $I \leq n$, then $\{U_i\}$ contains a subfamily $\{U_{i_{j_k}} : j_k \in J_k\}$ for which card $J_k \leq m$ and is a covering of Y_k. The union of these families is a b-θ-open subfamily of $\{U_i\}$ which covers Y and its cardinality is at most m. □

Definition 4.10. A point $x \in X$ is said to be an m-b-θ-accumulation point of a set S in X if for every b-θ-open set U_x containing x, we have card $(U_x \cap S) > m$.

It should be noted that if $m = 0, 1$ or ω, then the relation card $(U_x \cap S) > m$ means that $U_x \cap S \neq \emptyset$, not finite or not countable.

Theorem 4.11. *Lex X be a topological space and $S \subset X$ and card $S > m$. If X is b-θ-(m, n)-compact for some $n > m$, then S has a b-θ-accumulation point in X. If X is b-θ-(m, ∞)-compact, then S has an m-b-θ-accumulation point in X.*

Proof. Let $S \subset X$ and S be the cardinality at most n which has no b-θ-accumulation points in X. Then for each $x \in X$, there is a b-open set U_x such that at most one point of S belong to U_x. Suppose U is the union of all sets U_x which contain no points of S. Let U_s denote the union of all sets U_x which contain the point $s \in S$. Then U and U_s are b-θ-open sets. Therefore $\{U, U_s\}$ is a b-θ-open cover of X of cardinality at most n. If X is b-θ-(m, n)-compact, then this cover contains a subcover of cardinality at most m. But this subcover must contain every U_s since $s \in S$ is covered only by U_s. Hence card $S \leq m$. If the cardinality of a set S is greater than m, then S has at least one b-θ-accumulation point in X. The two other cases can be proved similarly with a little modification. □

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