PROPERTIES OF $b$-$\theta$-COMPACT SPACES

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Abstract: In this paper, we present and study the notion of firm $b$-$\theta$-continuity to investigate $b$-$\theta$-compactness. We also present some properties of $b$-$\theta$-compactness in terms of nets and ultranets.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset $A$ of a topological space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure...
of A and the interior of A, respectively. A set A is called \( b \)-open \([1]\) (\( \gamma \)-open \([2]\)) if \( A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A)) \). The complement of \( b \)-open set is called \( b \)-closed. The intersection of \( b \)-closed sets of \( X \) containing A is called \( b \)-closure \([1]\) of A and is denoted by \( b \text{Cl}(A) \). A set A is \( b \)-closed if and only if \( A = b \text{Cl}(A) \). The \( b \)-\( \theta \)-closure \([3]\), denoted by \( b \text{Cl}_\theta(A) \), is the set of all \( x \in X \) such that \( b \text{Cl}(U) \cap A \neq \emptyset \) for every \( b \)-open set \( U \) containing \( x \). A subset \( A \) is called \( b \)-\( \theta \)-closed \([3]\) if \( A = b \text{Cl}_\theta(A) \). By \([3]\), it is proved that, for a subset \( A \), \( b \text{Cl}_\theta(A) \) is the intersection of all \( b \)-\( \theta \)-closed sets containing \( A \). The complement of a \( b \)-\( \theta \)-closed set is called \( b \)-\( \theta \)-open. The family of all \( b \)-\( \theta \)-open (resp. \( b \)-\( \theta \)-closed) sets of \( (X, \tau) \) is denoted by \( B\theta O(X, \tau) \) (resp. \( B\theta C(X, \tau) \)). In this paper, we present and study the notion of firm \( b \)-\( \theta \)-continuity to investigate \( b \)-\( \theta \)-compactness. We also present some properties of \( b \)-\( \theta \)-compactness in terms of nets and ultranets. Moreover, we introduce and investigate some basic properties of \( b \)-\( \theta \)-(\( m, n \))-compact spaces.

2. Characterization of \( b \)-\( \theta \)-Compact Spaces

**Definition 2.1.** A subset \( K \) of a nonempty set \( X \) is said to be \( b \)-\( \theta \)-compact relative to \( (X, \tau) \) if every cover of \( K \) by \( b \)-\( \theta \)-open sets of \( X \) has a finite subcover. We say that \( (X, \tau) \) is \( b \)-\( \theta \)-compact if \( X \) is \( b \)-\( \theta \)-compact.

**Definition 2.2.** A function \( f : X \to Y \) is said to have property \( \mathcal{P} \) if for every \( b \)-\( \theta \)-open cover \( \nabla \) of \( Y \) there exists a finite cover (the members of which need not be necessarily \( b \)-\( \theta \)-open) \( \{A_1, A_2, ..., A_n\} \) of \( X \) such that for each \( i \in \{1, 2, ..., n\} \), there exists \( U_i \in \nabla \) such that \( f(A_i) \subset U_i \).

Recall that a function \( f : X \to Y \) is said to be quasi \( b \)-\( \theta \)-continuous if \( f^{-1}(V) \) is \( b \)-\( \theta \)-open in \( X \) for every \( b \)-\( \theta \)-open set \( V \) of \( Y \).

**Theorem 2.3.** A topological space \( X \) is \( b \)-\( \theta \)-compact if and only if for every topological space \( Y \) and every quasi \( b \)-\( \theta \)-continuous function \( f : X \to Y \), \( f \) has the property \( \mathcal{P} \).

**Proof.** Let the topological space \( X \) be \( b \)-\( \theta \)-compact and the function \( f : X \to Y \) be quasi \( b \)-\( \theta \)-continuous. Suppose that \( \Omega \) be a \( b \)-\( \theta \)-open cover of \( Y \). The set \( f(X) \) is \( b \)-\( \theta \)-compact relative to \( Y \). This means that there exists a finite subfamily \( \{U_1, U_2, ..., U_n\} \) of \( \Omega \) which cover \( f(X) \). Then the sets \( A_1 = f^{-1}(U_1), A_2 = f^{-1}(U_2), ..., A_n = f^{-1}(U_n) \) from a cover of \( X \) such that \( f(A_i) \subset U_i \) for each \( i \in \{1, 2, ..., n\} \). Conversely, suppose that \( X \) is a topological space such that for every topological space \( Y \) and every quasi \( b \)-\( \theta \)-continuous function \( f : X \to Y \), \( f \) has property \( \mathcal{P} \). It follows that the identity function \( id_X : X \to X \) has the
property \( P \). Hence, for every \( b\)-\( \theta \)-open cover \( \Omega \) of \( X \), there exists a finite cover \( A_1, A_2, \ldots, A_n \) of \( X \) such that for each \( i \in \{1, 2, \ldots, n\} \) there exists a set \( U_i \in \Omega \) such that \( A_i = id_X(A_i) \subset U_i \). Then \( U_1, U_2, \ldots, U_n \) are finite \( b\)-\( \theta \)-subcover of \( \Omega \). Since \( \Omega \) was an arbitrary \( b\)-\( \theta \)-open cover of \( X \), the space \( X \) is \( b\)-\( \theta \)-compact.

**Theorem 2.4.** A function \( f : X \to Y \) is called firmly \( b\)-\( \theta \)-continuous if for every \( b\)-\( \theta \)-open cover \( \nabla \) of \( Y \) there exists a finite \( b\)-\( \theta \)-open cover \( \Omega \) of \( X \) such that for every \( U \in \Theta \), there exists a set \( G \in \Omega \) such that \( f(U) \subset G \).

**Remark 2.5.** It should be noted that if the topological space, then every quasi \( b\)-\( \theta \)-continuous function \( f : X \to Y \) is firmly \( b\)-\( \theta \)-continuous.

**Lemma 2.6.** Let \( X, Y, Z \) and \( W \) be topological spaces. Let \( g : X \to Y \) and \( h : Z \to W \) be quasi \( b\)-\( \theta \)-continuous functions and let \( f : Y \to Z \) be firmly \( b\)-\( \theta \)-continuous. Then the functions \( f \circ g : X \to Z \) and \( h \circ f : Y \to W \) are firmly \( b\)-\( \theta \)-continuous.

**Lemma 2.7.** Let \( X \) and \( Y \) be topological spaces. Suppose that \( f : X \to Y \) is a quasi \( b\)-\( \theta \)-continuous function which has the property \( P \). Then \( f \) is firmly \( b\)-\( \theta \)-continuous.

**Theorem 2.8.** The following statements are equivalent for a topological space \( (X, \tau) \):

(i) \( X \) is \( b\)-\( \theta \)-compact.

(ii) The identity function \( id_X : X \to X \) is firmly \( b\)-\( \theta \)-continuous;

(iii) Every quasi \( b\)-\( \theta \)-continuous function from \( X \) to \( X \) is firmly \( b\)-\( \theta \)-continuous;

(iv) Every quasi \( b\)-\( \theta \)-continuous function from \( X \) to a topological space \( Y \) is firmly \( b\)-\( \theta \)-continuous;

(v) Every quasi \( b\)-\( \theta \)-continuous function from \( X \) to a topological space \( Y \) has the property \( P \);

(vi) For each topological space \( Y \) and each quasi \( b\)-\( \theta \)-continuous function \( f : Y \to X \), \( f \) is firmly \( b\)-\( \theta \)-continuous.

**Proof.** (i)\( \Rightarrow \) (ii): Let \( X \) be a \( b\)-\( \theta \)-compact space. The identity function \( id_X : X \to X \) is quasi \( b\)-\( \theta \)-continuous and by Remark 2.5 \( id_X \) is firmly \( b\)-\( \theta \)-continuous.

(ii)\( \Rightarrow \) (iii): Let \( f : X \to X \) is any quasi \( b\)-\( \theta \)-continuous function. By (ii), the identity function \( id_X : X \to X \) is firmly \( b\)-\( \theta \)-continuous Therefore by Lemma 2.6 \( f = id_X : X \to X \) is firmly \( b\)-\( \theta \)-continuous.
(iii)⇒(iv): Suppose that \( f : X \to Y \) is any quasi \( b\theta \)-continuous function. The identity function \( id_X : X \to X \) is firmly \( b\theta \)-continuous and by (iii) \( id_X \) is firmly \( b\theta \)-continuous. As a consequence of Lemma 2.6, We have that \( f = f \circ id_X : X \to Y \) is firmly \( b\theta \)-continuous.

(iv)⇒(v): Obvious.

(v)⇒(i): This is an immediate consequence of Lemma 2.3.

(vi)⇒(ii): Suppose that \( id : X \to X \) is the identity function. Then \( id \) is quasi \( b\theta \)-continuous and by (vi) \( id \) is firmly \( b\theta \)-continuous.

(i)⇒(vi): Suppose that \( \nabla \) is \( b\theta \)-open cover of \( X \). Since \( X \) is \( b\theta \)-compact, then there is a finite \( b\theta \)-subcover \( U_1, U_2, \ldots, U_n \) of \( \nabla \). Let \( A_i = f^{-1}(U_i) \) for \( i \in \{1, 2, \ldots, n\} \). We have that \( f(A_i) \subset U_i \) for every \( i \in \{1, 2, \ldots, n\} \). Therefore, \( f \) is firmly \( b\theta \)-continuous. \( \Box \)

**Definition 2.9.** A topological space \( (X, \tau) \) is said to be \( b\theta-T_1 \) if for each pair of distinct points \( x \) and \( y \) of \( X \), there exists \( b\theta \)-open sets \( U \) and \( V \) of \( X \) such that \( x \in U \) and \( y \notin U \) and \( y \in V \) and \( x \notin V \).

**Theorem 2.10.** If \( f : X \to Y \) is a firmly \( b\theta \)-continuous function, where \( X \) is a topological space and \( Y \) is \( b\theta-T_1 \) space, then \( f \) is quasi \( b\theta \)-continuous.

**Proof.** Let \( x \) be an arbitrary point of \( X \) and \( V \) be a \( b\theta \)-open set of \( Y \) containing \( f(x) \). We define a \( b\theta \)-open cover \( \Omega \) of \( Y \) such that \( \Omega = \{V, Y \setminus f(x)\} \). Since \( f \) is firmly \( b\theta \)-continuous, it follows that there exists a finite \( b\theta \)-open cover \( \{P_1, P_2, \ldots, P_n\} \) of \( X \) such that \( f(P_i) \subset V \) or \( f(P_i) \subset Y \setminus f(x) \) for every \( i \in \{1, 2, \ldots, n\} \). Let \( x \in P_j \) for some index \( j \). Since \( f(P_j) \) contains \( f(x) \), so it follows that \( f(P_j) \subset V \). This shows that \( f \) is quasi \( b\theta \)-continuous. \( \Box \)

3. Properties of \( b\theta \)-Compact Spaces in Terms of Nets and Ultranets

**Definition 3.1.** Let \( (X, \tau) \) be a topological space, \( x \in X \) and \( \{x_\ell : \ell \in L\} \) be a net of \( X \). We say that a net \( \{x_\ell : \ell \in L\} \) \( b\theta \)-converges to \( x \) if for each \( b\theta \)-open set \( U \) containing \( x \), there exists an element \( \ell_0 \in L \) such that \( \ell \geq \ell_0 \) implies \( x_\ell \in U \).

**Definition 3.2.** Let \( (X, \tau) \) be a topological space, \( G = \{F_i : i \in I\} \) be a filterbase of \( X \) and \( x \in X \). A filterbase \( G \) is said to be \( b\theta \)-converges to \( x \) if there exists a member \( F_i \in G \) such that \( F_i \subseteq U \) for each \( b\theta \)-open set containing \( x \).
Proposition 3.3. If \( x \in U \) and \( U \in B\theta C(X, \tau) \), then there exists a net \( \{x_i\}_{i \in I} \) that \( \theta \)-converges to \( x \) and \( x_i \in U \) for each \( i \in I \).

Proof. Suppose that \( x \in U \) and \( U \in B\theta C(X, \tau) \) which means \( U = \text{b Cl}_\theta(U) \). This means that if \( x \in N \) and \( N \in B\theta O(X, \tau) \) then \( N \cap U \neq \emptyset \). It follows that there exists an element \( x_N \in N \cap U \). This implies that \( \{x_N\}_{N \in I} \) \( \theta \)-converges to \( x \).

\[ \square \]

Proposition 3.4. Let \( \{x_i\}_{i \in I} \) be a net in \( (X, \tau) \) and \( U \in B\theta C(X, \tau) \), where \( x_i \in U \) for each \( i \in I \). If \( \{x_i\}_{i \in I} \) \( \theta \)-converges to \( x \), then \( x \in U \).

Proof. Assume that \( \{x_i\}_{i \in I} \) \( \theta \)-converges to \( x \), then \( x \not\in U \). Then there exists a \( \theta \)-open set \( N \) such that \( x \in N \) and \( N \cap U = \emptyset \). This means that there exists \( i_0 \in I \) such that \( x_i \in N \) for each \( i \geq i_0 \). Then \( x_i \) is not an element of \( U \) for each \( i \geq i_0 \). But this is a contradiction and hence the result. \[ \square \]

Definition 3.5. A point \( y \) is a \( \theta \)-cluster point of \( \{x_i\}_{i \in I} \) if for each \( i_0 \in I \) and \( U \in B\theta O(X, \tau) \) such that \( y \in U \), there exists an \( i_1 \geq i_0 \) such that \( x_{i_1} \in U \).

Proposition 3.6. Let \( (\ell_i)_{i \in I} \) be an ultranet and \( y \) be a \( \theta \)-cluster point of the net. Then the ultranet \( (\ell_i)_{i \in I} \) \( \theta \)-converges to \( y \).

Proof. Suppose that \( (\ell_i)_{i \in I} \) is an ultranet in a topological space \( (X, \tau) \) and \( y \) is a \( \theta \)-cluster point of the net \( (\ell_i)_{i \in I} \). Suppose that, \( (\ell_i)_{i \in I} \) does not \( \theta \)-converges to \( y \). This means that there exists \( U \in B\theta O(X, \tau) \) such that \( y \in U \) and \( \ell_i \) is not an element of \( U \) for any \( i \in I \). So \( y \) is not a \( \theta \)-cluster point of \( (\ell_i)_{i \in I} \).

\[ \square \]

Proposition 3.7. Let \( (\ell_i)_{i \in I} \) be a net in a topological space \( (X, \tau) \). Then \( y \in X \) is a \( \theta \)-cluster point of \( (\ell_i)_{i \in I} \), if and only if a subnet of \( (\ell_i)_{i \in I} \) \( \theta \)-converges to \( y \).

Proof. Let \( (\ell_i)_{i \in I} \) have a subnet \( (\ell_{k_j})_{j \in J} \) that \( \theta \)-converges to \( y \) and \( J \) be a directed set. Now suppose that \( y \in X \) is not a \( \theta \)-cluster point of \( (\ell_i)_{i \in I} \). This means that there exists \( U \in B\theta O(X, \tau) \) and \( i_0 \in I \) such that \( s_{i_0} \) is not an element of \( U \) for every \( i_1 \geq i_0 \). Then \( (\ell_{k_j})_{j \in J} \) does not \( \theta \)-converges to \( y \). Conversely, assume that \( y \) is a \( \theta \)-cluster point of \( (\ell_i)_{i \in I} \). \( J = \{(i, U) : i \in I, y \in U, U \in B\theta O(X, \tau) \text{ and } \ell_i \in U\} \) is a partially ordered set where \( (i, U) \leq (i, V) \), if \( i \leq i_1 \) and \( V \subseteq U \). (i) \( (i, U) \leq (i, U) \) for every \( (i, U) \in J \). Because, \( i \leq i \) and \( U \subseteq U \) for every \( i \in I \) and \( U \in B\theta O(X, \tau) \). (ii) Let \( (i, U) \leq (i_1, V) \) and \( (i_1, V) \leq (i, U) \). Then, \( i \leq i_1, V \subseteq U \) and \( i_1 \leq i, U \subseteq V \). This follows that \( i = i_1, V = U \). Then \( (i_1, V) = (i, U) \). (iii) Let \( (i, U), (i_1, V) \) and \( (i_2, W) \in J \).
such that \((i, U) \leq (i_1, V)\) and \((i_1, V) \leq (i_2, W)\). Since \(I\) is a directed set, \(i \leq i_2\) where \(i \leq i_1\) and \(i_1 \leq i_2\). Also, we know that \(W \subset U\) where \(V \subset U\) and \(W \subset V\). Then \((i, U) \leq (i_2, W)\) where \(i \leq i_2\) and \(W \subset U\). Consequently, \(J\) is a partially ordered set. Now let \((i, U), (i_1, V) \in J\). Then \(U \cap V \in B\theta O(X, \tau)\). We know that \(U \cap V \subset U\) and \(U \cap V \subset V\) and \(y \in U \cap V\). Since \(y\) is a \(b\)-\(\theta\)-cluster point of \((\ell_i)_{i \in I}\), there exists \(i_2 \in I\) such that \(i \leq i_2\), \(i_1 \leq i_2\) and \(s_{i_2} \subset U \cap V\). Then \((i_1, V) \leq (i_2, U \cap V)\) and \((i, U) \leq (i_2, U \cap V)\). This means that \(J\) is a directed set. Define \(k : J \to I\) by \(k(i, A) = i\). (a) \((i, U) \leq (i_1, V)\) means that \(i \leq i_1\). Then \(k(i, U) \leq k(i_1, V)\). (b) Let \(i, i_1 \in I\) and \(U \in B\theta O(X, \tau)\) which contains \(y\). Then there exists \(i_2 \in I\) such that \(i \leq i_2\), \(i_1 \leq i_2\) and \(s_{i_2} \subset U\). This means that \((i_2, U) \in J\), \(i \leq k(i_2, U)\) and \(i_1 \leq k(i_1, U)\). This follows that \(\{k(i_1, U)\}_{i \in I}\). Consider the set \(U \in B\theta O(X, \tau)\) which contains \(y\). There exists \(i_0 \in I\) such that \(\ell_{i_0} \subset U\). (ii) For any \(i, V) \in J\) that \((i_0, U) \leq (i, V), V \subset U\) and \(\ell_i \subset V\). This follows that \(\ell \subset U\) for every \((i_0, U) \leq (i, V)\). So the subnet, \{\ell_{k(i, U)}\}_{i \in I}\), \(b\)-\(\theta\)-converges to \(y\).

**Proposition 3.8.** Let \((X, \tau)\) be topological space. Then the following statements are equivalent:

(i) \((X, \tau)\) is \(b\)-\(\theta\)-compact.

(ii) For any family \(\Psi\) of \(b\)-\(\theta\)-closed subsets of \(X\) such that \(\cap_{k \in \Psi} K = \emptyset\), there exists a finite subfamily \(\Phi \subset \Psi\) such that \(\cap_{L \in \Phi} L = \emptyset\).

(iii) \(\cap_{k \in \Psi} K \neq \emptyset\) for any family \(\Psi\) of \(b\)-\(\theta\)-closed subsets of \(X\) such that \(\cap_{L \in \Phi} L \neq \emptyset\) where \(\Phi \subset \Psi\) is a finite subfamily.

**Proof.** (i) \(\Rightarrow\) (ii): Let \((X, \tau)\) be \(b\)-\(\theta\)-compact and \(\Psi\) be a family of \(b\)-\(\theta\)-closed subsets such that \(\cap_{k \in \Psi} K = \emptyset\). Then \([\cap_{k \in \Psi} K]^c = [\emptyset]^c\). This means \(\cup_{k \in \Psi} K^c = X\). There exists a finite subfamily \(\Phi \subset \Psi\) such that \(\cap_{L \in \Phi} L = \emptyset\).

(ii) \(\Rightarrow\) (iii): Let \(\Psi\) be a family of \(b\)-\(\theta\)-closed subsets of \(X\). From the assumption if \(\cap_{k \in \Psi} K \neq \emptyset\), then there exists a finite subfamily \(\Phi \subset \Psi\) such that \(\cap_{L \in \Phi} L = \emptyset\). This means that if \(\Psi\) does not have any finite subfamily \(\Phi\) such that \(\cap_{L \in \Phi} L = \emptyset\), then \(\cap_{k \in \Psi} K = \emptyset\).

(iii) \(\Rightarrow\) (ii): Let \(\Psi\) be a family of \(b\)-\(\theta\)-closed subsets of \(X\). From the assumption if \(\cap_{L \in \Phi} L \neq \emptyset\) for any subfamily \(\Phi \subset \Psi\), then \(\cap_{k \in \Psi} K \neq \emptyset\). This means that, if \(\cap_{k \in \Psi} K = \emptyset\), then there exists at least one subfamily \(\Phi \subset \Psi\) such that \(\cap_{L \in \Phi} L = \emptyset\).

(ii) \(\Rightarrow\) (i): Let \(\{U_i\}_{i \in I}\) be a \(b\)-\(\theta\)-open cover of \(X\). Then, \(\cup_{i \in I} U_i = X\). This means that \(\cap_{i \in I} U_i^c = \emptyset\) and \(U_i^c \in B\theta C(X, \tau)\) for each \(i \in I\). It follows from the
assumption that there exists a finite subfamily \( J \subset I \) such that \( \cap_{j \in J} U_j^c = \emptyset \). So \( \cup_{j \in J} U_j = X \). Therefore \((X, \tau)\) is \(b\theta\)-compact.

**Proposition 3.9.** A topological space \((X, \tau)\) is \(b\theta\)-compact if and only if every net has at least one \(b\theta\)-cluster point in the topological space.

Proof. Let \((X, \tau)\) be \(b\theta\)-compact and \(\{x_i\}_{i \in I}\) be any net in this space. Let us consider a family \(bCl_\theta(B_j)\) of subsets, where \(B_j = \{x_i : j \leq i\}\). Then, \(bCl_\theta(B_j) \in B\theta C(X, \tau)\) for any \(j \in I\) and the intersection of finitely many of \(bCl_\theta(B_j)\) is nonempty. It follows from proposition 3.8 that \(\cap_{j \in J} bCl_\theta(B_j) \neq \emptyset\) for \((X, \tau)\) is \(b\theta\)-compact. Let \(y \in \cap_{j \in J} bCl_\theta(B_j)\). Then \(y \in bCl_\theta(B_j)\) for any \(j \in I\). Consider \(y \in U, U \in B\theta C(X, \tau)\) and \(r \in I\). Then \(U \cap B_r \neq \emptyset\). So \(U \cap B_k \neq \emptyset\) for any \(k \in I\) such that \(k \geq r\). Consequently \(y\) is a \(b\theta\)-cluster point of \(\{x_i\}_{i \in I}\). Now suppose that every net in \((X, \tau)\) has at least one \(b\theta\)-cluster point. Let \(\{F_i\}_{i \in I}\) be a family of \(b\theta\)-closed sets where intersection of finitely many of \(F_i\)'s is nonempty. Consider the set \(J = \{\cap_{j=1}^n G_i : \{G_i\}_{j=1}^n \subset \{F_i\}_{i \in I}\}\) and the relation \(\subseteq\), where \(A \subseteq B\) whenever \(B \subseteq A\) and \(A, B \in J\). (i) \(A \subseteq A\) for every set \(A \in J\). This means that \(A \subseteq A\) for every set \(A \in J\). (ii) We know that if \(A \supseteq B\) and \(B \supseteq A\), then \(A = B\). So \(A \subseteq B\) and \(B \subseteq A\) then \(A = B\). (iii) We know that if \(C \supseteq B\) and \(B \supseteq A\), then \(C \supseteq A\). So, if \(C \supseteq B\) and \(B \subseteq A\), then \(C \subseteq A\). This means that \((J, \subseteq)\) is a directed set and partially ordered. Let us consider the function \(\ell : J \to X\) such that \(\ell(A) \in A\) for every \(A \in J\). Then \(\{\ell(A)\} \cap J\) is a net in \(X\) and by the assumption has a \(b\theta\)-cluster point. Let \(y\) be the \(b\theta\)-cluster point of \(\{\ell(A)\} \cap J\). We know that if \(A \in J\) and \(F_k \subseteq A, \text{ then } A \subseteq F_k\), where \(F_k \in \{F_i\}_{i \in I}\). So \(\ell_B \in F_k\) whenever \(A \subseteq B\). Then, \(\{\ell(A)\} \cap J\) is residually in \(F_k\). By proposition 3.7, since \(y\) is a \(b\theta\)-cluster point of \(\{\ell(A)\} \cap J\), a subset of \(\{\ell(A)\} \cap J\) \(b\theta\)-converges to \(y\). Since \(\{\ell(A)\} \cap J\) is residually in \(F_k\) for each \(k\), such a subnet would be residually in \(F_k\) for each \(k\). By Proposition 3.4, \(y \in F_k\) for each \(k\). So \(\cap_{i \in I} F_i \neq \emptyset\). By Proposition 3.8, \((X, \tau)\) is \(b\theta\)-compact.

**Proposition 3.10.** A topological space \((X, \tau)\) is \(b\theta\)-compact if and only if every ultra net in it is \(b\theta\)-convergent.

Proof. Suppose \((X, \tau)\) is \(b\theta\)-compact and \(\{\ell_i\}_{i \in I}\) is an ultra net in \((X, \tau)\). By Proposition 3.9, \(\{\ell_i\}_{i \in I}\) has at least one \(b\theta\)-cluster point. From Proposition 3.6, \(\{\ell_i\}_{i \in I}\) \(b\theta\)-converges to its \(b\theta\)-cluster point. Hence, \(\{\ell_i\}_{i \in I}\) is \(b\theta\)-convergent. Conversely, assume that every ultra net in \((X, \tau)\) \(b\theta\)-convergent. Let \(\{\ell_i\}_{i \in I}\) be a net in \((X, \tau)\). Since every net has a subnet which is an ultra net, so there exists a subnet of \(\{\ell_i\}_{i \in I}\) which is an ultra net. This ultra net \(b\theta\)-converges to a point which is \(b\theta\)-cluster point of \(\{\ell_i\}_{i \in I}\).
4. $b$-$\theta$-($m, n$)-Compact Spaces

Definition 4.1. A space $(X, \tau)$ is said to be $b$-$\theta$-($m, n$)-compact if from every $b$-$\theta$-open covering $\{U_i : i \in I\}$ of $X$ whose cardinality $I$, denoted by $\text{card} \ I$, is at most $n$, one can select a subcovering $\{U_j : j \in J\}$ of $X$ whose card $J$ is at most $m$.

Definition 4.2. A subset $A$ of a space $(X, \tau)$ is said to be $b$-$\theta$-($m, n$)-compact subspace if the subspace $A$ is $b$-$\theta$-($m, n$)-compact.

Definition 4.3. A space $(X, \tau)$ is said to be a completely $b$-$\theta$-($m, n$)-compact if every subspace $X$ is $b$-$\theta$-($m, n$)-compact.

Remark 4.4. It should be noted that a $b$-$\theta$-($1, n$)-compact space is a $b$-$\theta$-($n$)-compact space and $b$-$\theta$-($1, \infty$)-compact space is the usual $b$-$\theta$-compact space. A $b$-$\theta$-($\omega, \infty$)-compact space is a $b$-$\theta$-Lindeloff space.

Definition 4.5. A family $\{U_i : i \in I\}$ of subsets of a set $X$ is said to have the $m$-intersection property if every subfamily of cardinality at most $m$ has a non-void intersection.

Theorem 4.6. A space $(X, \tau)$ is $b$-$\theta$-($m, n$)-compact if and only if every family $\{P_i\}$ of $b$-$\theta$-closed sets $P_i \subseteq X$ having the $m$-intersection property also has the $n$-intersection property.

Proof. The proof is a consequence of the following equivalent statements: (i) $X$ is $b$-$\theta$-($m, n$)-compact. (ii) If $\{U_i : i \in I\}$ is a $b$-$\theta$-open cover of $X$ such that $\text{card} \ I \leq n$, then there is a subcover $\{U_{ij} : j \in J\}$ of $X$ such that $\text{card} \ J \leq m$. (iii) If $\{U_i : i \in I\}$ is a $b$-$\theta$-open sets such that $\text{card} \ I \leq n$ and every subfamily $\{U_{ij}\}$ of cardinality $\text{card} \ J \leq m$ has the property $X \setminus (U_{ij} \cup U_{ij}) \neq \emptyset$, then $X \setminus (U_{i\in I} U_{ij}) \neq \emptyset$. (iv) If $\{U_i : i \in I\}$ is a family of $b$-$\theta$-open sets such that $X \setminus (U_{j\in J} U_{ij}) \neq \emptyset$ whenever $\text{card} \ J \leq m$, then $X \setminus (U_{j\in J} U_{ij}) \neq \emptyset$ whenever $\text{card} \ J \leq n$. (v) If $\{P_i : i \in I\}$ is a family of $b$-$\theta$-closed sets having the $m$-intersection property, then $\{P_i\}$ has also the $n$-intersection property. □

Theorem 4.7. If a space $X$ is $b$-$\theta$-($m, n$)-compact and if $Y$ is a $b$-$\theta$-closed subset of $X$, then $Y$ is a $b$-$\theta$-($m, n$)-compact subspace of $X$.

Proof. Suppose that $\{U_i : i \in I\}$ is a $b$-$\theta$-open cover of $Y$ such that $\text{card} \ I \leq n$. By adding $U_j = X \setminus Y$, we obtain a $b$-$\theta$-open cover of $X$ with cardinality at most $n$. By eliminating $U_j$, we have a subcover of $\{U_j\}$ whose cardinality is at most $m$. □
Theorem 4.8. If $X$ is a space such that every $b$-$\theta$-open subset of $X$ is a $b$-$\theta$-$(m,n)$-compact subspace of $X$, then $X$ is completely $b$-$\theta$-$(m,n)$-compact.

Proof. Let $Y \subset X$ and $\{U_i : i \in I\}$ be $b$-$\theta$-open cover of $Y$ such that card $I \leq n$. Then the family $\{U_i : i \in I\}$ is a $b$-$\theta$-open cover of the $b$-$\theta$-open set $\cup_i U_i$. Then there is a subfamily $\{U_{i_j} : j \in J\}$ of card $J \leq m$ which covers $\cup_i U_i$. This subfamily also covers the set $Y$ and therefore $Y$ is $b$-$\theta$-$(m,n)$-compact. \hfill $\blacksquare$

Theorem 4.9. Let $X$ be a topological space and $\{Y_k : k \in K\}$ be a family of subsets of $X$. If every $Y_k$ is $b$-$\theta$-$(m,n)$-compact for some $m \geq \text{card} K$, then $U_{k \in K} Y_k$ is a $b$-$\theta$-$(m,n)$-compact subspace of $X$.

Proof. If $\{U_i : i \in I\}$ is a $b$-$\theta$-open cover of $Y = \cup K Y_k$, then it is a $b$-$\theta$-open cover of $Y_k$ for every $k \in K$. If card $I \leq n$, then $\{U_i\}$ contains a subfamily $\{U_{i_{k_j}} : j_k \in J_k\}$ for which card $J_k \leq m$ and is a covering of $Y_k$. The union of these families is a $b$-$\theta$-open subfamily of $\{U_i\}$ which covers $Y$ and its cardinality is at most $m$. \hfill $\blacksquare$

Definition 4.10. A point $x \in X$ is said to be an $m$-$b$-$\theta$-accumulation point of a set $S$ in $X$ if for every $b$-$\theta$-open set $U_x$ containing $x$, we have card $(U_x \cap S) > m$.

It should be noted that if $m = 0, 1$ or $\omega$, then the relation card $(U_x \cap S) > m$ means that $U_x \cap S \neq \emptyset$, not finite or not countable.

Theorem 4.11. Lex $X$ be a topological space and $S \subset X$ and card $S > m$. If $X$ is $b$-$\theta$-$(m,n)$-compact for some $n > m$, then $S$ has a $b$-$\theta$-accumulation point in $X$. If $X$ is $b$-$\theta$-$(m,\infty)$-compact, then $S$ has an $m$-$b$-$\theta$-accumulation point in $X$.

Proof. Let $S \subset X$ and $S$ be the cardinality at most $n$ which has no $b$-$\theta$-accumulation points in $X$. Then for each $x \in X$, there is a $b$-open set $U_x$ such that at most one point of $S$ belong to $U_x$. Suppose $U$ is the union of all sets $U_x$ which contain no points of $S$. Let $U_s$ denote the union of all sets $U_x$ which contain the point $s \in S$. Then $U$ and $U_s$ are $b$-$\theta$-open sets. Therefore $\{U, U_s\}$ is a $b$-$\theta$-open cover of $X$ of cardinality at most $n$. If $X$ is $b$-$\theta$-$(m,n)$-compact, then this cover contains a subcover of cardinality at most $m$. But this subcover must contain every $U_s$ since $s \in S$ is covered only by $U_s$. Hence card $S \leq m$. If the cardinality of a set $S$ is greater than $m$, then $S$ has at least one $b$-$\theta$-accumulation point in $X$. The two other cases can be proved similarly with a little modification. \hfill $\blacksquare$
References

