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SOME INTERESTING RESULTS ON 7-CORE PARTITIONS

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Abstract: If $p \geq 5$ is a prime with -7 is a quadratic non-residue modulo p then for any non-negative integers n, 7-core partitions of n satisfy several interesting results, for example

$$a_7(\Big(49 \cdot p^{2k}n + 7(r+1)p^{2k} - 2\Big) = 49a_7(\Big(7 \cdot p^{2k}n + (r+1)p^{2k} - 2\Big)$$

where $r \in \{2, 4, 5\}$ and $a_7(n)$ denotes the number of 7-cores of n.

Key Words: t-core partition, theta function, dissection, congruence

1. Introduction

Throughout this paper, we assume that for any complex number a and $|q| < 1$,

$$(a;q)_\infty := \prod_{n=1}^{\infty} (1 - aq^{n-1}).$$

A partition of n is called a $t-$ core of n if none of the hook numbers is a multiple of t; or a partition λ is said to be a t-core if and only if it has no hook numbers that are multiples of t. If $a_t(n)$ denotes the number of partitions of n that are t-cores, then the generating function for $a_t(n)$ is given by [6, Equation 2.1]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1}$$

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In particular, for $t = 7$,

$$\sum_{n=0}^{\infty} a_7(n)q^n = \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}. \tag{2}$$

By using the theory of modular forms, Radu and Sellers [10] showed that

$$a_7(14n+7,9,13) \equiv 0 \pmod{8}.$$

In this paper, we prove these congruences by using Ramanujan's theta functions. Also, we prove the following new results for 7-core partitions by employing Ramanujan's theta functions and their dissections.

Theorem 1.1. *If $p \geq 5$ is a prime with $\left(\frac{-7}{p}\right) = -1$, then for non-negative integers n and for $k \geq 1$,*

$$\sum_{n=0}^{\infty} a_7\left(7\cdot p^{2k}n + 7\cdot p^{2k} - 2\right)q^n - 49\sum_{n=0}^{\infty} a_7\left(p^{2k}n + p^{2k} - 2\right)q^n = 7p^{2k} f_7^3 f_1^3. \tag{3}$$

Corollary 1.2. *If $p \geq 5$ is a prime with $\left(\frac{-7}{p}\right) = -1$, then for non-negative integers n and for $k \geq 1$,*

$$a_7\left(7\cdot p^{2k}n + 7\cdot p^{2k} - 2\right) \equiv 49\sum_{n=0}^{\infty} a_7\left(p^{2k}n + p^{2k} - 2\right) \ (mod\ p^{2k}). \tag{4}$$

Theorem 1.3. *For any prime $p \geq 5$ with $\left(\frac{-7}{p}\right) = -1$, then for all non-negative integers n,*

$$a_7(\left(49\cdot p^{2k}n + 7(r+1)p^{2k} - 2\right) = 49a_7(\left(7\cdot p^{2k}n + (r+1)p^{2k} - 2\right), \tag{5}$$

where $r \in \{2,4,5\}$.

In the next section we present background material on Ramanujan's theta functions and some preliminary lemmas.

2. Background

For $|ab| < 1$, Ramanujan's general theta-function $f(a, b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

In this notation, Jacobi's famous triple product identity [2, p. 35, Entry 19] takes the form

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{6}$$

One important special case of the above is

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{7}$$

The above equality is Euler's famous pentagonal number theorem. Throughout our discussion, we have taken $f_k := (q^k, q^k)_\infty$.

We will also need the following results.

Lemma 2.1. *[11, Lemma 1.2] Let p be a prime and α a positive integer. Then*

$$(q;q)_\infty^{p^\alpha} \equiv (q^p;q^p)_\infty^{p^{\alpha-1}} \pmod{p^\alpha}. \tag{8}$$

Lemma 2.2. *[12] For any prime $p \geq 5$, we have*

$$f^3(-q) = \sum_{\substack{k=0 \\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\frac{pn+2k+1}{2}} + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} f^3(-q^{p^2}), \tag{9}$$

Furthermore, if $k \neq \dfrac{p-1}{2}$, $0 \leq k \leq p-1$, then

$$\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}.$$

F. G. Garvan [6] defined an operator H_i, $0 \leq i \leq 6$, which act on a series of powers of q and simply pick out those terms in which the power of q is congruent to i modulo 7.

$$\xi(q) := \frac{(q;q)_\infty}{q^2(q^{49};q^{49})_\infty}, \quad T(q) := \frac{(q^7;q^7)_\infty^4}{q^7(q^{49};q^{49})_\infty^4}, \tag{10}$$

Garvan found the following result.

Lemma 2.3. *[6, Lemma 3.13] If ξ is as defined above, then*

$$H_3(\xi^3) = 0, \quad H_5(\xi^3) = 0, \quad H_6(\xi^3) = 0. \tag{11}$$

3. Main Results

Theorem 3.1. *If $p \geq 5$ is a prime with $\left(\frac{-7}{p}\right) = -1$, then for non-negative integers n and for $k \geq 1$,*

$$\sum_{n=0}^{\infty} a_7\left(7 \cdot p^{2k}n + 7 \cdot p^{2k} - 2\right) q^n - 49 \sum_{n=0}^{\infty} a_7\left(p^{2k}n + p^{2k} - 2\right) q^n = 7p^{2k} f_7^3 f_1^3. \tag{12}$$

Proof. We note that

$$\sum_{n=0}^{\infty} a_7(n)q^n = \frac{(q^7;q^7)_{\infty}^7}{(q;q)_{\infty}} = (q^7;q^7)_{\infty}^7 \sum_{n=0}^{\infty} p(n). \tag{13}$$

From [3, p. 40, Theorem 2.4.2], we have

$$\sum_{n=0}^{\infty} p\left(7n+5\right) q^n = 7\frac{(q^7;q^7)_{\infty}^3}{(q;q)_{\infty}^4} + 49q\frac{(q^7;q^7)_{\infty}^7}{(q;q)_{\infty}}. \tag{14}$$

Applying (14) in (13), we have

$$\sum_{n=0}^{\infty} a_7\left(7n+5\right) q^n = 7(q^7;q^7)_{\infty}^3 (q;q)_{\infty}^3 + 49q\frac{(q^7;q^7)_{\infty}^7}{(q;q)_{\infty}}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_7\left(7n+5\right) q^n - 49\sum_{n=0}^{\infty} a_7\left(n\right) q^{n+1} = 7(q^7;q^7)_{\infty}^3 (q;q)_{\infty}^3. \tag{15}$$

Applying p-dissection of $(q;q)_{\infty}^3$ from (9) in (15) and then consider the congruence

$$\frac{(k^2+k)}{2} + 7 \cdot \frac{(m^2+m)}{2} \equiv 8 \cdot \frac{(p^2-1)}{8} \pmod{p}. \tag{16}$$

Since the above congruence is equivalent to

$$(2k+1)^2 + 7\cdot(2m+1)^2 \equiv 0 \pmod{p},$$

and $\left(\dfrac{-7}{p}\right) = -1$, it follows that (16) has only one solution, namely $k = m = (p-1)/2$. Therefore, equating the terms involving q^{pn+p^2-1} from both sides of (15), we deduce that

$$\sum_{n=0}^{\infty} a_7\left(7\cdot pn + 7\cdot p^2 - 2\right)q^n - 49\sum_{n=0}^{\infty} a_7\left(pn + p^2 - 2\right)q^n = 7p^2 f_{7p}^3 f_p^3, \quad (17)$$

which can be written as

$$\sum_{n=0}^{\infty} a_7\left(7p^2n + 7p^2 - 2\right)q^n - 49\sum_{n=0}^{\infty} a_7\left(p^2n + p^2 - 2\right)q^n$$
$$= 7p^2(q;q)_\infty^3(q^7;q^7)_\infty^3. \quad (18)$$

This is the case $k = 1$.

Now suppose the result is true for all $k \geq 1$. i.e.,

$$\sum_{n=0}^{\infty} a_7\left(7\cdot p^{2k}n + 7\cdot p^{2k} - 2\right)q^n - 49\sum_{n=0}^{\infty} a_7\left(p^{2k}n + p^{2k} - 2\right)q^n$$
$$= 7p^{2k} f_7^3 f_1^3. \quad (19)$$

Again, applying p-dissection of $(q;q)_\infty^3$ from (9) in the above equation and extracting the terms involving $q^{pn+(p^2-1)}$ from both sides, we can easily obtain,

$$\sum_{n=0}^{\infty} a_7\left(7\cdot p^{2k}(pn + p^2 - 1) + 7\cdot p^{2k} - 2\right)q^n -$$
$$49\sum_{n=0}^{\infty} a_7\left(p^{2k}(pn + p^2 - 1) + p^{2k} - 2\right)q^n$$
$$= 7p^{2k+2} f_{7p}^3 f_p^3. \quad (20)$$

Extracting terms involving q^{pn} from both sides of of the above equation and replacing q^p by q we obtain,

$$\sum_{n=0}^{\infty} a_7\left(7\cdot p^{2k+2} + 7\cdot p^{2k+2} - 2\right)q^n - 49\sum_{n=0}^{\infty} a_7\left(p^{2k+2} + p^{2k+2} - 2\right)q^n$$

$$= 7p^{2k+2} f_7^3 f_1^3, \quad (21)$$

which is the case for $k+1$ of (12). Hence, by mathematical induction, we can conclude the that (12) is true for all non-negative integers n and k. □

From the above result, we can easily obtain the following one.

Corollary 3.2. *If $p \geq 5$ is a prime with $\left(\frac{-7}{p}\right) = -1$, then for non-negative integers n and for $k \geq 1$,*

$$a_7\left(7 \cdot p^{2k} n + 7 \cdot p^{2k} - 2\right) \equiv 49 \sum_{n=0}^{\infty} a_7\left(p^{2k} n + p^{2k} - 2\right) \pmod{p^{2k}}. \quad (22)$$

Theorem 3.3. *For any prime $p \geq 5$ with $\left(\frac{-7}{p}\right) = -1$, then for all non-negative integers n,*

$$a_7(\left(49 \cdot p^{2k} n + 7(r+1)p^{2k} - 2\right) = 49a_7(\left(7 \cdot p^{2k} n + (r+1)p^{2k} - 2\right), \quad (23)$$

where $r \in \{2, 4, 5\}$.

Proof. Applying (10) in (12), we find that

$$\sum_{n=0}^{\infty} a_7\left(7 \cdot p^{2k} n + 7 \cdot p^{2k} - 2\right) q^n - 49 \sum_{n=0}^{\infty} a_7\left(p^{2k} n + p^{2k} - 2\right) q^n$$
$$= 7q^6 p^{2k} (q^7; q^7)_\infty^3 (q^{49}; q^{49})_\infty^3 \xi^3. \quad (24)$$

Extracting the terms containing q^{7n+r}, where $r \in \{2, 4, 5\}$ we have

$$\sum_{n=0}^{\infty} a_7\left(7 \cdot p^{2k} n + 7 \cdot p^{2k} - 2\right) q^{7n+r} - 49 \sum_{n=0}^{\infty} a_7\left(p^{2k} n + p^{2k} - 2\right) q^{7n+r}$$
$$= 7q^6 p^{2k} (q^7; q^7)_\infty^3 (q^{49}; q^{49})_\infty^3 H_{r+1}(\xi^3). \quad (25)$$

Employing (11) in the above equation, we obtain

$$\sum_{n=0}^{\infty} a_7\left(7 \cdot p^{2k}(7n+r) + 7 \cdot p^{2k} - 2\right) q^{7n+r}$$
$$- 49 \sum_{n=0}^{\infty} a_7\left(p^{2k}(7n+r) + p^{2k} - 2\right) q^{7n+r} = 0. \quad (26)$$

Replacing q^{7n+r} by q^n, we obtain

$$a_7\left(49\cdot p^{2k}n+7\cdot p^{2k}(r+1)-2\right)$$
$$=49a_7\left(7\cdot p^{2k}n+p^{2k}(r+1)-2\right). \quad \square \quad (27)$$

The following result was proved by Radu and Sellers by using the theory of modular forms. Here, we prove the result by using the theory of theta functions.

Theorem 3.4. *[10, Theorem 1.4, equation 1.9] For any $n\geq 0$, we have*

$$a_7\left(14n+7,9,13\right)\equiv 0(\bmod\ 8). \quad (28)$$

Proof. We can write

$$\sum_{n=0}^{\infty}a_7(n)q^n=\frac{(q^7;q^7)_\infty^8}{(q;q)_\infty(q^7;q^7)_\infty}. \quad (29)$$

Taking $p=2$ and $\alpha=3$ in (11), we obtain,

$$(q;q)_\infty^8\equiv(q^2;q^2)_\infty^4\ (\bmod\ 8). \quad (30)$$

Replacing q by q^7 in the above congruence and applying in (29), we obtain

$$\sum_{n=0}^{\infty}a_7(n)q^n\equiv\frac{(q^{14};q^{14})_\infty^4}{(q;q)_\infty(q^7;q^7)_\infty}\ (\bmod\ 8). \quad (31)$$

From Baruah and Ojah's paper [1, Eq. 1.7], we recall that

$$\sum_{n=0}^{\infty}p_{[1^17^1]}(2n+1)q^n=\frac{(q^2;q^2)_\infty^2(q^{14};q^{14})_\infty^2}{(q;q)_\infty^3(q^7;q^7)_\infty^3}, \quad (32)$$

where $p_{[1^17^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty}p_{[1^17^1]}(n)q^n:=\frac{1}{(q;q)_\infty(q^7;q^7)_\infty}.$$

Extracting the terms involving q^{2n+1} from both sides of (31), employing (32), and replacing q^2 by q yields

$$\sum_{n=0}^{\infty}a_7(2n+1)q^n\equiv\frac{(q^7;q^7)_\infty(q^2;q^2)_\infty^2(q^{14};q^{14})_\infty^2}{(q;q)_\infty^3}\ (\bmod\ 8)$$

$$\equiv \frac{2(q^7;q^7)_\infty (q^2;q^2)^2_\infty (q^{14};q^{14})^2_\infty}{2(q;q)^3_\infty} \pmod 8. \tag{33}$$

Again, taking $p = 2$ and $\alpha = 2$ in (11),

$$(q;q)^4_\infty \equiv (q^2;q^2)^2_\infty \pmod 4$$
$$2(q;q)^4_\infty \equiv 2(q^2;q^2)^2_\infty \pmod 8. \tag{34}$$

Employing (34) in (33), we obtain

$$\sum_{n=0}^{\infty} a_7(2n+1)q^n \equiv (q^7;q^7)_\infty (q^{14};q^{14})^2_\infty (q;q)_\infty \pmod 8. \tag{35}$$

From [2, p. 303, Entry 17(v)], we recall that

$$(q;q)_\infty = (q^{49};q^{49})_\infty \left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)} \right), \tag{36}$$

where

$$A(q) := \frac{f(-q^3,-q^4)}{f(-q^2)},\ B(q) := \frac{f(-q^2,-q^5)}{f(-q^2)} \text{ and } C(q) := \frac{f(-q,-q^6)}{f(-q^2)}.$$

Applying (36) in (35) and extracting terms involving q^{7n+r}, where $r = 3, 4, 6$ we can easily arrive at (28). □

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