ON SUBMANIFOLDS OF GENERALIZED RECURRENT MANIFOLDS

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Abstract: In this paper we investigate some properties of submanifolds of generalized recurrent manifold (briefly, $GRM_n$). Firstly, we show that a totally geodesic hypersurface of $GRM_n$ is a $GRM_{n-1}$. Secondly, if $GRM_n$ is a Riemannian product manifold, then either one decomposition manifold is locally symmetric or the other decomposition manifold is a space of constant curvature. Thirdly, a Riemannian product manifold of a space of constant curvature with itself is a $GRM_n$.

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1. Introduction

As a generalization of the notion of a space of constant curvature, the notion of (locally) symmetric manifolds was introduced by Cartan [4] who obtained a classification of such manifolds. During the last six decades, the notion of
locally symmetric manifolds was weakened by many authors in various ways to different extent such as recurrent manifolds [10], conformally recurrent manifolds [1], 2-recurrent manifolds [7], Ricci recurrent manifolds [9], concircularly recurrent manifolds [8] and projectively recurrent manifolds [2]. As an extending notion of recurrent manifolds, the notion of generalized recurrent manifolds was introduced by Dubey [6] and this manifold has received a great deal of attention. In [3] Arslan et al studied such a manifold in considerable detail. A Riemannian manifold $(M^n, g)$ $(n \geq 3)$ is said to be generalized recurrent if its curvature tensor $R$ of type $(0,4)$ satisfies the condition

$$\langle \nabla_X R \rangle(Y, Z, V, W) = A(X)R(Y, Z, V, W) + B(X)(g \bullet g)(Y, Z, V, W), \quad (1.1)$$

where $\nabla$ denotes the Levi-Civita connection and $A, B$ are the associated 1-forms. Here the symbol $\bullet$ is the Nomizu-Kulkarni product of symmetric $(0,2)$-tensors generating a curvature type tensor:

$$(h \bullet k)(X, Y, Z, W) = h(X, Z)k(Y, W) + h(Y, W)k(X, Z)$$

$$- h(X, W)k(Y, Z) - h(Y, Z)k(X, W).$$

From now on, in this paper, an $n$-dimensional generalized recurrent manifold is denoted by $GRM_n$. In particular, if $B = 0$ in (1.1), then the manifold reduces to a recurrent manifold. The purpose of this paper is to investigate some properties of a Riemannian product $GRM_n$ and hypersurfaces of $GRM_n$.

### 2. Preliminaries

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; y^\alpha\}$ and $(\bar{M}^{n-1}, \bar{g})$ a hypersurface of $(M^n, g)$ covered by a system of coordinate neighborhoods $\{V; x^i\}$. Let $y^\alpha = y^\alpha(x^i)$ be the parametric representation of the hypersurface $\bar{M}^{n-1}$ in $M^n$, where Greek indices take the values $1, 2, ..., n$ and Latin indices take the values $1, 2, ..., n - 1$. Then we have

$$\bar{g}_{ij} = g_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Here we adopt the Einstein convention, that is, when an index variable appears once in an upper and once in a lower position in a term, it implies summation of that term over all the values of the index. Let $N^\alpha$ be a local unit normal to
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\( \bar{M}^{n-1}, \bar{g} \). Then we have the relations

\[
g_{\alpha\beta}N^{\alpha}\frac{\partial y^{\beta}}{\partial x^{j}} = 0, g_{\alpha\beta}N^{\alpha}N^{\beta} = 1, g^{\alpha\beta} = \bar{g}^{ij}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}} + N^{\alpha}N^{\beta}.
\]

The structure equations of Gauss and Codazzi for a hypersurface \((\bar{M}^{n-1}, \bar{g})\) of \((M^n, g)\) can be respectively written as

\[
\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}\frac{\partial y^{\delta}}{\partial x^{l}} + \bar{\omega}_{ik}\bar{\omega}_{jl} - \bar{\omega}_{ik}\bar{\omega}_{lj},
\]

\[
R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}N^{\delta} = \bar{\omega}_{jk;i} - \bar{\omega}_{ik;j},
\]

where \(\bar{R}_{ijkl}\) and \(R_{\alpha\beta\gamma\delta}\) are the curvature tensors of \((\bar{M}^{n-1}, \bar{g})\) and \((M^n, g)\) respectively, and \(\bar{\omega}_{ij}\) is the second fundamental form of \((\bar{M}^{n-1}, \bar{g})\).

The hypersurface \((\bar{M}^{n-1}, \bar{g})\) is said to be a totally umbilic hypersurface of \((M^n, g)\) [5] if its second fundamental form \(\bar{\omega}_{ij}\) satisfies

\[
\bar{\omega}_{ij} = H\bar{g}_{ij}, \left(\frac{\partial y^{\alpha}}{\partial x^{i}}\right)_{;j} = \bar{g}_{ij}HN^{\alpha},
\]

where \(H\) denotes the mean curvature of \((\bar{M}^{n-1}, \bar{g})\) defined by \(H = \frac{1}{n-1}\bar{g}^{ij}\bar{\omega}_{ij}\), and semicolon ”;” indicates covariant differentiation. In particular, if \(H=0\), then the totally umbilic hypersurface \((\bar{M}^{n-1}, \bar{g})\) is called a totally geodesic hypersurface of \((M^n, g)\) [5]. The equations of Weingarten, Gauss and Codazzi for a totally umbilic hypersurface \((\bar{M}^{n-1}, \bar{g})\) of \((M^n, g)\) are respectively obtained as

\[
N^{\alpha}_{;i} = -H\frac{\partial y^{\alpha}}{\partial x^{i}},
\]

\[
\bar{R}_{ijkl} = R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}\frac{\partial y^{\delta}}{\partial x^{l}} + H^2(\bar{g}_{ij}\bar{g}_{jk} - \bar{g}_{ik}\bar{g}_{jl}),
\]

\[
R_{\alpha\beta\gamma\delta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}}\frac{\partial y^{\gamma}}{\partial x^{k}}N^{\delta} = H_{;i}\bar{g}_{jk} - H_{;j}\bar{g}_{ik}.
\]

3. Hypersurfaces of \(GRM_n\)

In this section we deal with some hypersurfaces of \(GRM_n\). At first, concerning the covariant derivative of curvature tensor, we obtain
Lemma 3.1. Let \((M^{n-1}, \tilde{g})\) be a totally umbilic hypersurface of \((M^n, g)\). Then we have

\[
\tilde{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \frac{\partial}{\partial x^p} + HH;\alpha(i \tilde{g}_{lp} g_{jk} - g_{kp} \tilde{g}_{jl}) + HH;j(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;k(\tilde{g}_{jp} g_{li} - \tilde{g}_{lp} \tilde{g}_{lj}) + HH;l(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + 2HH;p(\tilde{g}_{il} g_{jk} - \tilde{g}_{ik} \tilde{g}_{jl}).
\]

Proof. Differentiating (2.6) covariantly, we have

\[
\tilde{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta}(\frac{\partial y^\alpha}{\partial x^i})_{;p} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + R_{\alpha\beta\gamma\delta}(\frac{\partial y^\alpha}{\partial x^i})_{;p} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \frac{\partial}{\partial x^p} + HH;\alpha(i \tilde{g}_{lp} g_{jk} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;j(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;k(\tilde{g}_{jp} g_{li} - \tilde{g}_{lp} \tilde{g}_{lj}) + HH;l(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + 2HH;p(\tilde{g}_{il} g_{jk} - \tilde{g}_{ik} \tilde{g}_{jl}).
\]

By virtue of (2.4) and the last relation, we obtain

\[
\tilde{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \tilde{g}_{ip} HR_{\alpha\beta\gamma\delta} N^\alpha \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \tilde{g}_{jp} HR_{\alpha\beta\gamma\delta} N^\alpha \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + HH;\alpha(i \tilde{g}_{lp} g_{jk} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;j(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;k(\tilde{g}_{jp} g_{li} - \tilde{g}_{lp} \tilde{g}_{lj}) + HH;l(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + 2HH;p(\tilde{g}_{il} g_{jk} - \tilde{g}_{ik} \tilde{g}_{jl}).
\]

It follows from (2.7) that the last relation reduces to

\[
\tilde{R}_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + HH;\alpha(i \tilde{g}_{lp} g_{jk} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;j(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + HH;k(\tilde{g}_{jp} g_{li} - \tilde{g}_{lp} \tilde{g}_{lj}) + HH;l(\tilde{g}_{ip} g_{kj} - \tilde{g}_{kj} \tilde{g}_{ip}) + 2HH;p(\tilde{g}_{il} g_{jk} - \tilde{g}_{ik} \tilde{g}_{jl}).
\]
On the other hand, differentiating (2.7) covariantly, we get
\[ R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \]
\[ + R_{\alpha\beta\gamma\delta} \left( \frac{\partial y^\alpha}{\partial x^i} \right)_p \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \left( \frac{\partial y^\beta}{\partial x^j} \right)_p \frac{\partial y^\gamma}{\partial x^k} N^\delta \]
\[ + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \left( \frac{\partial y^\gamma}{\partial x^k} \right)_p N^\delta + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \]
\[ = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}. \]

Taking account of (2.4), (2.5) and the last relation, we have
\[ R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\mu}{\partial x^p} \]
\[ + H (R_{\alpha\beta\gamma\delta} N^\alpha \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \bar{g}_{ip} + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} N^\beta \frac{\partial y^\gamma}{\partial x^k} N^\delta \bar{g}_{jp} \]
\[ + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta \bar{g}_{kp} - R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^p} \]
\[ = H_{;ip}\bar{g}_{jk} - H_{;jp}\bar{g}_{ik}. \]

This completes the proof.

**Theorem 3.2.** Let \((M^n, g)\) be a GRM\(_n\). If \((\bar{M}^{n-1}, \bar{g})\) is a totally geodesic hypersurface of \((M^n, g)\), then the manifold \((\bar{M}^{n-1}, \bar{g})\) is a GRM\(_{n-1}\).

**Proof.** By virtue of \(H = 0\), we have from (3.8)
\[ R_{ijkl;p} = R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\mu}{\partial x^p}. \]

Since \((M^n, g)\) is a GRM\(_n\), the last relation yields from (1.1)
\[ R_{ijkl;p} = A_{\mu} R_{\alpha\beta\gamma\delta;\mu} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\mu}{\partial x^p} + B_{\mu}(g \bullet g)_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^p}. \]

Because of (2.6) and \(H = 0\), the last relation reduces to
\[ R_{ijkl;p} = A_{p} \bar{R}_{ijkl} + B_{p}(\bar{g} \bullet \bar{g})_{ijkl}, \]
showing that the manifold is a GRM\(_{n-1}\). This completes the proof. \(\square\)
4. Riemannian Product $GRM_n$

Let $(M^n, g)$ be a Riemannian product manifold $(M^p \times M^{n-p}, \hat{g} + \tilde{g})$. In local coordinates, we adopt the Latin indices (resp. the Greek indices) for tensor components which are constructed on $(M^p, \hat{g})$ (resp. $(M^{n-p}, \tilde{g})$). Therefore, the Latin indices take the values from $1, \ldots, p$ whereas the Greek indices run over the range $p + 1, \ldots, n$. Now we can state the followings.

**Theorem 4.1.** Let a Riemannian product manifold $(M^p \times M^{n-p}, \hat{g} + \tilde{g})$ be a $GRM_n$. Then either one decomposition manifold $(M^p, \hat{g})$ is locally symmetric or the other decomposition manifold $(M^{n-p}, \tilde{g})$ is a space of constant curvature.

**Proof.** Since any tensor components of $R$ and its covariant derivatives with both Latin and Greek indices together should be zero, we have from (1.1) and $R_{\alpha\beta\gamma\delta;p} = 0$

$$0 = A_p R_{\alpha\beta\gamma\delta} + B_p (g \cdot g)_{\alpha\beta\gamma\delta}. \quad (4.1)$$

If we assume that $A_p = 0$, then from (4.10) and $(g \cdot g)_{\alpha\beta\gamma\delta} \neq 0$ it follows that $B_p = 0$, which yields from (1.1)

$$R_{ijkl;p} = 0, \quad (4.2)$$

showing that $(M^p, \hat{g})$ is locally symmetric.

On the other hand, if we assume that $A_p \neq 0$, then it follows from (4.10) that

$$R_{\alpha\beta\gamma\delta} = -\frac{B_p}{A_p} (g \cdot g)_{\alpha\beta\gamma\delta}, \quad (4.3)$$

showing that $(M^{n-p}, \tilde{g})$ is a space of constant curvature. This completes the proof. \(\square\)

**Theorem 4.2.** Let $(M^n, g_c)$ be a space of constant curvature. Then the Riemannian product manifold $(M^{2n}, g)$ of $(M^n, g_c)$ with itself is a $GRM_{2n}$.

**Proof.** Since $(M^n, g_c)$ is a space of constant curvature, we have

$$R^c_{ijkl} = \frac{s}{2n(n - 1)}(g_c \cdot g_c)_{ijkl} \quad (4.4)$$

and

$$R^c_{ijkl;p} = 0. \quad (4.5)$$
Here $R^c$ and $s$ denote the curvature tensor and the scalar curvature respectively on $(M^n, g_c)$. Therefore, from (4.13) and (4.14) it follows that

$$R^c_{ijkl;p} = R^c_{ijkl} - \frac{s}{2n(n-1)}(g_c \bullet g_c)_{ijkl}. \quad (4.6)$$

Now we consider the Riemannian product manifold $(M^{2n}, g)$ of a space of constant curvature $(M^n, g_c)$ with itself. Obviously the Riemannian curvature tensor $R$ of $(M^{2n}, g)$ satisfies

$$R_{ijkl} = R^c_{ijkl} + R^c_{ijkl}$$

and

$$R_{ijkl;p} = 0 = R^c_{ijkl;p} + R^c_{ijkl;p},$$

which yields from (4.15) and the last relations

$$R_{ijkl;p} = R_{ijkl} - \frac{s}{2n(n-1)}\left(\frac{1}{2}\right)(g \bullet g)_{ijkl}$$

because of

$$(g_c \bullet g_c) + (g_c \bullet g_c) = \frac{1}{2}(g_c + g_c) \bullet (g_c + g_c) = \frac{1}{2}g \bullet g.$$

Therefore we have

$$R_{ijkl;p} = A_p R_{ijkl} + B_p (g \bullet g)_{ijkl},$$

where $A_p = 1$ and $B_p = -\frac{s}{4n(n-1)}$, showing that the Riemannian product manifold $(M^{2n}, g)$ of a space of constant curvature $(M^n, g_c)$ with itself is a $GRM_{2n}$. This completes the proof.

References


