***bI*-OPEN SETS IN IDEAL BITOPOLOGICAL SPACES**

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Abstract: The aim of this article is to introduce and study the concept of *bI*-open sets with respect to an ideal in bitopological spaces and to investigate some properties. Moreover, the concept of *bI*-continuous functions have also been introduced.

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1. Introduction

The concept of bitopological spaces (X, τ_1, τ_2) , equipped with topologies τ_1 and τ_2 was introduced by Kelly [10]. The concept of ideals has been applied and studied by Kuratowski [11], Vaidyanath-asamy [15], Jankovic and Hamlett [9] and many others. An ideal I on a non-empty set X is a collection of subsets of X satisfying (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. If I is an ideal on X , then (X, τ_1, τ_2, I) is said to be an ideal bitopological space. Andrijevic [5] introduced the notion of *b*-open sets in topological spaces. Further, Al-Hawary and Al-Omari [4] extended this notion to bitopological spaces. In 2007, Caksu Guler and Aslim [6] has introduced the notion of *bI*-open sets and *bI*-continuous functions in topological spaces. After

that Akdag [3], Ekici [7] and many others studied some more properties of these concepts and obtained several characterizations.

In this paper, we introduced bI -open sets and bI -continuous functions in ideal bitopological spaces and established several properties.

2. Preliminaries

Throughout this paper, (X, τ_1, τ_2) denotes a bitopological space on which no separation axioms are assumed and (X, τ_1, τ_2, I) be an ideal bitopological space. $i\text{-int}(A)$ and $j\text{-cl}(A)$ denotes the i -interior and j -closure of A with respect to the topology τ_i and τ_j respectively, where $i, j \in \{1, 2\}, i \neq j$.

Let (X, τ, I) be an ideal topological space. If $P(X)$ is the set of all subsets of X , then the operator $(.)^* : P(X) \rightarrow P(X)$ is called the local function ([11]) of A with respect to the topology τ and ideal I defined by $A^*(\tau, I) = \{x \in X : B \cap A \notin I, \text{ for every } B \in \tau(x)\}$, where $A \subset X$ and $\tau(x) = \{B \in \tau : x \in B\}$. Simply we can write A^* instead of $A^*(\tau, I)$ in case there is no chance for confusion. A Kuratowski closure operator for $\tau^*(I)$ which is finer than τ is defined by $cl^*(A) = A \cup A^*$. $i\text{-int}^*(A)$ denotes the interior of A in $\tau_i^*(I)$ and $i\text{-int}(A_j^*)$ denotes the interior of A_j^* with respect to the topology τ_i , where $A_j^* = \{x \in X : B \cap A \notin I, \text{ for every } B \in \tau_j(x)\}$.

The following definitions are due to Al-Hawary and Al-Omari [4].

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i, j) - b -open if $A \subset i\text{-int}(j\text{-cl}(A)) \cup j\text{-cl}(i\text{-int}(A))$. The complement of an (i, j) - b -open set is (i, j) - b -closed.

Definition 2.2. A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is said to be (i, j) - b -continuous(respectively, (i, j) - b -irresolute) if $f^{-1}(V)$ is (i, j) - b -open in X , for every σ_i -open(respectively, (i, j) - b -open) set V of Y .

The following definition is due to Pervine [13].

Definition 2.3. A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is said to be pairwise continuous if the induced functions $f : (X, \tau_1) \rightarrow (X, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (X, \sigma_2)$ are both continuous.

3. (i, j) - bI -Open Sets

Definition 3.1. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) - bI -open if $A \subset i\text{-int}(j\text{-cl}^*(A)) \cup j\text{-cl}^*(i\text{-int}(A))$, where $i, j \in \{1, 2\}, i \neq j$.

We denote the family of all (i, j) - bI -open sets of (X, τ_1, τ_2, I) by (i, j) - $BIO(X)$. By (i, j) we mean the pair of topologies (τ_i, τ_j) .

Remark 3.1. $BIO(X, \tau_1, \tau_2, I) \neq BIO(X, \tau_1) \cup BIO(X, \tau_2)$. It is clear from the following example.

Example 3.1. Let $X = \{p, q, r\}$, $\tau_1 = \{\emptyset, \{p\}, X\}$, $\tau_2 = \{\emptyset, \{q\}, X\}$ and $I = \{\emptyset, \{p\}\}$.

It can be easily shown that τ_1 - bI -open sets are $\{\emptyset, \{p\}, \{p, q\}, \{p, r\}, X\}$ and τ_2 - bI -open sets are $\{\emptyset, \{q\}, \{p, q\}, \{q, r\}, X\}$. But, (τ_1, τ_2) - bI -open sets are $\{\emptyset, \{p\}, \{q\}, \{p, q\}, \{q, r\}, X\}$.

Remark 3.2. Every (i, j) - bI -open set is (i, j) - b -open. It can be easily proved by using the fact that $\tau^*(I)$ is finer than τ .

But, the converse may not be true in general as shown by the following example.

Example 3.2. Let $X = \{p, q, r\}$, $\tau_1 = \{\emptyset, \{q\}, X\}$, $\tau_2 = \{\emptyset, \{p\}, \{p, r\}, X\}$ and $I = \{\emptyset, \{p\}\}$. Now $\{p\}$ is $(1, 2)$ - b -open but not $(1, 2)$ - bI -open.

Remark 3.3. The intersection of two (i, j) - bI -open sets may not be a (i, j) - bI -open set is clear from the following example.

Example 3.3. Let $X = \{p, q, r\}$, $\tau_1 = \{\emptyset, \{p, r\}, X\}$, $\tau_2 = \{\emptyset, \{q, r\}, X\}$ and $I = \{\emptyset, \{p\}\}$. Then $\{p, q\}$ and $\{p, r\}$ is $(1, 2)$ - bI -open sets but $\{p, q\} \cap \{p, r\} = \{p\}$ is not $(1, 2)$ - bI -open.

Theorem 3.1. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A, B \subset X$. If A is (i, j) - bI -open set and $B \in \tau_1 \cap \tau_2$, then $A \cap B$ is also (i, j) - bI -open.

Proof. Let A is (i, j) - bI -open. Then $A \subset i\text{-int}(j\text{-cl}^*(A)) \cup j\text{-cl}^*(i\text{-int}(A))$.
Now

$$\begin{aligned}
 A \cap B &\subset \{i\text{-int}(j\text{-cl}^*(A)) \cup j\text{-cl}^*(i\text{-int}(A))\} \cap B \\
 &= \{i\text{-int}(j\text{-cl}^*(A)) \cap B\} \cup \{j\text{-cl}^*(i\text{-int}(A)) \cap B\} \\
 &= \{i\text{-int}(A \cup A_j^*) \cap B\} \cup \{((i\text{-int}(A)) \cup (i\text{-int}(A))_j^*) \cap B\} \\
 &\subset \{i\text{-int}((A \cap B) \cup (A_j^* \cap B))\} \cup \{((i\text{-int}(A)) \cap B) \cup ((i\text{-int}(A)) \cap B)_j^*\} \\
 &\subset \{i\text{-int}((A \cap B) \cup (A \cap B)_j^*)\} \cup \{(i\text{-int}(A \cap B)) \cup (i\text{-int}(A \cap B))_j^*\} \\
 &= i\text{-int}(j\text{-cl}^*(A \cap B)) \cup j\text{-cl}^*(i\text{-int}(A \cap B)).
 \end{aligned}$$

Hence $A \cap B$ is (i, j) - bI -open.

Theorem 3.2. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If $A_\alpha \in (i, j)\text{-BIO}(X)$ for each $\alpha \in \Lambda$, then $\bigcup\{A_\alpha : \alpha \in \Lambda\} \in (i, j)\text{-BIO}(X)$ where Λ an index set.

Proof. Let $A_\alpha \in (i, j)\text{-BIO}(X)$. Then

$$A_\alpha \subset i - \text{int}(j - cl^*(A_\alpha)) \cup j - cl^*(i - \text{int}(A_\alpha)),$$

for each $\alpha \in \Lambda$.

Thus

$$\begin{aligned} \bigcup_{\alpha \in \Lambda} A_\alpha &\subset \bigcup_{\alpha \in \Lambda} \{i - \text{int}(j - cl^*(A_\alpha)) \cup j - cl^*(i - \text{int}(A_\alpha))\} \\ &\subset \bigcup_{\alpha \in \Lambda} \{i - \text{int}(A_\alpha \cup (A_\alpha)_j^*)\} \cup \{(i - \text{int}(A_\alpha)) \cup (i - \text{int}(A_\alpha))_j^*\} \\ &\subset \{i - \text{int}((\bigcup_{\alpha \in \Lambda} A_\alpha) \cup (\bigcup_{\alpha \in \Lambda} (A_\alpha)_j^*))\} \\ &\quad \cup \{(i - \text{int}(\bigcup_{\alpha \in \Lambda} A_\alpha)) \cup (\bigcup_{\alpha \in \Lambda} (i - \text{int}(A_\alpha))_j^*)\} \\ &\subset \{i - \text{int}((\bigcup_{\alpha \in \Lambda} A_\alpha) \cup (\bigcup_{\alpha \in \Lambda} A_\alpha)_j^*)\} \\ &\quad \cup \{i - \text{int}(\bigcup_{\alpha \in \Lambda} A_\alpha) \cup (i - \text{int}(\bigcup_{\alpha \in \Lambda} A_\alpha))_j^*\} \\ &\subset i - \text{int}(j - cl^*(\bigcup_{\alpha \in \Lambda} A_\alpha)) \cup j - cl^*(i - \text{int}(\bigcup_{\alpha \in \Lambda} A_\alpha)). \end{aligned}$$

Hence $\bigcup_{\alpha \in \Lambda} A_\alpha$ is $(i, j)\text{-bI-open}$.

Lemma 3.1. Let (X, τ_1, τ_2, I) be an ideal bitopological space and A, B are subsets of X such that $B \subset A$. Then $B_i^*(\tau_i|_A, I|_A) = B_i^*(\tau_i, I) \cap A$, for $i = 1, 2$.

Here we denote that for any subset A of (X, τ_1, τ_2, I) , $\tau_i|_A$ is the relative topology on A where $i = 1, 2$ and $I|_A = \{A \cap I : I \in I\}$ is obviously an ideal on A .

Theorem 3.3. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If $A \in (i, j)\text{-BIO}(X)$ and $B \in \tau_1 \cap \tau_2$, then $A \cap B \in \text{BIO}(B, \tau_1|_B, \tau_2|_B, I|_B)$.

Proof. Since $B \in \tau_1 \cap \tau_2$, therefore $i\text{-int}_B(P) = i\text{-int}(P)$, for any subset P of B and $i = 1, 2$. By using the fact and Lemma 3.1, we have

$$A \cap B \subset \{i - \text{int}(j - cl^*(A)) \cup j - cl^*(i - \text{int}(A))\} \cap B$$

$$\begin{aligned}
 &= \{i - \text{int}(j - \text{cl}^*(A)) \cap B\} \cup \{j - \text{cl}^*(i - \text{int}(A)) \cap B\} \\
 &= \{i - \text{int}(A \cup A_j^*) \cap B\} \cup \{((i - \text{int}(A)) \cup (i - \text{int}(A))^*_j) \cap B\} \\
 &\subset \{((i - \text{int}(A \cup A_j^*)) \cap B) \cap B\} \cup \{((i - \text{int}(A)) \cap B) \\
 &\quad \cup ((i - \text{int}(A))^*_j \cap B) \cap B\} \\
 &\subset \{((i - \text{int}(A \cup A_j^*)) \cap B) \cap B\} \cup \{((i - \text{int}(A)) \cap B) \\
 &\quad \cup ((i - \text{int}(A)) \cap B)^*_j \cap B\} \\
 &\subset \{((i - \text{int}((A \cap B) \cup (A \cap B)^*_j)) \cap B)\} \cup \{((i - \text{int}_B(A \cap B)) \cap B) \\
 &\quad \cup ((i - \text{int}_B(A \cap B))^*_j \cap B)\} \\
 &\subset \{((i - \text{int}((A \cap B) \cup (A \cap B)^*_j)) \cap B)\} \cup \{((i - \text{int}_B(A \cap B)) \\
 &\quad \cup (i - \text{int}_B(A \cap B))^*_j)_{(\tau_1|_B, \tau_2|_B, I|_B)}\} \\
 &= \{i - \text{int}_B(j - \text{cl}_B^*(A \cap B))\} \cup \{j - \text{cl}_B^*(i - \text{int}_B(A \cap B))\}.
 \end{aligned}$$

Hence $A \cap B \in BIO(B, \tau_1|_B, \tau_2|_B, I|_B)$.

Definition 3.2. A subset A of an ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) -bI-closed if its complement is (i, j) -bI-open.

Theorem 3.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then a subset A of (X, τ_1, τ_2, I) is (i, j) -bI-closed if $i\text{-cl}(j\text{-int}^*(A)) \cap j\text{-int}^*(i\text{-cl}(A)) \subset A$, where $i, j \in \{1, 2\}, i \neq j$.

Proof. It is clear from the definition.

Theorem 3.5. Let (X, τ_1, τ_2, I) be an ideal bitopological space. If a subset A of X is (i, j) -bI-closed, then $i\text{-cl}^*(j\text{-int}(A)) \cap j\text{-int}(i\text{-cl}^*(A)) \subset A$, where $i, j \in \{1, 2\}, i \neq j$.

Proof. Let A is (i, j) -bI-closed. Then $X \setminus A$ is (i, j) -bI-open. Since τ_i^* is finer than τ_i ($i = 1, 2$), we have

$$\begin{aligned}
 X \setminus A &\subset i - \text{int}(j - \text{cl}^*(X \setminus A)) \cup j - \text{cl}^*(i - \text{int}(X \setminus A)) \\
 &\subset i - \text{int}(j - \text{cl}(X \setminus A)) \cup j - \text{cl}(i - \text{int}(X \setminus A)) \\
 &= \{X \setminus (i - \text{cl}(j - \text{int}(A)))\} \cup \{X \setminus (j - \text{int}(i - \text{cl}(A)))\} \\
 &\subset \{X \setminus (i - \text{cl}^*(j - \text{int}(A)))\} \cup \{X \setminus (j - \text{int}(i - \text{cl}^*(A)))\} \\
 &= X \setminus \{(i - \text{cl}^*(j - \text{int}(A))) \cap (j - \text{int}(i - \text{cl}^*(A)))\}.
 \end{aligned}$$

Hence $i\text{-cl}^*(j\text{-int}(A)) \cap j\text{-int}(i\text{-cl}^*(A)) \subset A$.

Theorem 3.6. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then:

- (a) If $I = \emptyset$, then A is (i, j) - bI -open if and only if A is (i, j) - b -open.
- (b) If $I = P(X)$, then A is (i, j) - bI -open if and only if $A \in \tau_1 \cap \tau_2$.

Proof. (a) It follows from the fact that if $I = \emptyset$, then $A^* = cl(A)$, for every subset A of X .

(b) It follows from the fact that for every subset A of X , if $I = P(X)$ then $A^* = \emptyset$.

Definition 3.3. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then

- (a) The (i, j) - bI -interior of A , is defined as the union of all (i, j) - bI -open sets contained in A and is denoted by (i, j) - bI - $int(A)$.
- (b) The (i, j) - bI -closure of A , is defined as the intersection of all (i, j) - bI -closed sets containing A and is denoted by (i, j) - bI - $cl(A)$.

Theorem 3.7. Let A be a subset of an ideal bitopological space (X, τ_1, τ_2, I) . Then:

- (a) (i, j) - bI - $int(A)$ is (i, j) - bI -open.
- (b) (i, j) - bI - $cl(A)$ is (i, j) - bI -closed.
- (c) A is (i, j) - bI -open if and only if $A = (i, j)$ - bI - $int(A)$.
- (d) A is (i, j) - bI -closed if and only if $A = (i, j)$ - bI - $cl(A)$.

Proof. The proof of (a) – (d) are obvious.

Theorem 3.8. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then a point $x \in (i, j)$ - bI - $cl(A)$ if and only if $A \cap B \neq \emptyset$, for every (i, j) - bI -open set B of X containing x .

Proof. Suppose that $x \in (i, j)$ - bI - $cl(A)$ and B be a (i, j) - bI -open set containing x . Assume that $A \cap B = \emptyset$. Then $A \subset X \setminus B$, where $X \setminus B$ is (i, j) - bI -closed set. This implies that $x \in (i, j)$ - bI - $cl(A) \subset (i, j)$ - bI - $cl(X \setminus B) = X \setminus B$. Thus $x \in X \setminus B$, which is a contradiction. Hence $A \cap B \neq \emptyset$.

Conversely suppose that $A \cap B \neq \emptyset$, for every (i, j) - bI -open set B of X containing x . Let $x \notin (i, j)$ - bI - $cl(A)$. Then there exists a (i, j) - bI -closed set G of X such that $A \subset G$ and $x \notin G$. Therefore $x \in X \setminus G$, where $X \setminus G$ is (i, j) - bI -open and $(X \setminus G) \cap A = \emptyset$. Which is a contradiction to the assumption. Hence $x \in (i, j)$ - bI - $cl(A)$.

Theorem 3.9. Let A be a subset of an ideal bitopological space (X, τ_1, τ_2, I) . Then

$$(a) (i, j)\text{-}bI\text{-}cl(X \setminus A) = X \setminus (i, j)\text{-}bI\text{-}int(A).$$

$$(b) (i, j)\text{-}bI\text{-}int(X \setminus A) = X \setminus (i, j)\text{-}bI\text{-}cl(A).$$

Proof. (a) Let $x \notin (i, j)\text{-}bI\text{-}cl(X \setminus A)$. Then there exists an $(i, j)\text{-}bI$ -open set B of X containing x such that $B \cap (X \setminus A) = \emptyset$. We have $x \in B$, therefore $x \notin X \setminus A$ and so $x \in A$. Thus $x \in B \subset A$ and so $x \in (i, j)\text{-}bI\text{-}int(A)$. Which implies that $x \notin X \setminus (i, j)\text{-}bI\text{-}int(A)$. Hence $X \setminus (i, j)\text{-}bI\text{-}int(A) \subset (i, j)\text{-}bI\text{-}cl(X \setminus A)$. Next let $x \notin X \setminus (i, j)\text{-}bI\text{-}int(A)$. Then $x \in (i, j)\text{-}bI\text{-}int(A)$ and so there exists an $(i, j)\text{-}bI$ -open set B of X such that $x \in B \subset A$. Thus $B \cap (X \setminus A) = \emptyset$ and $x \notin (i, j)\text{-}bI\text{-}cl(X \setminus A)$. Therefore $(i, j)\text{-}bI\text{-}cl(X \setminus A) \subset X \setminus (i, j)\text{-}bI\text{-}int(A)$. Hence the result follows.

(b) Similar to the proof of (a).

Definition 3.4. Let (X, τ_1, τ_2, I) be an ideal bitopological space. Then a subset A of X is said to be an $(i, j)\text{-}bI$ -neighbourhood of a point x of X if there exists a $(i, j)\text{-}bI$ -open set B such that $x \in B \subset A$.

Theorem 3.10. Let (X, τ_1, τ_2, I) be an ideal bitopological space and $A \subset X$. Then A is $(i, j)\text{-}bI$ -open if and only if it is an $(i, j)\text{-}bI$ -neighbourhood of each of its points.

Proof. Let A be an $(i, j)\text{-}bI$ -open set of (X, τ_1, τ_2, I) . So by definition, we have A is an $(i, j)\text{-}bI$ -neighbourhood of each of its points, since for every $x \in A$, we have $x \in A \subset A$ and A is $(i, j)\text{-}bI$ -open. Conversely assume that A is an $(i, j)\text{-}bI$ -neighbourhood of each of its points. Then for every $x \in A$, there exists an $(i, j)\text{-}bI$ -open set B_x of (X, τ_1, τ_2, I) such that $x \in B_x \subset A$. Then $A = \bigcup \{B_x : x \in A\}$. Since arbitrary union of $(i, j)\text{-}bI$ -open sets is $(i, j)\text{-}bI$ -open, therefore we have A is $(i, j)\text{-}bI$ -open in X .

4. $(i, j)\text{-}bI$ -Continuous Functions

Definition 4.1. A function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)\text{-}bI$ -continuous if the inverse image of every σ_i -open set in Y is $(i, j)\text{-}bI$ -open in X , where $i, j = 1, 2$ and $i \neq j$.

Remark 4.1. Every $(i, j)\text{-}bI$ -continuous function is $(i, j)\text{-}b$ -continuous. But the converse may not be true in general as shown in the following example.

Example 4.1. Let $X = \{p, q, r\}$, $\tau_1 = \{\emptyset, \{q\}, X\}$, $\tau_2 = \{\emptyset, \{p\}, \{q, r\}, X\}$ and $I = \{\emptyset, \{p\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, I) \longrightarrow (X, \sigma_1, \sigma_2)$ is $(1, 2)\text{-}b$ -continuous but not $(1, 2)\text{-}bI$ -continuous.

Theorem 4.1. *The following statements are equivalent for the function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$.*

- (a) f is (i, j) - bI -continuous.
- (b) For all $x \in X$ and every σ_i -open set B of Y containing $f(x)$, there exists a (i, j) - bI -open set A of X containing x such that $f(A) \subset B$.
- (c) Inverse image of every σ_i -closed set of Y is (i, j) - bI -closed set in X .
- (d) $f((i, j)\text{-}bI\text{-}cl(A)) \subset \sigma_i\text{-}cl(f(A))$, for every subset A of X .
- (e) $(i, j)\text{-}bI\text{-}cl(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}cl(B))$, for every subset B of Y .
- (f) $f^{-1}(\sigma_i\text{-}int(G)) \subset (i, j)\text{-}bI\text{-}int(f^{-1}(G))$, for every subset G of Y .

Proof. (a) $\Rightarrow$ (b) Let B be a σ_i -open set in Y such that $f(x) \in B$. Since f is (i, j) - bI -continuous, therefore $f^{-1}(B)$ is (i, j) - bI -open in X . Let $A = f^{-1}(B)$. Then $f(x) \in f(A) \subset B$.

(b) $\Rightarrow$ (a) Let B be a σ_i -open set in Y and $x \in f^{-1}(B)$. Then we have $f(x) \in B$. By (b), there exists an (i, j) - bI -open set A_x in X containing x such that $f(A_x) \subset B$. Therefore $x \in A_x \subset f^{-1}(B)$. Consequently, $f^{-1}(B)$ is (i, j) - bI -open in X . Hence $f^{-1}(B)$ is (i, j) - bI -continuous.

(a) $\Rightarrow$ (c) It is obvious.

(c) $\Rightarrow$ (d) Let $A \subset X$. Since $\sigma_i\text{-}cl(f(A))$ is σ_i -closed set in Y , therefore by (c) we have $f^{-1}(\sigma_i\text{-}cl(f(A)))$ is (i, j) - bI -closed set in X . Also, $A \subset f^{-1}(\sigma_i\text{-}cl(f(A)))$ and $(i, j)\text{-}bI\text{-}cl(A)$ is the smallest (i, j) - bI -closed set containing A . Therefore $(i, j)\text{-}bI\text{-}cl(A) \subset f^{-1}(\sigma_i\text{-}cl(f(A)))$. This implies that $f((i, j)\text{-}bI\text{-}cl(A)) \subset \sigma_i\text{-}cl(f(A))$.

(d) $\Rightarrow$ (e) Let $B \subset Y$. Then $f^{-1}(B)$ is a subset of X . By (d), $f((i, j)\text{-}bI\text{-}cl(f^{-1}(B))) \subset \sigma_i\text{-}cl(f(f^{-1}(B))) \subset \sigma_i\text{-}cl(B)$. Hence $(i, j)\text{-}bI\text{-}cl(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}cl(B))$.

(e) $\Rightarrow$ (c) Let B be a σ_i -closed set in Y . By (e), $(i, j)\text{-}bI\text{-}cl(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-}cl(B)) = f^{-1}(B)$. Therefore $f^{-1}(B) = (i, j)\text{-}bI\text{-}cl(f^{-1}(B))$ and so $f^{-1}(B)$ is (i, j) - bI -closed in X .

(a) $\Rightarrow$ (f) Let G be a σ_i -open subset of Y . By (a), $f^{-1}(G) = f^{-1}(\sigma_i\text{-}int(G))$ is (i, j) - bI -open in X . Then $f^{-1}(\sigma_i\text{-}int(G)) \subset (i, j)\text{-}bI\text{-}int(f^{-1}(\sigma_i\text{-}int(G))) \subset (i, j)\text{-}bI\text{-}int(f^{-1}(G))$.

(f) $\Rightarrow$ (a) Let B be a σ_i -open subset of Y . Then $f^{-1}(B) = f^{-1}(\sigma_i\text{-}int(B)) \subset (i, j)\text{-}bI\text{-}int(f^{-1}(B))$. Therefore $f^{-1}(B)$ is (i, j) - bI -open in X and so f is (i, j) - bI -continuous.

Theorem 4.2. *Let*

$$f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2, J) \text{ and } g : (Y, \sigma_1, \sigma_2, J) \longrightarrow (Z, \theta_1, \theta_2)$$

be two functions, where I and J are two ideals in X and Y respectively. If f is (i, j) - bI -continuous and g is pairwise continuous, then g_0f is (i, j) - bI -continuous.

Proof. Let C be a θ_i -open set in Z . Since g is pairwise continuous, therefore $g^{-1}(C)$ is σ_i -open set in Y . Also, f is (i, j) - bI -continuous, so $(g_0f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is (i, j) - bI -open in X . Hence g_0f is (i, j) - bI -continuous.

Theorem 4.3. *Let $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -continuous and $A \subset X$. If $A \in \tau_1 \cap \tau_2$, then the restriction function $f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -continuous.*

Proof. Let B be a σ_i -open subset of Y . Since f is (i, j) - bI -continuous, therefore $f^{-1}(B)$ is (i, j) - bI -open set in X . Also, $A \in \tau_1 \cap \tau_2$, therefore by theorem 3.3, we have $A \cap f^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A)$. Again

$$(f|_A)^{-1}(B) = A \cap f^{-1}(B)$$

and so $(f|_A)^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A)$. Hence $f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \rightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -continuous.

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function. If a function $g : (X, \tau_1, \tau_2, I) \longrightarrow (X \times Y, \sigma_1 \times \sigma_2)$, defined by $g(x) = (x, f(x))$ for each $x \in X$ is (i, j) - bI -continuous, then f is (i, j) - bI -continuous.*

Proof. Suppose that g is (i, j) - bI -continuous. Let $x \in X$ and B be a σ_i -open subset of Y containing $f(x)$. Then $X \times B$ is $\tau_i \times \sigma_i$ -open in $X \times Y$. By the (i, j) - bI -continuity of g , there exists an (i, j) - bI -open set A in X containing x such that $f(A) \subset X \times Y$, by theorem 4.1. So $f(A) \subset B$. Hence f is (i, j) - bI -continuous.

Definition 4.2. (see [13]) A bitopological space (X, τ_1, τ_2) is said to be pairwise connected if it cannot be expressed as the union of two nonempty disjoint sets A and B such that A is τ_i -open and B is τ_j -open, where $i, j = \{1, 2\}$.

Definition 4.3. An ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) - bI -connected if there exists a nonempty (i, j) - bI -open set A and a nonempty (j, i) - bI -open set B in X such that $X \neq A \cup B$ and $A \cap B = \emptyset$.

Theorem 4.5. *Let $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function. If f is (i, j) - bI -continuous surjection and X is (i, j) - bI -connect-ed, then Y is pairwise connected.*

Proof. Assume that Y is not pairwise connected. Then there exists a nonempty σ_i -open set A and a nonempty σ_j -open set B such that $Y = A \cup B$ and $A \cap B = \emptyset$. Since f is (i, j) - bI -continuous, therefore $f^{-1}(A)$ is (i, j) - bI -open in X and $f^{-1}(B)$ is (j, i) - bI -open in X . Since $A \cap B = \emptyset$ and f is surjective, therefore $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus $f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$. This implies that $X = f^{-1}(A) \cup f^{-1}(B)$. Hence X is not (i, j) - bI -connected, which is a contradiction.

Lemma 4.1([12]). For any function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$, $f(I)$ is an ideal on Y .

Definition 4.4. An ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) - I -compact if for every open cover $\{C_\alpha : \alpha \in \Lambda\}$ by τ_i -open sets of X , there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\{C_\alpha : \alpha \in \Lambda\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.

Definition 4.5. An ideal bitopological space (X, τ_1, τ_2, I) is said to be (i, j) - bI -compact if for every open cover $\{C_\alpha : \alpha \in \Lambda\}$ by (i, j) - bI -open sets of X , there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\{C_\alpha : \alpha \in \Lambda\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.

Theorem 4.6. Let $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function. If $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -continuous surjection and X is (i, j) - bI -compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is (i, j) - $f(I)$ -compact.

Proof. Let $\{C_\alpha : \alpha \in \Lambda\}$ be an open cover of Y by σ_i -open sets in Y . Since f is (i, j) - bI -continuous, therefore $\{f^{-1}(C_\alpha) : \alpha \in \Lambda\}$ be an open cover of X by (i, j) - bI -open sets in X . Also, (X, τ_1, τ_2, I) is (i, j) - bI -compact. Therefore there exists a finite subset Λ_0 of Λ such that $X \setminus \bigcup\{f^{-1}(C_\alpha) : \alpha \in \Lambda_0\} \in I$. Thus $Y \setminus \bigcup\{C_\alpha : \alpha \in \Lambda_0\} = f(X \setminus \bigcup\{f^{-1}(C_\alpha) : \alpha \in \Lambda_0\}) \in f(I)$. Hence $(Y, \sigma_1, \sigma_2, f(I))$ is (i, j) - $f(I)$ -compact.

Definition 4.6. A function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be (i, j) - bI -irresolute if the inverse image of every (i, j) - b -open set in Y is (i, j) - bI -open in X , where $i, j = 1, 2$ and $i \neq j$.

Theorem 4.7. The following statements are equivalent for the function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$.

- (a) f is (i, j) - bI -irresolute.
- (b) For all $x \in X$ and every (i, j) - b -open set B of Y containing $f(x)$, there exists an (i, j) - bI -open set A of X containing x such that $f(A) \subset B$.
- (c) Inverse image of every (i, j) - b -closed set of Y is (i, j) - bI -closed set in X .

Theorem 4.8. *Let*

$$f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2, J) \text{ and } g : (Y, \sigma_1, \sigma_2, J) \longrightarrow (Z, \theta_1, \theta_2)$$

be two functions, where I and J are two ideals in X and Y respectively.

(a) *If f is (i, j) - bI -irresolute and g is (i, j) - b -irresolute, then g_0f is (i, j) - bI -irresolute.*

(b) *If f is (i, j) - bI -irresolute and g is (i, j) - b -continuous, then g_0f is (i, j) - bI -continuous.*

Proof. (a) Let C be an (i, j) - b -open set in Z . Since g is (i, j) - b -irresolute, therefore $g^{-1}(C)$ is (i, j) - b -open set in Y . Also, f is (i, j) - bI -irresolute, so $(g_0f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is (i, j) - bI -open in X . Hence g_0f is (i, j) - bI -irresolute.

(b) Let C be a θ_i -open set in Z . Since g is (i, j) - b -continuous, therefore $g^{-1}(C)$ is (i, j) - b -open set in Y . Also, f is (i, j) - bI -irresolute, so $(g_0f)^{-1}(C) = f^{-1}(g^{-1}(C))$ is (i, j) - bI -open in X . Hence g_0f is (i, j) - bI -continuous.

Theorem 4.9. *Let $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -irresolute and $A \subset X$. If $A \in \tau_1 \cap \tau_2$, then the restriction function $f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \longrightarrow (Y, \sigma_1, \sigma_2)$ is (i, j) - bI -irresolute.*

Proof. Let B be an (i, j) - b -open subset of Y . Since f is (i, j) - bI -irresolute, therefore $f^{-1}(B)$ is (i, j) - bI -open set in X . Also, $A \in \tau_1 \cap \tau_2$, therefore by theorem 3.3, we have $A \cap f^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A)$. Again

$$(f|_A)^{-1}(B) = A \cap f^{-1}(B)$$

and so

$$(f|_A)^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A).$$

Hence

$$f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \rightarrow (Y, \sigma_1, \sigma_2)$$

is (i, j) - bI -irresolute.

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