bI-OPEN SETS IN IDEAL BITOPOLOGICAL SPACES

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Abstract: The aim of this article is to introduce and study the concept of bI-open sets with respect to an ideal in bitopological spaces and to investigate some properties. Moreover, the concept of bI-continuous functions have also been introduced.

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1. Introduction

The concept of bitopological spaces $(X, \tau_1, \tau_2)$, equipped with topologies $\tau_1$ and $\tau_2$ was introduced by Kelly [10]. The concept of ideals has been applied and studied by Kuratowski [11], Vaidyanath-asamy [15], Jankovic and Hamlett [9] and many others. An ideal $I$ on a non-empty set $X$ is a collection of subsets of $X$ satisfying $(i)$ $A \in I$ and $B \subset A$ implies $B \in I$ and $(ii)$ $A \in I$ and $B \in I$ implies $A \cup B \in I$. If $I$ is an ideal on $X$, then $(X, \tau_1, \tau_2, I)$ is said to be an ideal bitopological space. Andrijevic [5] introduced the notion of $b$-open sets in topological spaces. Further, Al-Hawary and Al-Omari [4] extended this notion to bitopological spaces. In 2007, Caksu Guler and Aslim [6] has introduced the notion of $bI$-open sets and $bI$-continuous functions in topological spaces. After
that Akdag [3], Ekici [7] and many others studied some more properties of these concepts and obtained several characterizations.

In this paper, we introduced $bI$-open sets and $bI$-continuous functions in ideal bitopological spaces and established several properties.

2. Preliminaries

Throughout this paper, $(X, \tau_1, \tau_2)$ denotes a bitopological space on which no separation axioms are assumed and $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space. $i\text{-}int(A)$ and $j\text{-}cl(A)$ denotes the $i$-interior and $j$-closure of $A$ with respect to the topology $\tau_i$ and $\tau_j$ respectively, where $i, j \in \{1, 2\}, i \neq j$.

Let $(X, \tau, I)$ be an ideal topological space. If $P(X)$ is the set of all subsets of $X$, then the operator $()^*: P(X) \rightarrow P(X)$ is called the local function ([11]) of $A$ with respect to the topology $\tau$ and ideal $I$ defined by $A^*(\tau, I) = \{ x \in X : B \cap A \notin I, \text{for every } B \in \tau(x) \}$, where $A \subset X$ and $\tau(x) = \{ B \in \tau : x \in B \}$. Simply we can write $A^*$ instead of $A^*(\tau, I)$ in case there is no chance for confusion. A Kuratowski closure operator for $\tau^*(I)$ which is finer than $\tau$ is defined by $cl^*(A) = A \cup A^*$. $i\text{-}int^*(A)$ denotes the interior of $A$ in $\tau_i^*(I)$ and $i\text{-}int^*(A_j^*)$ denotes the interior of $A_j^*$ with respect to the topology $\tau_i$, where $A_j^* = \{ x \in X : B \cap A \notin I, \text{for every } B \in \tau_j(x) \}$.

The following definitions are due to Al-Hawary and Al-Omari [4].

**Definition 2.1.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is said to be $(i, j)$-b-open if $A \subset i\text{-}int(j\text{-}cl(A)) \cup j\text{-}cl(i\text{-}int(A))$. The complement of an $(i, j)$-b-open set is $(i, j)$-b-closed.

**Definition 2.2.** A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is said to be $(i, j)$-b-continuous(respectively, $(i, j)$-b-irresolute) if $f^{-1}(V)$ is $(i, j)$-b-open in $X$, for every $\sigma_i$-open(respectively, $(i, j)$-b-open) set $V$ of $Y$.

The following definition is due to Pervine [13].

**Definition 2.3.** A function $f : (X, \tau_1, \tau_2) \rightarrow (X, \sigma_1, \sigma_2)$ is said to be pairwise continuous if the induced functions $f : (X, \tau_1) \rightarrow (X, \sigma_1)$ and $f : (X, \tau_2) \rightarrow (X, \sigma_2)$ are both continuous.

3. $(i, j)$-bI-Open Sets

**Definition 3.1.** A subset $A$ of an ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(i, j)$-bI-open if $A \subset i\text{-}int(j\text{-}cl^*(A)) \cup j\text{-}cl^*(i\text{-}int(A))$, where $i, j \in \{1, 2\}, i \neq j$. 
We denote the family of all \((i, j)\)-bI-open sets of \((X, \tau_1, \tau_2, I)\) by \((i, j)\)-\textit{BIO}(X). By \((i, j)\) we mean the pair of topologies \((\tau_i, \tau_j)\).

**Remark 3.1.** \textit{BIO}(X, \tau_1, \tau_2, I) \neq \textit{BIO}(X, \tau_1) \cup \textit{BIO}(X, \tau_2).** It is clear from the following example.

**Example 3.1.** Let \(X = \{p, q, r\}, \tau_1 = \{\emptyset, \{p\}, X\}, \tau_2 = \{\emptyset, \{q\}, X\}\) and \(I = \{\emptyset, \{p\}\}\).

It can be easily shown that \(\tau_1\)-bI-open sets are \(\{\emptyset, \{p\}, \{p, q\}, \{p, r\}, X\}\) and \(\tau_2\)-bI-open sets are \(\{\emptyset, \{q\}, \{p, q\}, \{q, r\}, X\}\). But, \((\tau_1, \tau_2)\)-bI-open sets are \(\{\emptyset, \{p\}, \{q\}, \{p, q\}, \{q, r\}, X\}\).

**Remark 3.2.** Every \((i, j)\)-bI-open set is \((i, j)\)-\textit{bI-open}. It can be easily proved by using the fact that \(\tau^*(I)\) is finer than \(\tau\).

But, the converse may not be true in general as shown by the following example.

**Example 3.2.** Let \(X = \{p, q, r\}, \tau_1 = \{\emptyset, \{q\}, X\}, \tau_2 = \{\emptyset, \{p\}, \{p, r\}, X\}\) and \(I = \{\emptyset, \{p\}\}\). Now \(\{p\}\) is \((1, 2)\)-\textit{bI-open} but not \((1, 2)\)-bI-open.

**Remark 3.3.** The intersection of two \((i, j)\)-bI-open sets may not be a \((i, j)\)-bI-open set is clear from the following example.

**Example 3.3.** Let \(X = \{p, q, r\}, \tau_1 = \{\emptyset, \{p, r\}, X\}, \tau_2 = \{\emptyset, \{q, r\}, X\}\) and \(I = \{\emptyset, \{p\}\}\). Then \(\{p, q\}\) and \(\{p, r\}\) is \((1, 2)\)-bI-open sets but \(\{p, q\} \cap \{p, r\} = \{p\}\) is not \((1, 2)\)-bI-open.

**Theorem 3.1.** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A, B \subset X\). If \(A\) is \((i, j)\)-bI-open set and \(B \in \tau_1 \cap \tau_2\), then \(A \cap B\) is also \((i, j)\)-bI-open.

**Proof.** Let \(A\) is \((i, j)\)-bI-open. Then \(A \subset i\text{-int}(j\text{-cl}^*(A)) \cup j\text{-cl}^*(i\text{-int}(A))\). Now

\[
A \cap B \subset \{i \text{-int}(j - \text{cl}^*(A)) \cup j - \text{cl}^*(i - \text{int}(A))\} \cap B \\
= \{i \text{-int}(j - \text{cl}^*(A)) \cap B\} \cup \{j - \text{cl}^*(i - \text{int}(A)) \cap B\} \\
= \{i \text{-int}(A \cup A^*_j) \cap B\} \cup \{(i - \text{int}(A)) \cap (i - \text{int}(A))^*_j\} \cap B \\
\subset \{i \text{-int}((A \cap B) \cup (A^*_j \cap B))\} \cup \{(i - \text{int}(A) \cap B) \cup (i - \text{int}(A)) \cap B^*_j\} \\
\subset \{i \text{-int}((A \cap B) \cup (A \cap B^*_j))\} \cup \{(i - \text{int}(A \cap B)) \cup (i - \text{int}(A \cap B))^*_j\} \\
= i - \text{int}(j - \text{cl}^*(A \cap B)) \cup j - \text{cl}^*(i - \text{int}(A \cap B)).
\]

Hence \(A \cap B\) is \((i, j)\)-bI-open.
Theorem 3.2. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. If \(A_\alpha \in (i, j)-\text{BIO}(X)\) for each \(\alpha \in \wedge\), then \(\bigcup \{A_\alpha : \alpha \in \wedge\} \in (i, j)-\text{BIO}(X)\) where \(\wedge\) is an index set.

Proof. Let \(A_\alpha \in (i, j)-\text{BIO}(X)\). Then

\[ A_\alpha \subset i - \text{int}(j - \cl^*(A_\alpha)) \cup j - \cl^*(i - \text{int}(A_\alpha)), \]

for each \(\alpha \in \wedge\).

Thus

\[
\bigcup_{\alpha \in \wedge} A_\alpha \subset \bigcup_{\alpha \in \wedge} \{i - \text{int}(j - \cl^*(A_\alpha)) \cup j - \cl^*(i - \text{int}(A_\alpha))\}
\]

\[
\subset \bigcup_{\alpha \in \wedge} \{i - \text{int}(A_\alpha \cup (A_\alpha)_j)\} \cup \{(i - \text{int}(A_\alpha)) \cup (i - \text{int}(A_\alpha))_j^*\}
\]

\[
\subset \{i - \text{int}((\bigcup_{\alpha \in \wedge} A_\alpha) \cup (\bigcup_{\alpha \in \wedge} (A_\alpha)_j^*))\}
\]

\[
\cup \{(i - \text{int}(\bigcup_{\alpha \in \wedge} A_\alpha)) \cup (\bigcup_{\alpha \in \wedge} (i - \text{int}(A_\alpha))_j^*)\}
\]

\[
\subset \{i - \text{int}((\bigcup_{\alpha \in \wedge} A_\alpha) \cup (\bigcup_{\alpha \in \wedge} A_\alpha)^*_j)\}
\]

\[
\cup \{i - \text{int}(\bigcup_{\alpha \in \wedge} A_\alpha) \cup (i - \text{int}(\bigcup_{\alpha \in \wedge} A_\alpha))_j^*\}
\]

\[
\subset i - \text{int}(j - \cl^*(\bigcup_{\alpha \in \wedge} A_\alpha)) \cup j - \cl^*(i - \text{int}(\bigcup_{\alpha \in \wedge} A_\alpha)).
\]

Hence \(\bigcup_{\alpha \in \wedge} A_\alpha\) is \((i, j)-bI\)-open.

Lemma 3.1. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A, B\) are subsets of \(X\) such that \(B \subset A\). Then \(B_i^*(\tau_i|_A, I|_A) = B_i^*(\tau_i, I) \cap A\), for \(i = 1, 2\).

Here we denote that for any subset \(A\) of \((X, \tau_1, \tau_2, I)\), \(\tau_i|_A\) is the relative topology on \(A\) where \(i = 1, 2\) and \(I|_A = \{A \cap I : I \in I\}\) is obviously an ideal on \(A\).

Theorem 3.3. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. If \(A \in (i, j)-\text{BIO}(X)\) and \(B \in \tau_1 \cap \tau_2\), then \(A \cap B \in \text{BIO}(B, \tau_1|_B, \tau_2|_B, I|_B)\).

Proof. Since \(B \in \tau_1 \cap \tau_2\), therefore \(i\text{-int}_B(P) = i\text{-int}(P)\), for any subset \(P\) of \(B\) and \(i = 1, 2\). By using the fact and Lemma 3.1, we have

\[ A \cap B \subset \{i - \text{int}(j - \cl^*(A)) \cup j - \cl^*(i - \text{int}(A))\} \cap B \]
= \{ i - \text{int}(j - \text{cl}^* (A)) \cap B \} \cup j - \text{cl}^* (i - \text{int}(A)) \cap B \\
= \{ i - \text{int}(A \cup A^*_j) \} \cap B \cup \{(i - \text{int}(A)) \cup (i - \text{int}(A))_j^* \} \cap B \\
\subset \{(i - \text{int}(A \cup A^*_j)) \} \cap B \} \cup \{(i - \text{int}(A)) \} \cap B \\
\cup \{(i - \text{int}(A))_j^* \} \cap B \\
\subset \{(i - \text{int}((A \cap B) \cup (A \cap B)_j^*)) \} \cap B \} \cup \{(i - \text{int}_B(A \cap B)) \} \cap B \\
\cup \{(i - \text{int}_B(A \cap B))\}_j^*) \} \cap B \\
\subset \{(i - \text{int}((A \cap B) \cup (A \cap B)_j^*)) \} \cap B \} \cup \{(i - \text{int}_B(A \cap B)) \} \cap B \\
\cup \{(i - \text{int}_B(A \cap B))\}_j^*) \} \cap B \\
\subset \{ i - \text{int}_B(j - \text{cl}^* (A \cap B)) \} \cup \{ j - \text{cl}^* (i - \text{int}_B(A \cap B)) \}. \\

Hence A \cap B \in BIO(bI|B, \tau_1|B, \tau_2|B, I|B).

**Definition 3.2.** A subset A of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be \((i, j)\)-bI-closed if its complement is \((i, j)\)-bI-open.

**Theorem 3.4.** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. Then a subset A of \((X, \tau_1, \tau_2, I)\) is \((i, j)\)-bI-closed if \(i\)-cl\((j-int^*(A)) \cap j-int^*(i-cl(A)) \subset A\), where \(i, j \in \{1, 2\}, i \neq j\).

**Proof.** It is clear from the definition.

**Theorem 3.5.** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. If a subset A of X is \((i, j)\)-bI-closed, then \(i\)-cl\((j-int^*(A)) \cap j-int(i-cl^*(A)) \subset A\), where \(i, j \in \{1, 2\}, i \neq j\).

**Proof.** Let A is \((i, j)\)-bI-closed. Then \(X \setminus A\) is \((i, j)\)-bI-open. Since \(\tau_i^*\) is finer than \(\tau_i\) \((i = 1, 2)\), we have

\[
\begin{align*}
X \setminus A &\subset i - \text{int}(j - \text{cl}^*(X \setminus A)) \cup j - \text{cl}^* (i - \text{int}(X \setminus A)) \\
&\subset i - \text{int}(j - \text{cl}(X \setminus A)) \cup j - \text{cl}(i - \text{int}(X \setminus A)) \\
&= \{X \setminus (i - \text{cl}(j - \text{int}(A)))\} \cup \{X \setminus (j - \text{int}(i - \text{cl}(A)))\} \\
&\subset \{X \setminus (i - \text{cl}^*(j - \text{int}(A)))\} \cup \{X \setminus (j - \text{int}(i - \text{cl}^*(A)))\} \\
&= X \setminus \{(i - \text{cl}^*(j - \text{int}(A))) \cap (j - \text{int}(i - \text{cl}^*(A)))\}.
\end{align*}
\]

Hence \(i\)-cl\((j-int(A)) \cap j-int(i-cl^*(A)) \subset A\).

**Theorem 3.6.** Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A \subset X\). Then:
(a) If \( I = \emptyset \), then \( A \) is \((i, j)\)-bI-open if and only if \( A \) is \((i, j)\)-b-open.

(b) If \( I = P(X) \), then \( A \) is \((i, j)\)-bI-open if and only if \( A \in \tau_1 \cap \tau_2 \).

Proof. (a) It follows from the fact that if \( I = \emptyset \), then \( A^* = \text{cl}(A) \), for every subset \( A \) of \( X \).

(b) It follows from the fact that for every subset \( A \) of \( X \), if \( I = P(X) \) then \( A^* = \emptyset \).

Definition 3.3. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \( A \subseteq X \). Then

(a) The \((i, j)\)-bI-interior of \( A \), is defined as the union of all \((i, j)\)-bI-open sets contained in \( A \) and is denoted by \((i, j)\)-bI-int\((A)\).

(b) The \((i, j)\)-bI-closure of \( A \), is defined as the intersection of all \((i, j)\)-bI-closed sets containing \( A \) and is denoted by \((i, j)\)-bI-cl\((A)\).

Theorem 3.7. Let \( A \) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\). Then:

- (a) \((i, j)\)-bI-int\((A)\) is \((i, j)\)-bI-open.
- (b) \((i, j)\)-bI-cl\((A)\) is \((i, j)\)-bI-closed.
- (c) \( A \) is \((i, j)\)-bI-open if and only if \( A = (i, j)\)-bI-int\((A)\).
- (d) \( A \) is \((i, j)\)-bI-closed if and only if \( A = (i, j)\)-bI-cl\((A)\).

Proof. The proof of (a) – (d) are obvious.

Theorem 3.8. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \( A \subseteq X \). Then a point \( x \in (i, j)\)-bI-cl\((A)\) if and only if \( A \cap B \neq \emptyset \), for every \((i, j)\)-bI-open set \( B \) of \( X \) containing \( x \).

Proof. Suppose that \( x \in (i, j)\)-bI-cl\((A)\) and \( B \) be a \((i, j)\)-bI-open set containing \( x \). Assume that \( A \cap B = \emptyset \). Then \( A \subseteq X \setminus B \), where \( X \setminus B \) is \((i, j)\)-bI-closed set. This implies that \( x \in (i, j)\)-bI-cl\((A)\) \( \subseteq (i, j)\)-bI-cl\((X \setminus B)\) = \( X \setminus B \). Thus \( x \in X \setminus B \), which is a contradiction. Hence \( A \cap B \neq \emptyset \).

Conversely suppose that \( A \cap B \neq \emptyset \), for every \((i, j)\)-bI-open set \( B \) of \( X \) containing \( x \). Let \( x \notin (i, j)\)-bI-cl\((A)\). Then there exists a \((i, j)\)-bI-closed set \( G \) of \( X \) such that \( A \subseteq G \) and \( x \notin G \). Therefore \( x \in X \setminus G \), where \( X \setminus G \) is \((i, j)\)-bI-open and \((X \setminus G) \cap A = \emptyset \). Which is a contradiction to the assumption. Hence \( x \in (i, j)\)-bI-cl\((A)\).

Theorem 3.9. Let \( A \) be a subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\). Then
(a) \((i, j)-bI-cl(X \setminus A) = X \setminus (i, j)-bI-int(A)\).

(b) \((i, j)-bI-int(X \setminus A) = X \setminus (i, j)-bI-cl(A)\).

Proof. (a) Let \(x \notin (i, j)-bI-cl(X \setminus A)\). Then there exists an \((i, j)-bI\)-open set \(B\) of \(X\) containing \(x\) such that \(B \cap (X \setminus A) = \emptyset\). We have \(x \in B\), therefore \(x \notin X \setminus A\) and so \(x \in A\). Thus \(x \in B \subset A\) and so \(x \in (i, j)-bI-int(A)\). Which implies that \(x \notin (i, j)-bI-int(A)\). Hence \((i, j)-bI-int(A) \subset (i, j)-bI-cl(X \setminus A)\).

Next let \(x \notin X \setminus (i, j)-bI-int(A)\). Then \(x \in (i, j)-bI-int(A)\) and so there exists an \((i, j)-bI\)-open set \(B\) of \(X\) such that \(x \in B \subset A\). Thus \(B \cap (X \setminus A) = \emptyset\) and \(x \notin (i, j)-bI-cl(X \setminus A)\). Therefore \((i, j)-bI-cl(X \setminus A) \subset X \setminus (i, j)-bI-int(A)\).

Hence the result follows.

(b) Similar to the proof of (a).

Definition 3.4. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. Then a subset \(A\) of \(X\) is said to be an \((i, j)-bI\)-neighbourhood of a point \(x\) of \(X\) if there exists a \((i, j)-bI\)-open set \(B\) such that \(x \in B \subset A\).

Theorem 3.10. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(A \subset X\). Then \(A\) is \((i, j)-bI\)-open if and only if it is an \((i, j)-bI\)-neighbourhood of each of its points.

Proof. Let \(A\) be an \((i, j)-bI\)-open set of \((X, \tau_1, \tau_2, I)\). So by definition, we have \(A\) is an \((i, j)-bI\)-neighbourhood of each of its points, since for every \(x \in A\), we have \(x \in A \subset A\) and \(A\) is \((i, j)-bI\)-open. Conversely assume that \(A\) is an \((i, j)-bI\)-neighbourhood of each of its points. Then for every \(x \in A\), there exists an \((i, j)-bI\)-open set \(B_x\) of \((X, \tau_1, \tau_2, I)\) such that \(x \in B_x \subset A\). Then \(A = \bigcup\{B_x : x \in A\}\). Since arbitrary union of \((i, j)-bI\)-open sets is \((i, j)-bI\)-open, therefore we have \(A\) is \((i, j)-bI\)-open in \(X\).

4. \((i, j)-bI\)-Continuous Functions

Definition 4.1. A function \(f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)\) is said to be \((i, j)-bI\)-continuous if the inverse image of every \(\sigma_i\)-open set in \(Y\) is \((i, j)-bI\)-open in \(X\), where \(i, j = 1, 2, i \neq j\).

Remark 4.1. Every \((i, j)-bI\)-continuous function is \((i, j)-b\)-continuous. But the converse may not be true in general as shown in the following example.

Example 4.1. Let \(X = \{p, q, r\}, \tau_1 = \{\emptyset, \{q\}, X\}, \tau_2 = \{\emptyset, \{p\}, \{q, r\}, X\}\) and \(I = \{\emptyset, \{p\}\}\). Then the identity function \(f : (X, \tau_1, \tau_2, I) \to (X, \sigma_1, \sigma_2)\) is \((1, 2)-b\)-continuous but not \((1, 2)-bI\)-continuous.
**Theorem 4.1.** The following statements are equivalent for the function $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$.

(a) $f$ is $(i,j)$-bl-continuous.

(b) For all $x \in X$ and every $\sigma_i$-open set $B$ of $Y$ containing $f(x)$, there exists a $(i,j)$-bl-open set $A$ of $X$ containing $x$ such that $f(A) \subset B$.

(c) Inverse image of every $\sigma_i$-closed set of $Y$ is $(i,j)$-bl-continuous in $X$.

(d) $f((i,j)\text{-bl-cl}(A)) \subset \sigma_i\text{-cl}(f(A))$, for every subset $A$ of $X$.

(e) $(i,j)$-bl-cl$(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-cl}(B))$, for every subset $B$ of $Y$.

(f) $f^{-1}(\sigma_i\text{-int}(G)) \subset (i,j)\text{-bl-int}(f^{-1}(G))$, for every subset $G$ of $Y$.

*Proof.* (a) $\Rightarrow$ (b) Let $B$ be a $\sigma_i$-open set in $Y$ such that $f(x) \in B$. Since $f$ is $(i,j)$-bl-continuous, therefore $f^{-1}(B)$ is $(i,j)$-bl-open in $X$. Let $A = f^{-1}(B)$. Then $f(x) \in f(A) \subset B$.

(b) $\Rightarrow$ (a) Let $B$ be a $\sigma_i$-open set in $Y$ and $x \in f^{-1}(B)$. Then we have $f(x) \in B$. By (b), there exists an $(i,j)$-bl-open set $A_x$ in $X$ containing $x$ such that $f(A_x) \subset B$. Therefore $x \in A_x \in f^{-1}(B)$. Consequently, $f^{-1}(B)$ is $(i,j)$-bl-open in $X$. Hence $f^{-1}(B)$ is $(i,j)$-bl-continuous.

(a) $\Rightarrow$ (c) It is obvious.

(c) $\Rightarrow$ (d) Let $A \subset X$. Since $\sigma_i\text{-cl}(f(A))$ is $\sigma_i$-closed set in $Y$, therefore by (c) we have $f^{-1}(\sigma_i\text{-cl}(f(A)))$ is $(i,j)$-bl-continuous set in $X$. Also, $A \subset f^{-1}(\sigma_i\text{-cl}(f(A)))$ and $(i,j)$-bl-cl$(A)$ is the smallest $(i,j)$-bl-continuous set containing $A$. Therefore $(i,j)$-bl-cl$(A) \subset f^{-1}(\sigma_i\text{-cl}(f(A)))$. This implies that $f((i,j)\text{-bl-cl}(A)) \subset \sigma_i\text{-cl}(f(A))$.

(d) $\Rightarrow$ (e) Let $B \subset Y$. Then $f^{-1}(B)$ is a subset of $X$. By (d), $f((i,j)\text{-bl-cl}(f^{-1}(B))) \subset \sigma_i\text{-cl}(f(f^{-1}(B))) \subset \sigma_i\text{-cl}(B)$. Hence $(i,j)\text{-bl-cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-cl}(B))$.

(e) $\Rightarrow$ (c) Let $B$ be a $\sigma_i$-closed set in $Y$. By (e), $(i,j)\text{-bl-cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-cl}(B)) = f^{-1}(B)$. Therefore $f^{-1}(B) = (i,j)\text{-bl-cl}(f^{-1}(B))$ and so $f^{-1}(B)$ is $(i,j)$-bl-closed in $X$.

(a) $\Rightarrow$ (f) Let $G$ be a $\sigma_i$-open subset of $Y$. By (a), $f^{-1}(G) = f^{-1}(\sigma_i\text{-int}(G))$ is $(i,j)$-bl-open in $X$. Then $f^{-1}(\sigma_i\text{-int}(G)) \subset (i,j)\text{-bl-int}(f^{-1}(\sigma_i\text{-int}(G))) \subset (i,j)\text{-bl-int}(f^{-1}(G))$.

(f) $\Rightarrow$ (a) Let $B$ be a $\sigma_i$-open subset of $Y$. Then $f^{-1}(B) = f^{-1}(\sigma_i\text{-int}(B)) \subset (i,j)\text{-bl-int}(f^{-1}(B))$. Therefore $f^{-1}(B)$ is $(i,j)$-bl-open in $X$ and so $f$ is $(i,j)$-bl-continuous.
bI-open sets in ideal bitopological spaces

**Theorem 4.2.** Let

\[ f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2, J) \quad \text{and} \quad g : (Y, \sigma_1, \sigma_2, J) \longrightarrow (Z, \theta_1, \theta_2) \]

be two functions, where \( I \) and \( J \) are two ideals in \( X \) and \( Y \) respectively. If \( f \) is \((i, j)\)-bI-continuous and \( g \) is pairwise continuous, then \( g_0f \) is \((i, j)\)-bI-continuous.

**Proof.** Let \( C \) be a \( \theta_i \)-open set in \( Z \). Since \( g \) is pairwise continuous, therefore \( g^{-1}(C) \) is \( \sigma_i \)-open in \( Y \). Also, \( f \) is \((i, j)\)-bI-continuous, so \((g_0f)^{-1}(C) = f^{-1}(g^{-1}(C))\) is \((i, j)\)-bI-open in \( X \). Hence \( g_0f \) is \((i, j)\)-bI-continuous.

**Theorem 4.3.** Let \( f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2) \) be \((i, j)\)-bI-continuous and \( A \subset X \). If \( A \in \tau_1 \cap \tau_2 \), then the restriction function \( f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-bI-continuous.

**Proof.** Let \( B \) be a \( \sigma_i \)-open subset of \( Y \). Since \( f \) is \((i, j)\)-bI-continuous, therefore \( f^{-1}(B) \) is \((i, j)\)-bI-open set in \( X \). Also, \( A \in \tau_1 \cap \tau_2 \), therefore by theorem 3.3, we have \( A \cap f^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A) \). Again

\[ (f|_A)^{-1}(B) = A \cap f^{-1}(B) \]

and so \((f|_A)^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A)\). Hence \( f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \rightarrow (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-bI-continuous.

**Theorem 4.4.** Let \( f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2) \) be a function. If a function \( g : (X, \tau_1, \tau_2, I) \longrightarrow (X \times Y, \sigma_1 \times \sigma_2) \), defined by \( g(x) = (x, f(x)) \) for each \( x \in X \) is \((i, j)\)-bI-continuous, then \( f \) is \((i, j)\)-bI-continuous.

**Proof.** Suppose that \( g \) is \((i, j)\)-bI-continuous. Let \( x \in X \) and \( B \) be a \( \sigma_i \)-open subset of \( Y \) containing \( f(x) \). Then \( X \times B \) is \( \tau_i \times \sigma_i \)-open in \( X \times Y \). By the \((i, j)\)-bI-continuity of \( g \), there exists an \((i, j)\)-bI-open set \( A \) in \( X \) containing \( x \) such that \( f(A) \subset X \times Y \), by theorem 4.1. So \( f(A) \subset B \). Hence \( f \) is \((i, j)\)-bI-continuous.

**Definition 4.2.** (see [13]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise connected if it cannot be expressed as the union of two nonempty disjoint sets \( A \) and \( B \) such that \( A \) is \( \tau_i \)-open and \( B \) is \( \tau_j \)-open, where \( i, j = \{1, 2\} \).

**Definition 4.3.** An ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be \((i, j)\)-bI-connected if there exists a nonempty \((i, j)\)-bI-open set \( A \) and a nonempty \((j, i)\)-bI-open set \( B \) in \( X \) such that \( X \neq A \cup B \) and \( A \cap B = \emptyset \).

**Theorem 4.5.** Let \( f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2) \) be a function. If \( f \) is \((i, j)\)-bI-continuous surjection and \( X \) is \((i, j)\)-bI-connected, then \( Y \) is pairwise connected.
Proof. Assume that $Y$ is not pairwise connected. Then there exists a nonempty $\sigma_i$-open set $A$ and a nonempty $\sigma_j$-open set $B$ such that $Y = A \cup B$ and $A \cap B = \emptyset$. Since $f$ is $(i, j)$-$bI$-continuous, therefore $f^{-1}(A)$ is $(i, j)$-$bI$-open in $X$ and $f^{-1}(B)$ is $(j, i)$-$bI$-open in $X$. Since $A \cap B = \emptyset$ and $f$ is surjective, therefore $f^{-1}(A) \cap f^{-1}(B) = \emptyset$. Thus $f^{-1}(Y) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$. This implies that $X = f^{-1}(A) \cup f^{-1}(B)$. Hence $X$ is not $(i, j)$-$bI$-connected, which is a contradiction.

Lemma 4.1([12]). For any function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$, $f(I)$ is an ideal on $Y$.

Definition 4.4. An ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(i, j)$-$I$-compact if for every open cover $\{C_\alpha : \alpha \in \wedge\}$ by $\tau_i$-open sets of $X$, there exists a finite subset $\wedge_0$ of $\wedge$ such that $X \setminus \bigcup\{C_\alpha : \alpha \in \wedge\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.

Definition 4.5. An ideal bitopological space $(X, \tau_1, \tau_2, I)$ is said to be $(i, j)$-$bI$-compact if for every open cover $\{C_\alpha : \alpha \in \wedge\}$ by $(i, j)$-$bI$-open sets of $X$, there exists a finite subset $\wedge_0$ of $\wedge$ such that $X \setminus \bigcup\{C_\alpha : \alpha \in \wedge\} \in I$, where $i, j = \{1, 2\}$ and $i \neq j$.

Theorem 4.6. Let $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a function. If $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $(i, j)$-$bI$-continuous surjection and $X$ is $(i, j)$-$bI$-compact, then $(Y, \sigma_1, \sigma_2, f(I))$ is $(i, j)$-$f(I)$-compact.

Proof. Let $\{C_\alpha : \alpha \in \wedge\}$ be an open cover of $Y$ by $\sigma_i$-open sets in $Y$. Since $f$ is $(i, j)$-$bI$-continuous, therefore $\{f^{-1}(C_\alpha) : \alpha \in \wedge\}$ be an open cover of $X$ by $(i, j)$-$bI$-open sets in $X$. Also, $(X, \tau_1, \tau_2, I)$ is $(i, j)$-$bI$-compact. Therefore there exists a finite subset $\wedge_0$ of $\wedge$ such that $X \setminus \bigcup\{f^{-1}(C_\alpha) : \alpha \in \wedge_0\} \in I$. Thus $Y \setminus \bigcup\{C_\alpha : \alpha \in \wedge_0\} = f(X \setminus \bigcup\{f^{-1}(C_\alpha) : \alpha \in \wedge_0\} \in f(I)$. Hence $(Y, \sigma_1, \sigma_2, f(I))$ is $(i, j)$-$f(I)$-compact.

Definition 4.6. A function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-$bI$-irresolute if the inverse image of every $(i, j)$-$bI$-open set in $Y$ is $(i, j)$-$bI$-open in $X$, where $i, j = 1, 2$ and $i \neq j$.

Theorem 4.7. The following statements are equivalent for the function $f : (X, \tau_1, \tau_2, I) \longrightarrow (Y, \sigma_1, \sigma_2)$.

(a) $f$ is $(i, j)$-$bI$-irresolute.

(b) For all $x \in X$ and every $(i, j)$-$bI$-open set $B$ of $Y$ containing $f(x)$, there exists an $(i, j)$-$bI$-open set $A$ of $X$ containing $x$ such that $f(A) \subset B$.

(c) Inverse image of every $(i, j)$-$bI$-closed set of $Y$ is $(i, j)$-$bI$-closed set in $X$. 
**Theorem 4.8.** Let

\[ f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2, J) \] and \[ g : (Y, \sigma_1, \sigma_2, J) \to (Z, \theta_1, \theta_2) \]

be two functions, where \( I \) and \( J \) are two ideals in \( X \) and \( Y \) respectively.

(a) If \( f \) is \((i, j)\)-\( bI \)-irresolute and \( g \) is \((i, j)\)-\( b \)-irresolute, then \( g_0f \) is \((i, j)\)-\( bI \)-irresolute.

(b) If \( f \) is \((i, j)\)-\( bI \)-irresolute and \( g \) is \((i, j)\)-\( b \)-continuous, then \( g_0f \) is \((i, j)\)-\( bI \)-continuous.

**Proof.** (a) Let \( C \) be an \((i, j)\)-\( b \)-open set in \( Z \). Since \( g \) is \((i, j)\)-\( b \)-irresolute, therefore \( g^{-1}(C) \) is \((i, j)\)-\( b \)-open set in \( Y \). Also, \( f \) is \((i, j)\)-\( bI \)-irresolute, so \((g_0f)^{-1}(C) = f^{-1}(g^{-1}(C)) \) is \((i, j)\)-\( bI \)-open in \( X \). Hence \( g_0f \) is \((i, j)\)-\( bI \)-irresolute.

(b) Let \( C \) be a \( \theta_1 \)-open set in \( Z \). Since \( g \) is \((i, j)\)-\( b \)-continuous, therefore \( g^{-1}(C) \) is \((i, j)\)-\( b \)-open set in \( Y \). Also, \( f \) is \((i, j)\)-\( bI \)-irresolute, so \((g_0f)^{-1}(C) = f^{-1}(g^{-1}(C)) \) is \((i, j)\)-\( bI \)-open in \( X \). Hence \( g_0f \) is \((i, j)\)-\( bI \)-continuous.

**Theorem 4.9.** Let \( f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2) \) be \((i, j)\)-\( bI \)-irresolute and \( A \subseteq X \). If \( A \in \tau_1 \cap \tau_2 \), then the restriction function \( f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \to (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-\( bI \)-irresolute.

**Proof.** Let \( B \) be a \((i, j)\)-\( b \)-open subset of \( Y \). Since \( f \) is \((i, j)\)-\( bI \)-irresolute, therefore \( f^{-1}(B) \) is \((i, j)\)-\( bI \)-open set in \( X \). Also, \( A \in \tau_1 \cap \tau_2 \), therefore by theorem 3.3, we have \( A \cap f^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A) \). Again

\[ (f|_A)^{-1}(B) = A \cap f^{-1}(B) \]

and so

\[ (f|_A)^{-1}(B) \in BIO(A, \tau_1|_A, \tau_2|_A, I|_A). \]

Hence

\[ f|_A : (A, \tau_1|_A, \tau_2|_A, I|_A) \to (Y, \sigma_1, \sigma_2) \]

is \((i, j)\)-\( bI \)-irresolute.

**References**


