THE TOPOLOGICAL INDICES OF
NON-COMMUTING GRAPH OF A FINITE GROUP

M. Jahandideh¹§, N.H. Sarmin², S.M.S. Omer³

¹Department of Mathematics
Shahid Chamran University of Ahvaz
Ahvaz, IRAN

²Department of Mathematical Sciences
Faculty of Science
University Teknologi Malaysia
81310 UTM Johor Bahru, Johor, MALAYSIA

³Department of Mathematics
Faculty of Science
University of Benghazi
Benghazi, LIBYA

Abstract: Assume G is a non-abelian finite group. The non-commuting graph Γ_G of G is defined as a graph with vertex set G \(- Z(G) in which Z(G) is the center of G and two distinct vertices x and y are joined if and only if xy \neq yx. Various topological indices have been determined for simple and connected graphs. Since non-commuting graph is a simple and connected graph, topological indices could be defined for it. The main objective of this article is to calculate various topological indices including the Szeged index, Edge-Wiener index, the first Zagreb index and the second Zagreb index for the non-commuting graph of G.

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§Correspondence author
1. Introduction

In this paper, $G$ is a non-abelian finite group. Various graphs could be attributed to $G$, one of which is the non-commuting graph, denoted by $\Gamma_G$. The set of vertices and edges of $\Gamma_G$ are $V(\Gamma_G)$ and $E(\Gamma_G)$, respectively so that $V(\Gamma_G) = G - Z(G)$ in which $Z(G)$ is the center of $G$ and for every $x, y \in V(\Gamma_G)$ we have $\{x, y\} \in E(\Gamma_G) \iff xy \neq yx$. The centralizer of $x$ within $G$ which is denoted by $C_G(x)$ is a subset of $G$ which is defined as $\{g \in G : gx = xg\}$. According to [3], the non-commuting graph of a finite group $G$ was first introduced by Paul Erdos.

Assume that $G = (V, E)$ is a graph in which $V$ is the set of vertices and $E$ is the set of edges. This graph is a finite graph whenever $|V|$ and $|E|$ are finite. The distance between two vertices $x$ and $y$ is denoted by $d(x, y)$, which the length of the shortest path between the two vertices $x$ and $y$. The degree of the vertex $x$ is denoted by $\deg(x)$, equal to the number of edges through $x$. The diameter of $G$ is defined as follows:

$$\text{diam}(G) = \max\{d(x, y) : x, y \in V(\Gamma_G)\}.$$

The Szeged index of the graph $G = (V, E)$ is defined as follows: This index is a recently introduced invariant of a graph which is based on the distances of the vertices of the graph [5] and [6]. Let $e = xy$ be an edge of $G$. We define the following sets:

$$N_x(e|G) = \{w \in V : d(w, x) < d(w, y)\},$$
$$N_y(e|G) = \{w \in V : d(w, y) < d(w, x)\}.$$

Hence $N_x(e|G)$ is the set of all vertices of $G$ which are closer to $x$ than $y$ and $N_y(e|G)$ is the set of all vertices of $G$ which are closer to $y$ than $x$. The size of $N_x(e|G)$ are $N_y(e|G)$ are denoted by $n_x(e|G)$ and $n_y(e|G)$, respectively.

The Szeged index of the graph $G$ is defined by

$$Sz(G) = \sum_{e=xy \in E(G)} n_x(e|G) \cdot n_y(e|G).$$

Let $G$ be a connected graph. The Edge-Wiener index of $G$ is defined as follows:

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} d(e, f).$$

Where $e, f$ are two edges in $G$ and $d(e, f)$ is the distance between two vertices in the line-graph. In view of the above definition $W_e(G) = W(\overline{G})$ ($\overline{G}$ is the
line-graph of $G$). For more details, refer to the [4]. The first Zagreb index of $G$ is denoted by $Z_1(G)$ and is defined by:

$$Z_1(G) = \sum_{x \in V} (\text{deg}(x))^2.$$ 

The second Zagreb index of the graph $G$ is defined by:

$$Z_2(G) = \sum_{\{x,y\} \subseteq V} \text{deg}(x) \cdot \text{deg}(y).$$

The readers can refer to [7] for more details. Our main goal is to calculate the above mentioned indices for the non-commuting graph of $G$ in terms of the order of $G$, $Z(G)$ and the number of conjugacy classes of $G$. The following lemmas will be used repeatedly.

**Lemma 1.** [1]. Let $G$ be a finite group. Then $\text{diam}(\Gamma_G) = 2$.

**Lemma 2.** [1]. Let $G$ be a finite group and $k(G)$ the number of conjugacy classes of $G$, then

$$|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G)).$$

**Lemma 3.** [1]. Let $G$ be a finite group. If $x$ be one of the vertices of $\Gamma_G$, then

$$\text{deg}(x) = |G| - |C_G(x)|.$$ 

2. The Szeged Index of a Non-Commuting Graph

In this section, we find the Szeged index for the non-commuting graph of a finite group.

**Lemma 4.** Let $G$ be a finite group. Then

$$\sum_{x \notin Z(G)} |C_G(x)| = |G|(k(G) - |Z(G)|).$$

**Proof.** We know that $G$ is the union of its conjugacy classes. Assume that $\{x_i\}_{i=1}^k$ are the representative of the conjugacy classes and $\text{class}(x_i)$ denotes the conjugacy class of $x_i$ and $G = \bigcup_{i=1}^k \text{class}(x_i)$.

Now, let $\{x_i\}_{i=1}^t \notin Z(G)$, thus we have $k(G) = t + |Z(G)|$. Every $x$ which is not
placed within \( Z(G) \) would be placed within one of class\( (x_i) \)s in which \( 1 \leq i \leq t \). Therefore we have:

\[
\sum_{x \notin Z(G)} |C_G(x)| = \sum_{i=1}^{t} |\text{class}(x_i)||C_G(x_i)| = |G|t = |G|(k(G) - |Z(G)|).
\]

\[
\Box
\]

In the next theorem, we calculate the Szeged index of \( \Gamma_G \).

**Theorem 5.** Assume \( G \) is a finite group and \( \Gamma_G \) its non-commuting graph. Then the Szeged index of \( \Gamma_G \) is

\[
Sz(\Gamma_G) = \frac{1}{2} \left( \sum_{i=1}^{n} \left( \sum_{x_j \notin C_G(x_i)} \left( |C_G(x_i) \cap C_G(x_j)| \right)^2 \right) + \sum_{i=1}^{n} \deg(x_i) \left( \sum_{x_j \notin C_G(x_i)} +2|C_G(x_i) \cap C_G(x_j)| - |C_G(x_j)| \right) \right. \\
\left. + |G| \left( \sum_{i=1}^{n} \left( \sum_{x_j \notin C_G(x_i)} -2|C_G(x_i) \cap C_G(x_j)| + |C_G(x_j)| \right) \right) \right).
\]

**Proof.** Assume that \( x \) and \( y \) are two arbitrary vertices of the graph \( \Gamma_G \) that are joined together by \( e \) (where \( e \) is one of the edges of the non-commuting graph). Now we calculate \( n_x(e|\Gamma_G) \) and \( n_y(e|\Gamma_G) \):

\[
N_x(e|\Gamma_G) = \{w \in V(\Gamma_G) : d(w, x) < d(w, y)\}.
\]

According to Lemma 1, we have:

If \( d(w, y) = 1 \) then \( d(w, x) = 0 \) and \( w = x \). If \( d(w, y) = 2 \) then \( d(w, x) = 0 \) or 1. So

\[
n_x(e|\Gamma_G) = (|C_G(y)| - 1) - |C_G(x) \cap C_G(y)| + 1 \\
= |C_G(y)| - |C_G(x) \cap C_G(y)|
\]

In order to

\[
n_y(e|\Gamma_G) = (|C_G(x)| - 1) - |C_G(x) \cap C_G(y)| + 1 \\
= |C_G(x)| - |C_G(x) \cap C_G(y)|.
\]
\[ S_{2}(\Gamma_{G}) = \sum_{e = xy \in E} n_{x}(e|\Gamma_{G}) \cdot n_{y}(e|\Gamma_{G}) \]

\[ = \sum_{e = xy \in E} (|C_{G}(y)| - |C_{G}(x) \cap C_{G}(y)|)(|C_{G}(x)| - |C_{G}(x) \cap C_{G}(y)|) \]

\[ = \sum_{e = xy \in E} |C_{G}(x) \cap C_{G}(y)|^2 \]

\[ - \sum_{e = xy \in E} (|C_{G}(x)| + |C_{G}(y)|)|C_{G}(x) \cap C_{G}(y)| + \sum_{e = xy \in E} |C_{G}(x)||C_{G}(y)| \]

Now, we have to calculate the all of summations. Letting \(|G| - |Z(G)| = n\), we obtain

\[ \sum_{e = xy \in E} |C_{G}(x) \cap C_{G}(y)|^2 = \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{x \in G - Z(G)} \sum_{j \notin C_{G}(x)} (|C_{G}(x_i) \cap C_{G}(x_j)|)^2 \right) \]

So we can gain

\[ \sum_{e = xy \in E} (|C_{G}(x)| + |C_{G}(y)|)|C_{G}(x) \cap C_{G}(y)| = \sum_{x \in G - Z(G)} |C_{G}(x)||C_{G}(x) \cap C_{G}(y)| \]

\[ = \sum_{i=1}^{n} |C_{G}(x_i)| \left( \sum_{x \notin C_{G}(x_i)} |C_{G}(x_i) \cap C_{G}(x_j)| \right) \]

\[ = \sum_{i=1}^{n} (|G| - \deg(x_i)) \left( \sum_{x \notin C_{G}(x_i)} |C_{G}(x_i) \cap C_{G}(x_j)| \right) \]

\[ = \sum_{i=1}^{n} - \deg(x_i) \left( \sum_{x \notin C_{G}(x_i)} |C_{G}(x_i) \cap C_{G}(x_j)| \right) \]

\[ + |G| \sum_{i=1}^{n} \left( \sum_{x \notin C_{G}(x_i)} |C_{G}(x_i) \cap C_{G}(x_j)| \right). \]

Now, calculating

\[ \sum_{e = xy \in E} |C_{G}(x)||C_{G}(y)|. \]

\[ \sum_{e = xy \in E} |C_{G}(x)||C_{G}(y)| = \frac{1}{2} \sum_{i=1}^{n} \left( |C_{G}(x_i)| \sum_{x \notin C_{G}(x_i)} |C_{G}(x_j)| \right) \]
\[
= \frac{1}{2} \sum_{i=1}^{n} \left( |G| - \deg(x_i) \sum_{x_j \not\in C_G(x_i)} |C_G(x_j)| \right)
\]
\[
= -\frac{1}{2} \sum_{i=1}^{n} \left( \deg(x_i) \sum_{x_j \not\in C_G(x_i)} |C_G(x_j)| \right)
\]
\[
+ \frac{|G|}{2} \sum_{i=1}^{n} \left( \sum_{x_j \not\in C_G(x_i)} |C_G(x_j)| \right).
\]

Now, the Szeged index is equal to
\[
S_Z(\Gamma_G) = \sum_{e=xy \in E} |C_G(x) \cap C_G(y)|^2 - \sum_{e=xy \in E} (|C_G(x)| + |C_G(y)|)|C_G(x) \cap C_G(y)|
\]
\[
+ \sum_{e=xy \in E} |C_G(x)||C_G(y)|
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{x_j \not\in C_G(x_i)} \left( |C_G(x_i) \cap C_G(x_j)| \right)^2 \right)
\]
\[
+ \sum_{i=1}^{n} \deg(x_i) \left( \sum_{x_j \not\in C_G(x_i)} |C_G(x_i) \cap C_G(x_j)| \right)
\]
\[
- |G| \sum_{i=1}^{n} \left( \sum_{x_j \not\in C_G(x_i)} |C_G(x_i) \cap C_G(x_j)| \right)
\]
\[
- \frac{1}{2} \sum_{i=1}^{n} \left( \deg(x_i) \sum_{x_j \not\in C_G(x_i)} |C_G(x_j)| \right)
\]
\[
+ \frac{|G|}{2} \sum_{i=1}^{n} \left( \sum_{x_j \not\in C_G(x_i)} |C_G(x_j)| \right)
\]
\[
= \frac{1}{2} \left( \sum_{x_j \not\in C_G(x_i)} \left( |C_G(x_i) \cap C_G(x_j)| \right)^2 \right)
\]
\[
+ \sum_{i=1}^{n} \deg(x_i) \left( \sum_{x_j \not\in C_G(x_i)} +2|C_G(x_i) \cap C_G(x_j)| - |C_G(x_j)| \right)
\]
+ |G| \left( \sum_{i=1}^{n} \left( \sum_{x_j \notin C_G(x_i)} -2|C_G(x_i) \cap C_G(x_j)| + |C_G(x_j)| \right) \right) \right). \]

3. The Edge-Wiener Index of a Non-Commuting Graph

In this section, we find the Edge-Wiener index of a non-commuting graph. We start with a couple of lemmas.

Lemma 6. [2]. Assume $G$ is a finite group and $\Gamma_G$ its non-commuting graph. If $\Gamma_G$ is a line-graph then,

$$|V(\Gamma_G)| = |E(\Gamma_G)|, \quad |E(\Gamma_G)| = \sum_{x \in V(\Gamma_G)} \left( \frac{\deg(x)}{2} \right).$$

Lemma 7. Assume $G$ is a finite group and $\Gamma_G$ a line-graph of $\Gamma_G$. Then $\Gamma_G$ is a connected graph and $\text{diam}(\Gamma_G) = 2$.

Proof. First, we prove that there is a path between two vertices of $\Gamma_G$. Assume that two arbitrary vertices $e$ and $f$ belong to $\Gamma_G$, thus $e$ is an edge in $\Gamma_G$, so there are two vertices $x$ and $y$ of $\Gamma_G$ that are joined together by $e$. Furthermore, there are two vertices $x_1$ and $y_1$ that are connected together by $f$. We know that $\text{diam}(\Gamma_G) = 2$, thus there is at least an edge between all mentioned vertices. It means: there is a path between two edges. Now, we prove that $\text{diam}(\Gamma_G) = 2$. Suppose that $\text{diam}(\Gamma_G) = 1$, then $\Gamma_G$ is a complete graph. Next

$$\exists \ x \in G \ \exists \ y \neq x^{-1} \Rightarrow \exists \ y \in G \ \exists \ x \xrightarrow{e} y \xrightarrow{f} x^{-1},$$

$$G \neq C_G(x) \cup C_G(y) \Rightarrow \exists \ z \in G - C_G(x) \cup C_G(y).$$

Therefore, we have $z \xrightarrow{h} x \xrightarrow{e} y \xrightarrow{g} z$, but $\Gamma_G$ is a complete graph, so $h$ and $f$ are joined together, which is impossible. Since $z \neq x, y$ and $x \neq y, x^{-1}$. Thus $\text{diam}(\Gamma_G) \neq 1$. Hence $\text{diam}(\Gamma_G) = 2$. \qed

Theorem 8. Let $G$ be a finite group and $\Gamma_G$ a line-graph of $\Gamma_G$. Then

$$W_e(\Gamma_G) = |E(\Gamma_G)|^2 + |G|^2 \left( k(G) - \frac{1}{2} |Z(G)| - \frac{1}{2} |G| \right) - \frac{1}{2} \sum_{x \in G - Z(G)} |C_G(x)|^2.$$
Proof. By definition, \( W_e(\overline{\Gamma_G}) = \sum_{\{e,f\} \subseteq E(\Gamma_G)} d(e, f) = \frac{1}{2} \sum_{e \in E(\Gamma_G)} d(e) \) where
\[
d(e) = \sum_{f \in E(\Gamma_G)} d(e, f).
\]

First we compute \( d(e) \) for an arbitrary vertex of the graph \( \overline{\Gamma_G} \). According to Lemma 7, \( d(e) = \sum_{f \in E(\Gamma_G)} d(e, f) = 2 \) (the number of vertices whose distance from \( e \) is 2) + 1 (the number of vertices whose distance from \( e \) is 1). Let \( x \) and \( y \) be joined together by \( e \). Then
\[
d(e) = \sum_{f \in E(\Gamma_G)} d(e, f)
= 1((\deg(x) - 1) + (\deg(y) - 1)) + 2(|E(\Gamma_G)| - \deg(x) - \deg(y) + 1)
= 2|E(\Gamma_G)| - (\deg(x) + \deg(y)).
\]

Using the above formula, we can calculate \( W_e(\overline{\Gamma_G}) \):
\[
W_e(\overline{\Gamma_G}) = \frac{1}{2} \sum_{e \in E(\Gamma_G)} d(e)
= \frac{1}{2} \sum_{e \in E(\Gamma_G)} 2|E(\Gamma_G)| - (\deg(x) + \deg(y))
= |E(\Gamma_G)|^2 - \frac{1}{2} \sum_{e \in E(\Gamma_G)} (\deg(x) + \deg(y))
= |E(\Gamma_G)|^2 - \frac{1}{2} \sum_{x \in G - Z(G)} (|G| - |C_G(x)|)^2
= |E(\Gamma_G)|^2 - \frac{1}{2} |G|^2 (|G| - |Z(G)|) + |G| \sum_{x \in G - Z(G)} |C_G(x)|
- \frac{1}{2} \sum_{x \in G - Z(G)} |C_G(x)|^2
= |E(\Gamma_G)|^2 - \frac{1}{2} |G|^2 (|G| - |Z(G)|) + |G|^2 (k(G) - |Z(G)|)
- \frac{1}{2} \sum_{x \in G - Z(G)} |C_G(x)|^2.
\[ |E(\Gamma_G)|^2 + |G|^2 \left( k(G) - \frac{1}{2}|Z(G)| - \frac{1}{2}|G| \right) \]
\[ - \frac{1}{2} \sum_{x \in G - Z(G)} |C_G(x)|^2. \]

\[ \square \]

4. The First Zagreb Index of a Non-Commuting Graph

In this section, the first Zagreb index of a non-commuting graph is computed.

**Theorem 9.** Let \( G \) be a finite group and \( \Gamma_G \) its non-commuting graph. Then
\[ Z_1(\Gamma_G) = |G|^2 (|G| + |Z(G)| - 2k(G)) + \sum_{x \in G - Z(G)} |C_G(x)|^2. \]

**Proof.** Using the definition of \( Z_1(\Gamma_G) \), we have
\[ Z_1(\Gamma_G) = \sum_{x \in G - Z(G)} \text{deg}(x)^2 \]
\[ = \sum_{x \in G - Z(G)} (|G| - |C_G(x)|)^2 \]
\[ = |G|^2 (|G| - |Z(G)|) - 2|G| \sum_{x \in G - Z(G)} |C_G(x)| + \sum_{x \in G - Z(G)} |C_G(x)|^2 \]
\[ = |G|^2 (|G| + |Z(G)| - 2k(G)) + \sum_{x \in G - Z(G)} |C_G(x)|^2. \]
\[ \square \]

5. The Second Zagreb Index of a Non-Commuting Graph

In this section, we calculate the second Zagreb index of a non-commuting graph.

**Theorem 10.** Let \( G \) be a finite group and \( \Gamma_G \) its non-commuting graph. Then
\[ Z_2(\Gamma_G) = \frac{1}{2} \left( |G|^2 (|G| - k(G))^2 + |G|^2 (k(G) - |Z(G)|) - \sum_{1 \leq i \leq n} |C_G(x_i)|^2 \right). \]
Next, we calculate the second Zagreb index of the non-commuting graph. Let
\[
\sum_{x \neq y \in G - Z(G)} \deg(x) \cdot \deg(y) = \sum_{x \neq y \in G - Z(G)} (|G| - |C_G(x)|)(|G| - |C_G(y)|)
\]
\[
= \sum_{x \neq y \in G - Z(G)} (|G| - |C_G(x)|)|G|
\]
\[
- \sum_{x \neq y \in G - Z(G)} (|G| - |C_G(x)|)|C_G(y)|
\]
\[
= |G|(|G| - |Z(G)| - 1)(|G| - |C_G(x)|)
\]
\[
- (|G| - |C_G(x)|) \sum_{x \neq y \in G - Z(G)} |C_G(y)|.
\]
We know that \( \sum_{y \in G - Z(G)} |C_G(y)| = |G|(k(G) - |Z(G)|) \), thus \( \sum_{x \neq y \in G - Z(G)} |C_G(y)|. \)

Can be found as follows:
\[
\sum_{x \neq y \in G - Z(G)} |C_G(y)| = |G|(k(G) - |Z(G)|) - |C_G(x)| \quad \text{where}
\]
\[
\sum_{x \neq y \in G - Z(G)} \deg(x) \cdot \deg(y) = (|G| - |C_G(x)|)(|G|(|G| - k(G)) + |C_G(x)| - |G|).
\]

Next, we calculate the second Zagreb index of the non-commuting graph. Let
\( G - Z(G) = \{x_1, x_2, \ldots, x_n\} \). Then
\[
Z_2(G) = \sum_{\{x, y\} \subseteq V} \deg(x) \cdot \deg(y)
\]
\[
= \frac{1}{2} \left( \sum_{x_1 \neq y \in G - Z(G)} \deg(x_1) \cdot \deg(y) \right)
\]
\[
+ \sum_{x_2 \neq y \in G - Z(G)} \deg(x_2) \cdot \deg(y)
\]
\[
+ \ldots + \sum_{x_n \neq y \in G - Z(G)} \deg(x_n) \cdot \deg(y)
\]
\[
= \frac{1}{2} \left[ (|G| - |C_G(x_1)|)(|G|(|G| - k(G)) + |C_G(x_1)| - |G|) 
\right.
\]
\[
+ (|G| - |C_G(x_2)|)(|G|(|G| - k(G)) + |C_G(x_2)| - |G|)
\]
\[
+ \ldots + (|G| - |C_G(x_n)|)(|G|(|G| - k(G)) + |C_G(x_n)| - |G|)
\]
\[
= \frac{1}{2}(|G|^2(|G| - k(G))(|G| - |Z(G)|) - |G|(|G| - k(G)) \sum_{1 \leq i \leq n} |C_G(x_i)| \\
+ |G| \sum_{1 \leq i \leq n} |C_G(x_i)| - \sum_{1 \leq i \leq n} |C_G(x_i)|^2 \\
- |G|^2(|G| - |Z(G)|) + |G| \sum_{1 \leq i \leq n} |C_G(x_i)|)
\]

\[
= \frac{1}{2} \left( |G|^2(|G| - k(G))^2 + |G|^2(k(G) - |Z(G)|) - \sum_{1 \leq i \leq n} |C_G(x_i)|^2 \right).
\]

\[ \square \]

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