ALMOST CONTINUITY VIA TOPOLOGICAL GRILLS

N. Karthikeyan¹ §, N. Rajesh²

¹Department of Mathematics

Jeppiaar Engineering College

Chennai, 600119, TamilNadu, INDIA

²Department of Mathematics

Rajah Serfoji Govt. College

Thanjavur, 613005, TamilNadu, INDIA

Abstract: The aim of this paper is to introduce and characterize a new class of functions called almost $\mathcal{G}$ -precontinuous functions in ideal topological spaces by using $\mathcal{G}$ -preopen sets.

AMS Subject Classification: 54D10

Key Words: ideal topological spaces, $\mathcal{G}$ -preopen sets, almost $\mathcal{G}$ -precontinuous functions

1. Introduction

The idea of grills on a topological space was first introduced by Choquet [8]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [6],[7],[21] for details). In [17], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [10] have defined new classes of

Received: July 28, 2015

© 2015 Academic Publications, Ltd.
url: www.acadpubl.eu

§Correspondence author

sets in a grill topological space and obtained a new decomposition of continuity in terms of grills. The aim of this paper is to introduce and characterize a new class of functions called almost $\mathcal{G}$ -precontinuous functions in ideal topological spaces by using $\mathcal{G}$ -preopen sets.

2. Preliminaries

Let A be a subset of a topological space (X, τ) . We denote the closure of A and the interior of A by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be regular open [20] if $A = \text{Int}(\text{Cl}(A))$. A set $A \subset X$ is said to be δ -open [22] if it is the union of regular open sets of X . The complement of a regular open (resp. δ -open) set is called regular closed (resp. δ -closed). The intersection of all δ -closed sets of (X, τ) containing A is called the δ -closure [22] of A and is denoted by $\text{Cl}_\delta(A)$. A point $x \in X$ is called a θ -cluster point of A if $\text{Cl}(A) \cap A \neq \emptyset$ for every open set V of X containing x . The set of all θ -cluster points of A is called the θ -closure of A [22] and is denoted by $\text{Cl}_\theta(A)$. If $A = \text{Cl}_\theta(A)$, then A is said to be θ -closed [22]. The complement of θ -closed set is said to be θ -open [22]. A subset A of a topological space (X, τ) is said to be preopen [4] (resp. semiopen [14], preopen [15], β -open [2]) if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$ (resp. $A \subset \text{Cl}(\text{Int}(A))$, $A \subset \text{Int}(\text{Cl}(A))$, $A \subset \text{Cl}(\text{Int}(\text{Cl}(A)))$). The set of all regular open (resp. regular closed, δ -open, δ -closed, preopen, preopen) sets of (X, τ) is denoted by $RO(X)$ (resp. $RC(X)$, $\delta O(X)$, $\delta C(X)$, $BO(X)$, $PO(X)$). The definition of grill on a topological space, as given by Choquet [8], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological space (X, τ) is said to be a grill on X if

- (1) $\emptyset \notin \mathcal{G}$,
- (2) $A \in \mathcal{G}$ and $A \subset B$ implies that $B \in \mathcal{G}$,
- (3) $A, B \subset X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.1. [17] Let (X, τ) be a topological space and $\mathcal{G}$ a grill on X . A mapping $\Phi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for every open set } U \text{ containing } x\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ .

Definition 2.2. [17] Let $\mathcal{G}$ be a grill on a topological space (X, τ) . Then we define a map $\Psi : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on

a topological space (X, τ) , there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X \setminus U) = X \setminus U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}} \text{Cl}(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ) , $\tau \subset \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X , then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Definition 2.3. [10] A subset S of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ -preopen if $S \subset \text{Int}(\Psi(S))$. The complement of a $\mathcal{G}$ -preopen set is called a $\mathcal{G}$ -preclosed set.

Definition 2.4. The intersection of all $\mathcal{G}$ -preclosed sets containing $S \subset X$ is called the $\mathcal{G}$ -preclosure of S and is denoted by $p\text{Cl}_{\mathcal{G}}(S)$. The family of all $\mathcal{G}$ -preopen (resp. $\mathcal{G}$ -preclosed) sets of $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{GPO}(X)$ (resp. $\mathcal{GPC}(X)$). The family of all $\mathcal{G}$ -preopen (resp. $\mathcal{G}$ -preclosed) sets of $(X, \tau, \mathcal{G})$ containing a point $x \in X$ is denoted by $\mathcal{GPO}(X, x)$ (resp. $\mathcal{GPC}(X, x)$).

Definition 2.5. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

- (1) precontinuous [4] if $f^{-1}(V)$ is preopen in X for every open set V of Y ;
- (2) almost continuous [19] if $f^{-1}(V)$ is open in X for every regular open set V of Y ;
- (3) R -map [5] if $f^{-1}(V)$ is regular open in X for every regular open set V of Y ;
- (4) almost precontinuous [16] if $f^{-1}(V)$ is preopen in X for every regular open set V of Y .

Definition 2.6. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma, \mathcal{G})$ is said to be $\mathcal{G}$ -preirresolute if $f^{-1}(V)$ is $\mathcal{G}$ -preopen in X for every $\mathcal{G}$ -preopen subset V of Y .

Definition 2.7. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be:

- (1) $\mathcal{G}$ -precontinuous [10] if $f^{-1}(V)$ is $\mathcal{G}$ -preopen in X for every open set V of Y ,
- (2) weakly $\mathcal{G}$ -precontinuous [12] if for each $x \in X$ if for each open subset V in Y containing $f(x)$, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset \text{Cl}(V)$.

Definition 2.8. An ideal topological space $(X, \tau, \mathcal{G})$ is said to be:

- (1) $\mathcal{G}$ -pre- T_1 [13] (resp. r - T_1 [9]) if for each pair of distinct points x and y of X , there exist $\mathcal{G}$ -preopen (resp. regular open) sets U and V such that $x \in U, y \notin U$ and $x \notin V, y \in V$.

- (2) $\mathcal{G}$ -pre- T_2 [13] (resp. r - T_2 [9]) if for each pair of distinct points x and y of X , there exist $\mathcal{G}$ -preopen (resp. regular open) sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 2.9. *The following statements are true:*

- (1) Let A be a subset of a space (X, τ) . Then $A \in PO(X)$ if and only if $s\text{Cl}(A) = \text{Int}(\text{Cl}(A))$ [11].
- (2) A subset A of a space (X, τ) is β -open if and only if $\text{Cl}(A)$ is regular closed [3].

3. Almost $\mathcal{G}$ -Precontinuous Functions

Definition 3.1. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be:

- (1) almost $\mathcal{G}$ -precontinuous at a point $x \in X$ if for each open subset V of Y containing $f(x)$, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \text{Int}(\text{Cl}(V))$;
- (2) almost $\mathcal{G}$ -precontinuous if it has this property at each point of X .

Remark 3.2. Almost $\mathcal{G}$ -precontinuity implies weak $\mathcal{G}$ -precontinuity and it is obvious that almost $\mathcal{G}$ -precontinuity implied by $\mathcal{G}$ -precontinuity. However, the converse of these implications is not true in general as the following examples show.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset, \{a\}\}$. Define a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then f is almost $\mathcal{G}$ -precontinuous but not $\mathcal{G}$ -precontinuous.

Example 3.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{G}) \rightarrow (X, \sigma)$ is weakly $\mathcal{G}$ -precontinuous but not almost $\mathcal{G}$ -continuous.

Theorem 3.5. For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$, the following statements are equivalent:

- (1) f is almost $\mathcal{G}$ -precontinuous at $x \in X$;
- (2) $x \in \text{Int}(\Psi(f^{-1}(s\text{Cl}(V))))$ for every open set V of Y containing $f(x)$;
- (3) $f^{-1}(V) \subset p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V)))$ for every open set V of Y ;
- (4) $p\text{Cl}_{\mathcal{G}}(f^{-1}(s\text{Int}(F))) \subset f^{-1}(F)$ for every closed set F of Y

Proof. (1) $\Rightarrow$ (2): Let V be an open set of Y containing $f(x)$. Then there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset \text{Int}(\text{Cl}(V)) = s\text{Cl}(V)$. Then $U \subset f^{-1}(s\text{Cl}(V))$. Since $U \in \mathcal{GPO}(X, x)$, $x \in U \subset \text{Int}(\Psi(f^{-1}(U))) \subset \text{Int}(\Psi(f^{-1}(s\text{Cl}(V))))$

(2) $\Rightarrow$ (3): Let V be an open set of Y containing $f(x)$ and U an open set of X containing x . Since $x \in \text{Int}(\Psi(f^{-1}(s\text{Cl}(V))))$, we have $x \in f^{-1}(s\text{Cl}(V)) \cap (\text{Int}(\Psi(f^{-1}(s\text{Cl}(V)))) = p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V)))$. Hence

$$f^{-1}(V) \subset p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V))).$$

(3) $\Rightarrow$ (1): Let V be an open set of Y containing $f(x)$, then $x \in f^{-1}(V) \subset p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V)))$. Set $U = p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V)))$, then $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset s\text{Cl}(V)$. This shows that f is almost $\mathcal{G}$ -precontinuous at x .

(3) $\Rightarrow$ (4): Clear. □

Theorem 3.6. For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ the following statements are equivalent:

- (1) f is almost $\mathcal{G}$ -precontinuous;
- (2) $f^{-1}(\text{Int}(\text{Cl}(V))) \in \mathcal{GPO}(X)$ for every open set V of Y ;
- (3) $f^{-1}(\text{Cl}(\text{Int}(V))) \in \mathcal{GPO}(X)$ for every closed set V of Y ;
- (4) $f^{-1}(V) \in \mathcal{GPO}(X)$ for every $V \in RO(Y)$;
- (5) $f^{-1}(F) \in \mathcal{GPC}(X)$ for every $F \in RC(Y)$;
- (6) for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset s\text{Cl}(V)$;
- (7) $p\text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(F)))) \subset f^{-1}(F)$ for every closed set F of Y ;
- (8) $p\text{Cl}_{\mathcal{G}}(f^{-1}(A)) \subset f^{-1}(\text{Cl}(A))$ for every $A \in BO(Y)$;
- (9) $p\text{Cl}_{\mathcal{G}}(f^{-1}(A)) \subset f^{-1}(\text{Cl}(A))$ for every $A \in SO(Y)$;
- (10) $f^{-1}(V) \subset p\text{Int}_{\mathcal{G}}(f^{-1}(\text{Int}(\text{Cl}(V))))$ for every open set $V \in PO(Y)$;
- (11) $f(p\text{Cl}_{\mathcal{G}}(A)) \subset \text{Cl}_{\delta}(f(A))$ for every subset A of X ;
- (12) $p\text{Cl}_{\mathcal{G}}(f^{-1}(B)) \subset f^{-1}(\text{Cl}_{\delta}(B))$ for every subset B of Y ;
- (13) $f^{-1}(F) \in \mathcal{GPC}(X)$ for every $F \in \delta C(Y)$;
- (14) $f^{-1}(V) \in \mathcal{GPO}(X)$ for every $V \in \delta O(Y)$.

Proof. (4) $\Rightarrow$ (5): Let $F \in RC(Y)$. Then $Y \setminus F \in RO(Y)$. Take $x \in f^{-1}(Y \setminus F)$, then $f(x) \in Y \setminus F$ and since f is almost $b\mathcal{G}$ -precontinuous, there exists $W_x \in \mathcal{GPO}(X, x)$ such that $x \in W_x$ and $f(W_x) \subset Y \setminus F$. Then $x \in W_x \subset f^{-1}(Y \setminus F)$ so that $f^{-1}(Y \setminus F) = \bigcup_{x \in f^{-1}(Y \setminus F)} W_x$. Since any union of $\mathcal{G}$ -preopen sets is $\mathcal{G}$ -preopen [10], $f^{-1}(Y \setminus F)$ is $\mathcal{G}$ -preopen in X and hence $f^{-1}(F) \in \mathcal{GPC}(X)$.

(5) $\Rightarrow$ (11): Let A be a subset of X . Since $\text{Cl}_\delta(f(A))$ is δ -closed in Y , it is equal to $\cap \{F_\alpha : F_\alpha \text{ is regular closed in } Y, \alpha \in \Lambda\}$, where Λ is an index set. From (5), we have $A \subset f^{-1}(\text{Cl}_\delta(f(A))) = \cap \{f^{-1}(F_\alpha) : \alpha \in \Lambda\} \in \mathcal{GPC}(X)$ and hence $p\text{Cl}_\mathcal{G}(A) \subset f^{-1}(\text{Cl}_\delta(f(A)))$. Therefore, we obtain $f(p\text{Cl}_\mathcal{G}(A)) \subset \text{Cl}_\delta(f(A))$.

(11) $\Rightarrow$ (12): Set $A = f^{-1}(B)$ in (11), then

$$f(p\text{Cl}_\mathcal{G}(f^{-1}(B))) \subset \text{Cl}_\delta(f(f^{-1}(B))) \subset \text{Cl}_\delta(B)$$

and hence $p\text{Cl}_\mathcal{G}(f^{-1}(B)) \subset f^{-1}(\text{Cl}_\delta(B))$.

(12) $\Rightarrow$ (13): Let F be δ -closed set of Y , then $p\text{Cl}_\mathcal{G}(f^{-1}(F)) \subset f^{-1}(F)$ so $f^{-1}(F) \in \mathcal{GPC}(X)$.

(13) $\Rightarrow$ (14): Let V be δ -open set of Y , then $Y \setminus V$ is δ -closed set in Y . This gives $f^{-1}(Y \setminus V) \in \mathcal{GPC}(X)$ and hence $f^{-1}(V) \in \mathcal{GPO}(X)$.

(14) $\Rightarrow$ (1): Let V be any regular open set of Y . Since V is δ -open in Y , $f^{-1}(V) \in \mathcal{GPO}(X)$ and hence from $f(f^{-1}(V)) \subset V = \text{Int}(\text{Cl}(V))$. Then f is almost $\mathcal{G}$ -precontinuous.

(5) $\Rightarrow$ (8): Let A be any b -open set in Y . Since $\text{Cl}(A)$ is regular closed, $f^{-1}(\text{Cl}(A))$ is δ -closed and $f^{-1}(A) \subset f^{-1}(\text{Cl}(A))$. Hence, $p\text{Cl}_\mathcal{G}(f^{-1}(A)) \subset f^{-1}(\text{Cl}(A))$.

(8) $\Rightarrow$ (9): obvious.

(9) $\Rightarrow$ (10): Let V be a preopen set. Then we have $V \subset \text{Int}(\text{Cl}(V))$ and $\text{Cl}(\text{Int}(Y \setminus V)) \subset Y \setminus V$. Moreover, since the set $\text{Cl}(\text{Int}(Y \setminus V))$ is semiopen, it follows that

$$\begin{aligned} X \setminus p\text{Int}_\mathcal{G}(f^{-1}(\text{Int}(\text{Cl}(V)))) &= p\text{Cl}_\mathcal{G}(X \setminus f^{-1}(\text{Int}(\text{Cl}(V)))) \\ &= p\text{Cl}_\mathcal{G}(f^{-1}(Y \setminus \text{Int}(\text{Cl}(V)))) = p\text{Cl}_\mathcal{G}(f^{-1}(\text{Cl}(\text{Int}(Y \setminus V)))) \\ &\subset f^{-1}(\text{Cl}(\text{Int}(Y \setminus V))) \subset f^{-1}(Y \setminus V) \subset X \setminus f^{-1}(V). \end{aligned}$$

Hence, we obtain $f^{-1}(V) \subset p\text{Int}_\mathcal{G} \text{Int}(f^{-1}(\text{Int}(\text{Cl}(V))))$.

(10) $\Rightarrow$ (4): Let V be a regular open set. Since V is pre open, we get $f^{-1}(V) \subset p \text{Int}_{\mathcal{G}}(f^{-1}(\text{Int}(\text{Cl}(V)))) = p \text{Int}_{\mathcal{G}}(f^{-1}(V))$. Hence $f^{-1}(V) \in \mathcal{GPO}(X)$.

The other implications are obvious. $\square$

Theorem 3.7. *For a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$, the following statements are equivalent:*

- (1) f is almost $\mathcal{G}$ -precontinuous;
- (2) $p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(B))))) \subset f^{-1}(\text{Cl}(B))$ for every open subset B of Y ;
- (3) $p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(F)))) \subset f^{-1}(F)$ for every closed subset F of Y ;
- (4) $p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(V))) \subset f^{-1}(\text{Cl}(V))$ for every open subset V of Y ;
- (5) $f^{-1}(V) \subset p \text{Int}_{\mathcal{G}}(f^{-1}(s \text{Cl}(V)))$ for every open subset V of Y ;
- (6) $f^{-1}(V) \subset \text{Int}(\Psi(f^{-1}(s \text{Cl}(V))))$ for every open subset V of Y ;
- (7) $f^{-1}(V) \subset \text{Int}(\Psi(f^{-1}(\text{Int}(\text{Cl}(V)))))$ for every open subset V of Y .

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y . Assume that $x \in X \setminus f^{-1}(\text{Cl}(B))$. Then $f(x) \in Y \setminus \text{Cl}(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \emptyset$; hence $\text{Int}(\text{Cl}(V)) \cap \text{Cl}(\text{Int}(\text{Cl}(B))) = \emptyset$. Since f is almost $\mathcal{G}$ -precontinuous, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset \text{Int}(\text{Cl}(V))$.

Therefore, we have $U \cap f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(B)))) = \emptyset$ and hence

$$x \in X \setminus p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(B)))).$$

Thus, we obtain $p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(B))))) \subset f^{-1}(\text{Cl}(B))$.

(2) $\Rightarrow$ (3): Let F be any closed set of Y . Then we have

$$\begin{aligned} p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(\text{Int}(F))))) &= p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(F)))) \\ &\subset f^{-1}(\text{Cl}(\text{Int}(F))) \subset f^{-1}(F). \end{aligned}$$

(3) $\Rightarrow$ (4): For any open set V of Y , $\text{Cl}(V)$ is regular closed in Y and we have $p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(V))) = p \text{Cl}_{\mathcal{G}}(f^{-1}(\text{Cl}(\text{Int}(\text{Cl}(V))))) \subset f^{-1}(\text{Cl}(V))$.

(4) $\Rightarrow$ (5): Let V be any open set of Y . Then $Y \setminus \text{Cl}(V)$ is open in Y and we have $X \setminus p \text{Int}_{\mathcal{G}}(f^{-1}(s \text{Cl}(V))) = p \text{Cl}_{\mathcal{G}}(f^{-1}(Y \setminus (s \text{Cl}(V)))) \subset f^{-1}(\text{Cl}(Y \setminus \text{Cl}(V))) \subset X \setminus f^{-1}(V)$. Therefore, we obtain $f^{-1}(V) \subset p \text{Int}_{\mathcal{G}}(f^{-1}(s \text{Cl}(V)))$.

(5) $\Rightarrow$ (6): Let V be any open set of Y . Then we obtain $f^{-1}(V) \subset {}_p\text{Int}_{\mathcal{G}}(f^{-1}(s\text{Cl}(V))) \subset \text{Int}(\Psi(f^{-1}(s\text{Cl}(V))))$.

(6) $\Rightarrow$ (1): Let x be any point of X and V any open set of Y containing $f(x)$. Then $x \in f^{-1}(V) \subset \text{Int}(\Psi(f^{-1}(s\text{Cl}(V))))$. It follows from Theorem 3.5 that f is almost $\mathcal{G}$ -precontinuous at any point x of X . Therefore, f is almost $\mathcal{G}$ -precontinuous at any point x of X .

(6) $\Rightarrow$ (7): Clear. □

Theorem 3.8. (1) A function $f : (X, \tau, \mathcal{P}(X) \setminus \{\emptyset\}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous if and only if it is almost precontinuous. (2) A function $f : (X, \tau, \{X\}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous if and only if it is almost continuous.

Proof. The proof follows from Corollaries 3.3 and 3.4 of [1]. □

Definition 3.9. A filterbase Λ is said to be

(1) $\mathcal{G}$ -preconvergent to a point x in X if for any $U \in \mathcal{GPO}(X, x)$, there exists $B \in \Lambda$ such that $B \subset U$.

(2) r -convergent to a point x in X if for any regular open set U of X containing x , there exists $B \in \Lambda$ such that $B \subset U$.

Theorem 3.10. If a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous, then for each point $x \in X$ and each filter base Λ in X $\mathcal{G}$ -preconverging to x , the filter base $f(\Lambda)$ is r -convergent to $f(x)$.

Proof. Let $x \in X$ and Λ be any filter base in X $\mathcal{G}$ -preconverging to x . Since f is $\mathcal{G}$ -precontinuous, then for any open set V of (Y, σ) containing $f(x)$, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset V$. Since Λ is $\mathcal{G}$ -preconverging to x , there exists $B \in \Lambda$ such that $B \subset U$. This means that $f(B) \subset V$ and hence the filter base $f(\Lambda)$ is convergent to $f(x)$. □

Definition 3.11. A sequence (x_n) is said to be $\mathcal{G}$ -preconvergent to a point x if for every $\mathcal{G}$ -preopen set V containing x , there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$.

Theorem 3.12. If a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous, then for each point $x \in X$ and each net (x_n) which is $\mathcal{G}$ -preconvergent to x , the net $(f(x_n))$ is r -convergent to $f(x)$.

Proof. The proof is similar to that of Theorem 3.10. □

Theorem 3.13. *If an injective function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous and (Y, σ) is r - T_1 , then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ -pre T_1 .*

Proof. Suppose that Y is r - T_1 . For any distinct points x and y in X , there exist regular open sets V and W such that $f(x) \in V, f(y) \notin V, f(x) \notin W$ and $f(y) \in W$. Since f is almost $\mathcal{G}$ -precontinuous, $f^{-1}(V)$ and $f^{-1}(W)$ are $\mathcal{G}$ -preopen subsets of $(X, \tau, \mathcal{G})$ such that $x \in f^{-1}(V), y \notin f^{-1}(V), x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ -pre T_1 . $\square$

Theorem 3.14. *If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is a almost $\mathcal{G}$ -precontinuous injective function and (Y, σ) is r - T_2 , then (X, τ) is $\mathcal{G}$ -pre T_2 .*

Proof. For any pair of distinct points x and y in X , there exist disjoint regular open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is almost $\mathcal{G}$ -precontinuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $\mathcal{G}$ -preopen sets in X containing x and y , respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that $(X, \tau, \mathcal{G})$ is $\mathcal{G}$ -pre T_2 . $\square$

Theorem 3.15. *If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is a almost continuous function and $g : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is a almost $\mathcal{G}$ -precontinuous function and Y is a r - T_2 -space, then the set $E = \{x \in X : f(x) = g(x)\}$ is $\mathcal{G}$ -preclosed set in $(X, \tau, \mathcal{G})$.*

Proof. If $x \in X \setminus E$, then it follows that $f(x) \notin g(x)$. Since Y is r - T_2 , there exist disjoint regular open sets V and W of Y such that $f(x) \in V$ and $g(x) \in W$. Since f is almost continuous and g is almost $\mathcal{G}$ -precontinuous, then $f^{-1}(V)$ is open and $g^{-1}(W)$ is $\mathcal{G}$ -preopen in X with $x \in f^{-1}(V)$ and $x \in g^{-1}(W)$. Put $A = f^{-1}(V) \cap g^{-1}(W)$. By Corollary 3.1 of [1], A is $\mathcal{G}$ -preopen in X . Therefore, $f(A) \cap g(A) = \emptyset$ and it follows that $x \notin p\text{Cl}_{\mathcal{G}}(E)$. This shows that E is $\mathcal{G}$ -preclosed in X . $\square$

Definition 3.16. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be faintly $\mathcal{G}$ -precontinuous if for each $x \in X$ and each θ -open set V of Y containing $f(x)$, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset V$.

Theorem 3.17. *A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is faintly $\mathcal{G}$ -precontinuous if and only if for every θ -closed set V of Y $f^{-1}(V) \in \mathcal{GPC}(X)$.*

Theorem 3.18. *The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) hold for the following properties of a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$:*

(1) f is $\mathcal{G}$ -precontinuous.

- (2) $f^{-1}(\text{Cl}_\delta(B))$ is $\mathcal{G}$ -preclosed in X for every subset B of Y .
- (3) f is almost $\mathcal{G}$ -precontinuous.
- (4) f is weakly $\mathcal{G}$ -precontinuous.
- (5) f is faintly $\mathcal{G}$ -precontinuous.

If, in addition, Y is regular, then the five properties are equivalent of one another.

Proof. (1) $\Rightarrow$ (2): Since $\text{Cl}_\delta(B)$ is closed in Y for every subset B of Y , by Theorem 3.6, $f^{-1}(\text{Cl}_\delta(B))$ is $\mathcal{G}$ -preclosed in X .

(2) $\Rightarrow$ (3): For any subset B of Y , $f^{-1}(\text{Cl}_\delta(B))$ is $\mathcal{G}$ -preclosed in X and hence we have $p\text{Cl}_\mathcal{G}(f^{-1}(B)) \subset p\text{Cl}_\mathcal{G}(f^{-1}(\text{Cl}_\delta(B))) = f^{-1}(\text{Cl}_\delta(B))$. It follows from Theorem 3.6 that f is almost $\mathcal{G}$ -precontinuous.

(3) $\Rightarrow$ (4): This is obvious.

(4) $\Rightarrow$ (5): Let F be any θ -closed set of Y . It follows from 3.17 that $p\text{Cl}_\mathcal{G}(f^{-1}(F)) \subset f^{-1}(\text{Cl}_\theta(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is $\mathcal{G}$ -preclosed in X and hence f is faintly $\mathcal{G}$ -precontinuous.

Suppose that Y is regular. We prove that (5) $\Rightarrow$ (1). Let V be any open set of Y . Since Y is regular, V is θ -open in Y . By the faint pre continuity of f , f^{-1} is $\mathcal{G}$ -preopen in X . Therefore, f is $\mathcal{G}$ -precontinuous. $\square$

Definition 3.19. A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is said to be $\mathcal{G}$ -preopen if $f(U) \in PO(Y)$ for every $\mathcal{G}$ -preopen set U of X .

Theorem 3.20. If a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is $\mathcal{G}$ -preopen and weakly $\mathcal{G}$ -precontinuous, then f is almost $\mathcal{G}$ -precontinuous.

Proof. Let $x \in X$ and let V be an open set of Y containing $f(x)$. Since f is weakly $\mathcal{G}$ -precontinuous, there exists $U \in \mathcal{G}PO(X, x)$ such that $f(U) \subset \text{Cl}(V)$. Since f is $\mathcal{G}$ -preopen, $f(U) \subset \text{Int}(\text{Cl}(f(U))) \subset \text{Int}(\text{Cl}(V))$; hence f is almost $\mathcal{G}$ -precontinuous. $\square$

Theorem 3.21. Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ be a function and $g : X \rightarrow X \times Y$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is almost $\mathcal{G}$ -precontinuous if and only if f is almost $\mathcal{G}$ -precontinuous.

Proof. Let x be any point of X and V any regular open set of Y containing $f(x)$. Then we have $g(x) = (x, f(x)) \in X \times V$ is regular open in $X \times Y$. Since g is almost $\mathcal{G}$ -precontinuous, there exists $U \in \mathcal{GPO}(X, x)$ such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$; hence f is almost $\mathcal{G}$ -precontinuous. Conversely, let $x \in X$ and W be a regular open set of $X \times Y$ containing $g(x)$. There exist a regular open set U_1 in X and a regular open set V in Y such that $U_1 \times V \subset W$. Since f is almost $\mathcal{G}$ -precontinuous, there exist $U_2 \in \mathcal{GPO}(X, x)$ such that $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then we obtain $x \in U \in \mathcal{GPO}(X)$ and $g(U) \subset U \times V \subset W$. This shows that g is almost $\mathcal{G}$ -precontinuous. $\square$

Theorem 3.22. *Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma, \mathcal{G})$ and $g : (Y, \sigma, \mathcal{G}) \rightarrow (Z, \eta)$ be functions. Then the composition $g \circ f : (X, \tau, \mathcal{G}) \rightarrow (Z, \eta)$ is almost $\mathcal{G}$ -precontinuous if f and g satisfy one of the following conditions:*

- (1) f is almost $\mathcal{G}$ -precontinuous and g is R -map.
- (2) f is $\mathcal{G}$ -preirresolute and g is almost $\mathcal{G}$ -precontinuous.
- (3) f is $\mathcal{G}$ -precontinuous and g is almost continuous

Proof. Clear. $\square$

Definition 3.23. A topological space (X, τ) is said to be:

- (1) almost regular [18] if for any regular closed set F of X and any point $x \in X \setminus F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.
- (2) semi-regular if for any open set U of X and each point $x \in U$ there exists a regular open set V of X such that $x \in V \subset U$.

Theorem 3.24. *If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is a weakly $\mathcal{G}$ -precontinuous function and Y is almost regular, then f is almost $\mathcal{G}$ -precontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the almost regularity of Y , there exists a regular open set G of Y such that $f(x) \in G \subset \text{Cl}(G) \subset \text{Int}(\text{Cl}(V))$ [[18], Theorem 2.2]. Since f is weakly $\mathcal{G}$ -precontinuous, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \subset \text{Cl}(G) \subset \text{Int}(\text{Cl}(V))$. Therefore, f is almost $\mathcal{G}$ -precontinuous. $\square$

Theorem 3.25. *If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is an almost $\mathcal{G}$ -precontinuous function and Y is semi-regular, then f is $\mathcal{G}$ -precontinuous.*

Proof. Let $x \in X$ and let V be any open set of Y containing $f(x)$. By the semi-regularity of Y , there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is almost $\mathcal{G}$ -precontinuous, there exists $U \in \mathcal{GPO}(X, x)$ such that $f(U) \in \text{Int}(\text{Cl}(G)) = G \subset V$ and hence f is $\mathcal{G}$ -precontinuous. $\square$

Definition 3.26. A $\mathcal{G}$ -prefrontier of a subset A of $(X, \tau, \mathcal{G})$, denoted by $b\mathcal{G}Fr(A)$, is defined by $p\text{Cl}_{\mathcal{G}}(A) \cap p\text{Cl}_{\mathcal{G}}(X \setminus A)$.

Theorem 3.27. The set of all points $x \in X$ in which a function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is not almost $\mathcal{G}$ -precontinuous is identical with the union of $\mathcal{G}$ -prefrontier of the inverse images of regular open sets containing $f(x)$.

Proof. Suppose that f is not almost $\mathcal{G}$ -precontinuous at $x \in X$. Then there exists a regular open set V of Y containing $f(x)$ such that $U \cap (X \setminus f^{-1}(V)) \neq \emptyset$ for every $U \in \mathcal{GPO}(X, x)$. Therefore, we have $x \in p\text{Cl}_{\mathcal{G}}(X \setminus f^{-1}(V)) = X \setminus p\text{Int}_{\mathcal{G}}(f^{-1}(V))$ and $x \in f^{-1}(V)$. Thus, we obtain $x \in p_{\mathcal{G}}Fr(f^{-1}(U))$. Conversely, suppose that f is almost $\mathcal{G}$ -precontinuous at $x \in X$ and let V be a regular open set of Y containing $f(x)$. Then there exists $U \in \mathcal{GPO}(X, x)$ such that $U \subset f^{-1}(V)$. That is $x \in p\text{Int}_{\mathcal{G}}(f^{-1}(V))$. Therefore, $x \in X \setminus p_{\mathcal{G}}Fr(f^{-1}(V))$. $\square$

Theorem 3.28. If $g : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous and S is δ -closed set of $X \times Y$, then $p_X(S \cap \mathcal{G}(g))$ is $\mathcal{G}$ -preclosed in X , where p_X represents the projection of $X \times Y$ onto X and $\mathcal{G}(g)$ denotes the graph of g .

Proof. Let S be any δ -closed set of $X \times Y$ and $x \in p\text{Cl}_{\mathcal{G}}(p_X(S \cap \mathcal{G}(g)))$. Let U be any open set of X containing x and V any open set of Y containing $g(x)$. Since g is almost $\mathcal{G}$ -precontinuous, we have $x \in g^{-1}(V) \subset p\text{Int}_{\mathcal{G}}(g^{-1}(\text{Int}(\text{Cl}(V))))$ and $U \cap p\text{Int}_{\mathcal{G}}(g^{-1}(\text{Int}(\text{Cl}(V)))) \in \mathcal{GPO}(X, x)$. Since $x \in p\text{Cl}_{\mathcal{G}}(p_X(S \cap \mathcal{G}(g)))$, $(U \cap p\text{Int}_{\mathcal{G}}(g^{-1}(\text{Int}(\text{Cl}(V)))) \cap p_X(S \cap \mathcal{G}(g))$ contains some point u of X . This implies that $(u, g(u)) \in S$ and $g(u) \in \text{Int}(\text{Cl}(V))$. Thus, we have $\emptyset \neq (U \times \text{Int}(\text{Cl}(V))) \cap S \subset \text{Int}(\text{Cl}(U \times V)) \cap S$ and hence $(x, g(x)) \in \text{Cl}_{\delta}(S)$. Since S is δ -closed, $(x, g(x)) \in p_X(S \cap \mathcal{G}(g))$ and $x \in p_X(S \cap \mathcal{G}(g))$. Then $p_X(S \cap \mathcal{G}(g))$ is $\mathcal{G}$ -preclosed. $\square$

Corollary 3.29. If $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ has a δ -closed graph and $g : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is almost $\mathcal{G}$ -precontinuous, then the set $\{x \in X : f(x) = g(x)\}$ is $\mathcal{G}$ -preclosed in X .

Proof. Since $G(f)$ is δ -closed and $p_X(G(f) \cap G(g)) = \{x \in X : f(x) = g(x)\}$ it follows from Theorem 3.28 that $\{x \in X : f(x) = g(x)\}$ is $\mathcal{G}$ -preclosed in X . $\square$

Theorem 3.30. *If for each pair of distinct x_1 and x_2 in an ideal topological space $(X, \tau, \mathcal{G})$ there exists a function f of X into a Hausdorff space Y such that $f(x_1) \neq f(x_2)$, f is weakly $\mathcal{G}$ -precontinuous and f is almost $\mathcal{G}$ -precontinuous at x_2 , then X is $\mathcal{G}$ -pre- T_2 .*

Proof. Since Y is Hausdorff, if for each pair of distinct point x_1 and x_2 there exist disjoint open sets V_1 and V_2 of Y containing $f(x_1)$ and $f(x_2)$, respectively; hence $\text{Cl}(V_1) \cap \text{Int}(\text{Cl}(V_2)) = \emptyset$. Since f is weakly $\mathcal{G}$ -precontinuous at x_1 , there exists $U_1 \in \mathcal{GPO}(X, x_1)$ such that $f(U_1) \subset \text{Cl}(V_1)$. Since f is almost $\mathcal{G}$ -precontinuous at x_2 , there exists $U_2 \in \mathcal{GPO}(X, x_2)$ such that $f(U_2) \subset \text{Int}(\text{Cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. This shows that X is $\mathcal{G}$ -pre- T_2 . $\square$

References

- [1] A. Al-Omari and T. Noiri, Decomposition of continuity via grills, *Jordan J. Math and Stat.*, **4**, No. 1 (2011), 33-46.
- [2] M. E. Abd El-Monsef, S. N. El-Deep and R. A. Mahmoud, *beta*-open sets and *beta*-continuous functions *Bull. Fac. Sci. Assiut Univ. A*, **12** (1983), 77-90.
- [3] D. Andrijevic, Semi-preopen sets, *Math. Vesnik*, **38** (1986), 24-32.
- [4] D. Andrijevic, On preopen sets, *Math. Vesnik*, **48** (1996), 59-64.
- [5] D. Carnahan, Some properties related to compactness in topological spaces, Ph.D. Thesis, Univ. Arkansas (1973).
- [6] K. C. Chattopadhyay, O. Njastad and W. J. Thron, Merotopic spaces and extensions of closure spaces, *Can. J. Math.*, **35**, No. 4 (1983), 613-629.
- [7] K. C. Chattopadhyay and W. J. Thron, Extensions of closure spaces, *Can. J. Math.*, **29**, No. 6 (1977), 1277-1286.
- [8] G. Choquet, Sur les notions de filter et grill, *Comptes Rendus Acad. Sci. Paris*, **224** (1947), 171-173.
- [9] E. Ekici, Generalization of perfectly continuous, Regular set-connected and clopen functions, *Acta. Math. Hungar.*, **107**, No. 3 (2005), 193-206.
- [10] E. Hatir and S. Jafari, On some new classes of sets and a new decomposition of continuity via grills, *J. Adv. Math. Studies*, **3**, No. 1 (2010), 33-40.

- [11] D. S. Jankovic, A note on mappings of extremally disconnected spaces, *Acta Math. Hungar.*, **46** (1985), 8392.
- [12] N. Karthikeyan and N. Rajesh, On Weakly G-precontinuous functions (sub- mitted).
- [13] N. Karthikeyan and N. Rajesh, Some New separation axioms in ideal topolog- ical space (submitted).
- [14] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, **70** (1963), 36-41.
- [15] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, On precontin- uous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt*, **53** (1982), 47-53.
- [16] A. Nasif, T. Noiri, Some weak forms of almost continuity, *Acta Math. Hungar.*, **74** (1997), 211-219.
- [17] B. Roy and M. N. Mukherjee, On a typical topology induced by a grill, *Soochow J. Math.*, **33** (4) (2007), 771-786.
- [18] M. K. Singal and S. P. Arya, On almost regular spaces, *Glasnik Mat.*, **4**, No. 24 (1969), 89-99.
- [19] M. K. Singal and A. R. Singal, Almost-continuous mappings, *Yokohama Math. J.*, **16** (1968), 63-73.
- [20] M. Stone, Applications of the theory of boolean rings to general topology, *Trans. Amer. Math. Soc.*, **41** (1937), 374-381.
- [21] W. J. Thron, Proximity structure and grills, *Math. Ann.*, **206** (1973), 35-62.
- [22] N. V. Velicko, H-closed topological spaces, *Amer. Math. Soc. Transl.* **78**, No. 2 (1968), 103-118.