**LEGENDRE WAVELET OPERATIONAL MATRIX
OF FRACTIONAL DERIVATIVE
THROUGH WAVELET-POLYNOMIAL TRANSFORMATION
AND ITS APPLICATIONS IN SOLVING FRACTIONAL ORDER
DIFFERENTIAL EQUATIONS**

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Abstract: In this paper we present a numerical method for solving fractional differential equations(FDEs) based on Legendre wavelets. The known Legendre Wavelets is presented first then we derived the operational matrix of fractional order derivative through wavelet-polynomial transformation matrix which was utilised together with spectral and collocation methods to reduce the linear and non-linear FDEs to a system of algebraic equations respectively. The method presented represents a more simple technique of obtaining the operational matrix with straight forward applicability to the FDEs in comparison to the existing operational matrices, most of which obtained either by direct integration of the wavelet vector or through block pulse functions. Results obtained when solving different linear and non-linear FDEs confirm the applicability and accuracy of the proposed method.

AMS Subject Classification: 34K37, 42C40, 65T60

Key Words: shifted Legendre polynomials, Legendre wavelets, operational matrix, fractional derivatives

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Received: August 7, 2015

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1. Introduction

Fractional calculus which is a generalization of ordinary differentiation and integration to an arbitrary order, have been the focus of many studies and it was proved to be a valuable tool in the modelling of many phenomena in the fields of physics, chemistry, engineering, aerodynamics, electrodynamics of complex medium, polymer rheology etc,[1, 2] and [3] - [4]. Modelling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally lead to fractional differential equations and to the necessity of solving such equations. However, several analytic and numerical methods for solving FDEs are available in open literature, Adomian decomposition method [6], variational iteration method [7], homotopy perturbation method [8], Podlubny introduced a numerical method for arbitrary order derivative based on the relationship between the Grunwald-Letnikov and Riemann-Liouville derivative [5]. Diethelm et al. has presented predictor-corrector method for numerical solutions [9]. Recently, the operational matrices of fractional order integration for Haar wavelets, Legendre wavelets, Chebyshev wavelets and Bernoulli wavelets have been developed in [10, 11, 12, 13] respectively, to solve FDEs.

The motivation of this research is to propose more simple technique of obtaining the operational matrix with straight forward applicability to the FDEs in comparison to the existing operational matrices, most of which obtained either by direct integration of the wavelet vector or through block pulse functions[11, 12].

In this paper we introduced a new method by using wavelets based on shifted Legendre polynomials to derive the operational matrix of fractional order derivative(OMOFOD), through wavelet-polynomial transformation matrix, which is used to solve FDEs, the method reduces the FDEs to a system of algebraic equations. The paper is organised as follows. Section 2 introduce some basic definitions and mathematical preliminaries of fractional calculus, in Section 3 we first defined shifted Legendre polynomial then, described the basics of wavelets and Legendre wavelets, in Section 4 we introduce the shifted Legendre operational matrix of fractional order derivative and Legendre wavelet operational matrix of fractional order derivative, in Section 5 application of the Legendre wavelets operational matrix of fractional order derivative together with Tau and collocation methods is shown. In Section 6 the method is used on numerical examples to demonstrate the accuracy of the scheme.

2. Preliminaries

2.1. Fractional Derivative and Integral

Here, we recall some basic definitions and properties of fractional calculus that are used in this article. There are various definitions for fractional differentiation [2]. The Riemann-Liouville definition has certain disadvantage when we try to model a real-world phenomenon [2]. However, the Caputo's definition is more reliable in application and so we use this definition for fractional derivatives. But, the most frequently encountered definition of an integral of fractional order is the Riemann-Liouville integral, in which the fractional integral operator I of a function $f(t)$ is defined as:

Definition 2.1. The Riemann-Liouville integral I of fractional order α of $f(t)$ is given by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha \in \mathbb{R}^+, \quad (1)$$

where $\Gamma(\cdot)$ is the gamma function, its fractional derivative of order $\alpha > 0$ is given by

$$(D_t^\alpha f)(x) = \left(\frac{d}{dx} \right)^m (I^{m-\alpha} f)(x), \quad (\alpha > 0, \quad m-1 < \alpha < m).$$

Some properties of I^α are as follows:

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad \alpha > 0, \beta > 0, \quad (2)$$

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}. \quad (3)$$

Definition 2.2. The Caputo fractional derivative D^α of a function $f(t)$ is defined as:

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \quad (4)$$

The following are some properties of Caputo fractional derivatives

$$D^\alpha C = 0, \quad (C \text{ is constant}), \quad (5)$$

$$D^\alpha t^\beta = \begin{cases} 0, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < [\alpha] \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta \geq [\alpha] \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > [\alpha], \end{cases} \quad (6)$$

where $\lceil \alpha \rceil$ denote the smallest integer greater than or equal to α and $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α .

Similar to the integer order differentiation, the Caputo fractional differential operator is a linear operator, since,

$$D^\alpha(\lambda f(x) + \mu g(x)) = \lambda D^\alpha f(x) + \mu D^\alpha g(x) \quad (7)$$

where λ and μ are constants.

3. Legendre Polynomial and Legendre Wavelets

3.1. Shifted Legendre Polynomial

The well known Legendre polynomials of degree m are defined on the interval $[-1, 1]$ and can be determined with the aid of the following recurrence formulae

$$L_{m+1}(t) = \frac{2m+1}{m+1}tL_m(t) - \frac{m}{m+1}L_{m-1}(t), \quad m = 1, 2, \dots,$$

where $L_0(t) = 1$ and $L_1(t) = t$. For one to use these polynomials on the interval $[0, 1]$, we define the so called shifted Legendre polynomials by using the change of variable $t = 2x - 1$. Let the shifted Legendre polynomials $L_m(2x - 1)$ be denoted by $P_m(x)$. Then $P_m(x)$ can be obtained as follows:

$$P_{m+1}(x) = \frac{(2m+1)(2x-1)}{m+1}P_m(x) - \frac{m}{m+1}P_{m-1}(x), \quad m = 1, 2, \dots, \quad (8)$$

where $P_0(x) = 1$ and $P_1(x) = 2x - 1$. The analytical form of the shifted Legendre polynomials $P_m(x)$ of degree m is given by:

$$P_m(x) = \sum_{k=0}^m (-1)^{m+k} \frac{(m+k)!x^k}{(m-k)!(k!)^2}. \quad (9)$$

Note that $P_m(0) = (-1)^m$ and $P_m(1) = 1$. The orthogonality condition is

$$\int_0^1 P_m(x)P_n(x)dx = \begin{cases} \frac{1}{2m+1}, & \text{for } m = n, \\ 0, & \text{for } m \neq n. \end{cases} \quad (10)$$

3.2. Wavelets and Legendre Wavelets

Wavelets are family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously we have the following family of continuous wavelets as [14]

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right) \quad a, b \in \mathbb{R} \quad a \neq 0$$

If we restrict the parameters a and b to discrete values as $a = a_0^{-k}$, $b = nb_0 a_0^{-k}$, $a_0 > 1$, $b_0 > 1$, where n and k are positive integers, the family of discrete wavelets are defined as

$$\psi_{n,k}(t) = |a_0|^{\frac{k}{2}} \psi(a_0^k t - nb_0)$$

where $\psi_{n,k}$ form a wavelet basis for $L^2(\mathbb{R})$.

Legendre wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments: k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval $[0, 1]$ by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m + \frac{1}{2}} L_m(2^{k+1}t - (2n+1)), & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

where $m = 0, 1, \dots, M$ and $n = 0, 1, \dots, 2^k - 1$. The coefficient $\sqrt{m + \frac{1}{2}}$ is for the orthonormality and the $L_m(t)$ is the well known Legendre polynomial defined in Section 3.1

Using shifted Legendre polynomial $P_m(t)$ as defined in Section 3.1 one can write the Legendre Wavelets as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} \sqrt{m + \frac{1}{2}} P_m(2^k t - n), & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

with the same range of m and n as in (11).

3.3. Function Approximations

Any function $f(t)$ which is square integrable in the interval $[0, 1]$ can be expanded into Legendre wavelet series as [15]

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m}, \quad (13)$$

where the coefficient $C_{n,m}$ is given by

$$C_{n,m} = \langle f(t), \psi_{n,m}(t) \rangle.$$

The convergence of the Legendre wavelet series (13) is established in [16]. If the infinite series in (13) is truncated, then it can be written as

$$f(t) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^M C_{n,m} \psi_{n,m} = \mathbf{C}^T \Psi(t), \quad (14)$$

where $\mathbf{C}$ and Ψ are $2^k(M+1) \times 1$ matrices given by

$$\mathbf{C} = [c_{0,0}, c_{0,1}, \dots, c_{0,M}, c_{1,0}, c_{1,1}, \dots, c_{1,M}, \dots, c_{(2^k-1),0}, c_{(2^k-1),1}, \dots, c_{(2^k-1),M}]^T, \quad (15)$$

$$\Psi(t) = [\psi_{0,0}, \psi_{0,1}, \dots, \psi_{0,M}, \psi_{1,0}, \psi_{1,1}, \dots, \psi_{1,M}, \dots, \psi_{(2^k-1),0}, \psi_{(2^k-1),1}, \dots, \psi_{(2^k-1),M}]^T. \quad (16)$$

4. Legendre Wavelet Operational Matrix of Fractional Order Derivative

In this section, we derive the Legendre wavelet operational matrix of the fractional derivative by first transforming the wavelets to shifted Legendre polynomials, we then make use of the shifted Legendre operational matrix of the fractional derivative derived in [17], and finally we derive the Legendre wavelet operational matrix of the fractional derivative.

4.1. Transformation Matrix of the Legendre Wavelets to Legendre Polynomials

An arbitrary function $y(t) \in L^2[0,1]$ can be expanded into shifted Legendre polynomials as

$$y(x) = \sum_{m=0}^M r_m P_m(x) = R \Psi'(x),$$

where the shifted Legendre coefficient vector R and the shifted Legendre vector $\Psi'(x)$ are given by

$$R = [r_0, r_1, \dots, r_M], \quad (17)$$

$$\Psi'(x) = [P_0(x), P_1(x), \dots, P_M(x)]^T. \quad (18)$$

The Legendre wavelet may be expanded in to $(M + 1)$ -terms shifted Legendre polynomials as

$$\Psi_{2^k(M+1) \times 1}(t) = \Phi_{2^k(M+1) \times (M+1)} \Psi'_{(M+1) \times 1}, \quad (19)$$

where Φ is the transformation matrix of the Legendre wavelet to Legendre polynomial. E.g when $M = 2$ and $k = 1$ we have

$$\Psi(t) = [\psi_{0,0}(t), \psi_{0,1}(t), \psi_{0,2}(t), \psi_{1,0}(t), \psi_{1,1}(t), \psi_{1,2}(t)]^T,$$

$$\Psi'(x) = [P_0(x), P_1(x), P_2(x)]^T,$$

where

$$\begin{aligned} \psi_{0,0}(t) &= \sqrt{2} = \sqrt{2}P_0(t), \\ \psi_{0,1}(t) &= \sqrt{6}(4t - 1) = \sqrt{6}P_0(t) + 2\sqrt{6}P_1(t), \\ \psi_{0,2}(t) &= \sqrt{10}(24t^2 - 12t + 1) = 2\sqrt{10}P_0(t) \\ &\quad + 6\sqrt{10}P_1(t) + 4\sqrt{10}P_2(t), \end{aligned} \left\{ \begin{array}{l} 0 \leq t < \frac{1}{2}, \end{array} \right.$$

$$\begin{aligned} \psi_{1,0}(t) &= \sqrt{2} = \sqrt{2}P_0(t), \\ \psi_{1,1}(t) &= \sqrt{6}(4t - 3) = -\sqrt{6}P_0(t) + 2\sqrt{6}P_1(t), \\ \psi_{1,2}(t) &= \sqrt{10}(24t^2 - 36t + 13) = 3\sqrt{10}P_0(t) \\ &\quad - 6\sqrt{10}P_1(t) + 4\sqrt{10}P_2(t), \end{aligned} \left\{ \begin{array}{l} \frac{1}{2} \leq t < 1. \end{array} \right.$$

Thus, in this case

$$\Phi = \begin{cases} \Phi_1 = [a_{i,j}]_{6 \times 3}, & 0 \leq t < \frac{1}{2} \\ \Phi_2 = [b_{i,j}]_{6 \times 3}, & \frac{1}{2} \leq t < 1. \end{cases}$$

where

$$\Phi_1 = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{6} & 2\sqrt{6} & 0 \\ 3\sqrt{10} & 6\sqrt{10} & 4\sqrt{10} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ -\sqrt{6} & 2\sqrt{6} & 0 \\ 3\sqrt{10} & -6\sqrt{10} & 4\sqrt{10} \end{bmatrix}.$$

4.2. Shifted Legendre Operational Matrix of Fractional Order Derivative

The fractional derivative of order α of the vector $\Psi'(t)$ as shown in [17] can be expressed by

$$D^\alpha \Psi'(t) = \mathbf{F}^{(\alpha)} \Psi'(t), \quad (20)$$

where $\mathbf{F}^{(\alpha)}$ is the $(m+1) \times (m+1)$ operational matrix of fractional derivative of order α defined as:

$$\mathbf{F}^{(\alpha)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil \alpha \rceil}^{\lceil \alpha \rceil} \theta_{\lceil \alpha \rceil,0,k} & \sum_{k=\lceil \alpha \rceil}^{\lceil \alpha \rceil} \theta_{\lceil \alpha \rceil,1,k} & \cdots & \sum_{k=\lceil \alpha \rceil}^{\lceil \alpha \rceil} \theta_{\lceil \alpha \rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil \alpha \rceil}^i \theta_{i,0,k} & \sum_{k=\lceil \alpha \rceil}^i \theta_{i,1,k} & \cdots & \sum_{k=\lceil \alpha \rceil}^i \theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil \alpha \rceil}^m \theta_{m,0,k} & \sum_{k=\lceil \alpha \rceil}^m \theta_{m,1,k} & \cdots & \sum_{k=\lceil \alpha \rceil}^m \theta_{m,m,k} \end{pmatrix},$$

where $\theta_{i,j,k}$ is given by:

$$\theta_{i,j,k} = (2j+1) \sum_{l=0}^j \frac{(-1)^{i+j+k+l} (i+k)! (l+j)!}{(i-k)! k! \Gamma(k-\alpha+1) (j-l)! (l!)^2 (k+l-\alpha+1)} \quad (21)$$

Check [17] for more details on $\mathbf{F}^\alpha$.

4.3. Legendre Wavelet Operational Matrix of Fractional Order Derivative

We now derive Legendre wavelet operational matrix of fractional order derivative. Let

$$D^\alpha \Psi(t) = \mathbf{H}^{(\alpha)} \Psi(t), \quad (22)$$

where $\mathbf{H}^{(\alpha)}$ is the Legendre wavelet operational matrix of fractional derivative. Using Eq.(19) and (20) we get

$$D^\alpha \Psi(t) = D^\alpha \Phi \Psi'(t) = \Phi D^\alpha \Psi'(t) = \Phi \mathbf{F}^{(\alpha)} \Psi'(t), \quad (23)$$

from Eq. (22) and (23) we have

$$\mathbf{H}^{(\alpha)}\Psi(t) = \mathbf{H}^{(\alpha)}\Phi\Psi'(t) = \Phi\mathbf{F}^{(\alpha)}\Psi'(t). \quad (24)$$

Thus, the Legendre wavelet operational matrix of fractional derivative $\mathbf{H}^{(\alpha)}$ is given by

$$\mathbf{H}^{(\alpha)} = \Phi\mathbf{F}^{(\alpha)}\Phi^{-1}. \quad (25)$$

5. Applications of Legendre Wavelet Operational Matrix of Fractional Order Derivative to FDEs

Here we are going to apply the Legendre wavelet operational matrix of fractional derivative to solve linear and non-linear multi-order FDEs.

5.1. Linear Multi-Order Fractional Differential Equation

Consider the linear FDE

$$D^\alpha y(x) = a_1 D^{\mu_1} y(x) + \cdots + a_s D^{\mu_s} y(x) + a_{s+1} y(x) + a_{s+2} g(x), \quad (26)$$

with initial conditions

$$y^{(i)}(0) = d_i, \quad i = 0, 1, \dots, r \quad (27)$$

where a_j , for $j = 1, \dots, s+2$ are real constant coefficients and also $r < \alpha \leq r+1$, $0 < \mu_1 < \mu_2 < \cdots < \mu_s < \alpha$. D^α denotes the Caputo fractional derivative of order α .

To solve problem (26) and (27) we first approximate $y(x)$ and $g(x)$ by the Legendre wavelets as,

$$y(x) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^M c_{n,m} \psi_{n,m} = \mathbf{C}^T \Psi(t) \quad (28)$$

$$g(x) \approx \sum_{n=0}^{2^k-1} \sum_{m=0}^M z_{n,m} \psi_{n,m} = \mathbf{G}^T \Psi(t) \quad (29)$$

where

$$\mathbf{G} = [z_{0,0}, z_{0,1}, \dots, z_{0,M}, z_{1,0}, z_{1,1}, \dots, z_{1,M}, \dots, z_{(2^k-1),0}, z_{(2^k-1),1}, \dots, z_{(2^k-1),M}]^T$$

is known but $\mathbf{C}$ as defined in (15) is the unknown vector.

Now, using (22) and (28) we get

$$D^\alpha y(x) \approx \mathbf{C}^T D^\alpha \Psi(x) \approx \mathbf{C}^T \mathbf{H}^{(\alpha)} \Psi(x) \quad (30)$$

$$D^{\mu_j} y(x) \approx \mathbf{C}^T D^{\mu_j} \Psi(x) \approx \mathbf{C}^T \mathbf{H}^{(\mu_j)} \Psi(x). \quad (31)$$

Employing (28)-(31) the residual $R(x)$ for Eqn.(26) can be written as

$$\begin{aligned} R(x) \approx & (\mathbf{C}^T \mathbf{H}^{(\alpha)} \Psi(x) - a_1 \mathbf{C}^T \mathbf{H}^{(\mu_1)} \Psi(x) - \dots \\ & - a_k \mathbf{C}^T \mathbf{H}^{(\mu_k)} \Psi(x) - a_{k+1} \mathbf{C}^T \Psi(x) - a_{k+2} \mathbf{G}^T \Psi(x)). \end{aligned} \quad (32)$$

As in a typical tau method [18] we generate $2^k(M+1) - r$ linear equations by applying

$$\langle R(x), \Psi_j(x) \rangle = \int_0^1 R(x) \Psi_j(x) dx = 0, \quad j = 1, \dots, 2^k(M+1) - r. \quad (33)$$

Also, by substituting initial conditions (27) in to (28)-(31) we have

$$\begin{aligned} y(0) &\approx \mathbf{C}^T \Psi(0) = d_0 \\ y'(0) &\approx \mathbf{C}^T \mathbf{H}^{(1)} \Psi(0) = d_1 \\ &\vdots \\ y^{(r)}(0) &\approx \mathbf{C}^T \mathbf{H}^{(r)} \Psi(0) = d_r \end{aligned} \quad (34)$$

Eqs. (33)-(34) generate $2^k(M+1)$ set of linear equations. These linear equations can be solved for unknown coefficients of the vector $\mathbf{C}$. Consequently, $y(x)$ given in Eq. (28) can be calculated.

5.2. Non-Linear Multi-Order Fractional Differential Equation

Consider the non-linear multi-order fractional differential equation

$$D^\alpha y(x) = F(x, y(x), D^{\mu_1} y(x), \dots, D^{\mu_s} y(x)) \quad (35)$$

with initial conditions

$$y^{(i)}(0) = d_i, \quad i = 0, 1, \dots, r \quad (36)$$

where $r < \alpha \leq r+1$, $0 < \mu_1 < \mu_2 < \dots < \mu_s < \alpha$, D^α denotes the Caputo fractional derivative of order α and F can be non-linear in general.

To solve this problem we first approximate $y(x)$, $D^\alpha y(x)$, $D^{\mu_j} y(x)$ for $j = 1 \dots, s$ as (28), (30) and (31) respectively. And by substituting these equations in (35) we get

$$\mathbf{C}^T \mathbf{H}^{(\alpha)} \Psi(x) \approx F(x, \mathbf{C}^T \Psi(x), \mathbf{C}^T \mathbf{H}^{(\mu_1)} \Psi(x), \dots, \mathbf{C}^T \mathbf{H}^{(\mu_s)} \Psi(x)) \quad (37)$$

Also for the initial conditions we use the wavelets to approximate and we get (34).

Now to find the solution $y(x)$, we collocate (37) at $2^k(M+1) - r$ points. For suitable collocation points we use the first $2^k(M+1) - r$ shifted Legendre polynomial roots $P_{2^k(M+1)}(x)$, these equations together with (34) generates $2^k(M+1)$ non-linear equations which can be solved using Newton's iterative method. Consequently $y(x)$ given in (28) can be calculated.

6. Illustrative Examples

In this section, we demonstrate the effectiveness of the proposed Legendre wavelet method with numerical examples. It will be shown that the Legendre wavelet method is very efficient for solving both linear and non-linear fractional differential equations.

Example 6.1. First, let's consider the following initial value problem

$$D^2 y(x) + D^{\frac{1}{2}} y(x) + y(x) = g(x)$$

$$y(0) = 0, \quad y'(0) = 0.$$

with $g(x) = x^2 + 2 + \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}}$

The exact solution of this problem is $y(x) = x^2$. By applying the technique described in section 5.1 with $M = 2$ and $k = 0$ we approximate the following as

$$\begin{aligned} y(x) &= \mathbf{C}^T \Psi(x), \\ D^2 y(x) &= \mathbf{C}^T \mathbf{H}^{(2)} \Psi(x), \\ D^{\frac{1}{2}} y(x) &= \mathbf{C}^T \mathbf{H}^{(\frac{1}{2})} \Psi(x), \\ g(x) &= \mathbf{G}^T \Psi(x). \end{aligned}$$

Thus, we have

$$\mathbf{H}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{15} & 0 \end{pmatrix}, \quad \mathbf{H}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 12\sqrt{5} & 0 & 0 \end{pmatrix},$$

$$\mathbf{H}^{(\frac{1}{2})} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{8\sqrt{3}}{3\sqrt{\pi}} & \frac{8}{5\sqrt{\pi}} & \frac{-8\sqrt{15}}{105\sqrt{\pi}} \\ \frac{-8\sqrt{5}}{5\sqrt{\pi}} & \frac{8\sqrt{15}}{7\sqrt{\pi}} & \frac{8}{3\sqrt{\pi}} \end{pmatrix}.$$

Similarly, the right hand side, $g(x)$, of the problem can be expanded by Legendre wavelets as $g(x) = \begin{pmatrix} 2.934853328 & 0.7360932263 & 0.1367850083 \end{pmatrix} \Psi(x)$. Therefore, using (33) we obtain

$$12\sqrt{5}c_{02} + \frac{8\sqrt{3}}{3\sqrt{\pi}}c_{01} - \frac{8\sqrt{5}}{5\sqrt{\pi}}c_{02} + c_{00} = 2.934853328 \quad (38)$$

Then, applying (34) we get

$$\begin{aligned} c_{00} - \sqrt{3}c_{01} + \sqrt{5}c_{02} &= 0 \\ 2\sqrt{3}c_{01} - 6\sqrt{5}c_{02} &= 0 \end{aligned} \quad (39)$$

Solving (38) and (39) we have

$$c_{00} = 0.3333012817, \quad c_{01} = 0.2886473771, \quad c_{02} = 0.07452843228$$

Thus,

$$y(x) = [0.3333012817 \quad 0.2886473771 \quad 0.07452843228] \begin{pmatrix} 1 \\ \sqrt{3}(2x-1) \\ \sqrt{5}(6x^2-6x+1) \end{pmatrix} \approx x^2,$$

which is the exact solution.

Example 6.2. We consider the following linear initial value problem [9]

$$D^\alpha y(x) + y(x) = 0, \quad 0 < \alpha \leq 2,$$

$$y(0) = 1, \quad y'(0) = 0$$

The second initial condition is for $\alpha > 1$ only and the exact solution of this problem is given by

$$y(x) = \sum_{k=0}^{\infty} \frac{(-x^\alpha)^k}{\Gamma(\alpha k + 1)}$$

we solved the problem, by applying the technique described in Section 5.1. with $M = 8$, $k = 1$ and $\alpha = 1$ the result obtain fits exactly with the exact

solution, $y(x) = \exp(-x)$ [17], as shown in Fig.1. Also the results for $y(x)$ when $M = 8$, $k = 1$, $\alpha = 0.5, 0.75, 0.95$ and 1 are plotted in Fig. 2. Here we note that as α approaches 1 , the numerical solution converges to the analytical solution $y(x) = \exp(x)$, i.e. in the limit, the solution of the fractional differential equations approaches to that of the integer-order differential equations.

We also present the results for $\alpha > 1$, numerical results for $y(x)$ when $M = 9$, $k = 0$ and $\alpha = 1.5, 1.75, 1.95$, and 2 . For $\alpha = 2$, we compare our solution and the exact solution is given by $y(x) = \cos(x)$ as shown in Fig.3, also Fig.4 shows the behaviour of our solutions when $\alpha = 1.5, 1.75, 1.95$, and 2 together with the exact solution

Fig.1. Comparison of our solution and exact solution $y(x)$ when $\alpha=1$, $M=8$ and $k=1$ for example 6.2

Fig.2. comparison of $y(x)$ when $\alpha=0.5, 0.75, 0.95$ and 1 with $M=8$ and $k=1$ for example 6.2

Fig.3 Comparison of our solution and exact solution $y(x)$ with $M=9$ and $k=0$ when $\alpha=2$ for example 6.2

Fig.4. Comparison of exact solution and $y(x)$ when $\alpha=1.5, 1.75, 1.95$ and 2 , with $M=9$ and $k=0$ for example 6.2

Example 6.3. In this example we consider the following non-linear initial

value problem

$$D^3y(x) + D^{\frac{5}{2}}y(x) + y^2(x) = x^4$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 2$$

The exact solution is known to be $y(x) = x^2$ [17]. We use the technique described in section 5.2 to solve this problem with $M = 3$ and $k = 0$, first, we approximate the equation with Legendre wavelet as follows

$$\mathbf{C}^T \mathbf{H}^{(3)} \Psi(x) + \mathbf{C}^T \mathbf{H}^{(\frac{5}{2})} \Psi(x) + (\mathbf{C}^T \Psi(x))^2 - x^4 = 0. \quad (40)$$

Collocating (40) at the first root of 4th order shifted Legendre polynomial $x = 0.3300094782$ we obtain the following non-linear algebraic equation,

$$\begin{aligned} (120 + \frac{139.2191699}{\sqrt{\pi}}) \sqrt{7} c_{03} + (c_{00} - 0.3399810436 \sqrt{3} c_{01} - 0.326619335 \sqrt{5} c_{02} \\ + 0.411727999 \sqrt{7} c_{03})^2 = 0.01186057253. \end{aligned} \quad (41)$$

Also with the initial conditions the following equations are generated.

$$\begin{aligned} c_{00} - \sqrt{3} c_{01} + \sqrt{5} c_{02} - \sqrt{7} c_{03} &= 0 \\ 2\sqrt{3} c_{01} - 6\sqrt{5} c_{02} + 12\sqrt{7} c_{03} &= 0 \\ 12\sqrt{5} c_{02} - 60\sqrt{7} c_{03} &= 2 \end{aligned} \quad (42)$$

Thus, equation (41) together with (42) are solved and we get

$$c_{00} = 0.3333333333, \quad c_{01} = 0.2886751345, \quad c_{02} = 0.07453559928, \quad c_{03} = 0$$

Consequently,

$$y(x) =$$

$$[0.3333333333 \ 0.2886751345 \ 0.07453559928 \ 0] \begin{pmatrix} 1 \\ \sqrt{3}(2x-1) \\ \sqrt{5}(6x^2-6x+1) \\ \sqrt{7}(20x^3-30x^2+12x-1) \end{pmatrix}$$

$$= x^2$$

which is the exact solution.

Example 6.4. We finally consider the following non-linear initial value problem

$$D^4y(x) + D^{\frac{7}{2}}y(x) + y^3(x) = x^9$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y^{(3)}(0) = 6$$

The exact solution to this is known to be $y(x) = x^3$.

Similar to example 6.3. here we also use the technique described in Section 5.2. with $M = 4$ and $k = 1$ we approximate the equation as follows

$$\mathbf{C}^T \mathbf{H}^{(4)} \Psi(x) + \mathbf{C}^T \mathbf{H}^{(\frac{7}{2})} \Psi(x) + (\mathbf{C}^T \Psi(x))^3 - x^9 = 0 \quad (43)$$

(43) is collocated at the first six roots of 10th order shifted Legendre polynomial $P_{10}(x)$ and six non-linear algebraic equations are generated, the initial conditions also generates.

$$\begin{aligned} \mathbf{C}^T \Psi(0) &= 0 \\ \mathbf{C}^T \mathbf{H}^{(1)} \Psi(0) &= 0 \\ \mathbf{C}^T \mathbf{H}^{(2)} \Psi(0) &= 0 \\ \mathbf{C}^T \mathbf{H}^{(3)} \Psi(0) &= 6 \end{aligned} \quad (44)$$

The six non-linear equations generated by collocating (43) are solved together with (44) and we obtain

$$\begin{aligned} c_{00} &= \frac{1}{64}\sqrt{2}, \quad c_{01} = \frac{3}{320}\sqrt{6}, \quad c_{02} = \frac{1}{320}\sqrt{10}, \quad c_{03} = \frac{1}{2240}\sqrt{14}, \quad c_{04} = 0, \quad c_{10} = 0, \\ c_{11} &= 0, \\ c_{12} &= 0, \quad c_{13} = 0, \quad c_{14} = 0. \end{aligned}$$

Consequently

$$y(x) = \left[\frac{1}{64}\sqrt{2} \quad \frac{3}{320}\sqrt{6} \quad \frac{1}{320}\sqrt{10} \quad \frac{1}{2240}\sqrt{14} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \right] \\ \times \begin{pmatrix} \sqrt{2} \\ \sqrt{6}(4x-1) \\ \sqrt{10}(24x^2-12x+1) \\ \sqrt{14}(160x^3-120x^2+24x-1) \\ 3\sqrt{2}(1120x^4-1120x^3+360x^2-40x+1) \\ \sqrt{2} \\ \sqrt{6}(4x-3) \\ \sqrt{10}(24x^2-36x+13) \\ \sqrt{14}(160x^3-360x^2+264x-63) \\ 3\sqrt{2}(1120x^4-3360x^3+3720x^2-1800x+321) \end{pmatrix} = x^3,$$

which is the exact solution.

7. Conclusion

A general formulation for the Legendre wavelet operational matrix of fractional order derivative has been derived through wavelet-polynomial transformation, the procedure is easy to use and yet obtain the accurate result. This matrix is used to approximate numerical solution of FDEs. Our numerical findings are compared with exact solutions. The solutions obtained using the present method shows that this approach can solve problems effectively.

Acknowledgements

This work was supported in part by FRGS Grant Vot 1433. we also thank for financial support from UTHM through GIPS U060.

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