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THE FOURIER TRANSFORM OF P_+^λ and P_-^λ

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Abstract: We know from([5],page284) that the Fourier transform of P_+^λ and P_-^λ are given by the formulae(4) and(5) respectively. In this article using another method we obtain the Fourier transform of P_+^λ and P_-^λ, where $P = P(x)$ is defined by(1), P_+^λ by(8) and P_-^λ by(9). We prove that our formulae(44)and(82) are equivalent to formulae(4)and(5) respectively.

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1. Introduction

Let $x = (x_1, ...x_p, x_{p+1}, ...x_{p+q})$ be a point of R^n, where $p + q = n$ is the dimension of the space and let P be the quadratic form defined by

$$P = P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2, \tag{1}$$

where $p + q = n$ is the dimension of the space

The $P = 0$ hypersurface is a hypercone with a singular point(the vertex) at the origin.

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We defined the generalized function P_+^λ where λ is a complex number, by the following form

$$\left(P_+^\lambda, \varphi\right) = \int\limits_{P>0} (P(x)^\lambda \varphi(x) dx \tag{2}$$

where $dx = dx_1 ... dx_n, \varphi \in C_o^\infty$ is the space of infinitely differentiable function with compact support.For $\mathrm{Re}(\lambda) \geq 0$,the integral(3) converges and are analytic function of λ. Analytic continuation to $\mathrm{Re}(\lambda) < 0$ can be used to extend the definition $\left(P_+^\lambda, \varphi\right)$.

Similarly we defined the generalized fuction P_-^λ by the following form

$$\left(P_-^\lambda, \varphi\right) = \int\limits_{P<0} (-P(x)^\lambda \varphi(x) dx. \tag{3}$$

We know from([5],page284) that the Fourier transform of P_+^λ and P_-^λ are given by the following formula

$$\begin{aligned} F\left\{P_+^\lambda\right\} = {} & \pi^{\frac{n}{2}-1} 2^{2\lambda+n} \Gamma(\lambda + \frac{n}{2}) \Gamma(\lambda+1).(2i)^{-1} \\ & \times \left[e^{-\pi i(\lambda+\frac{q}{2})} (Q - i0)^{-\lambda-\frac{n}{2}} - e^{\pi i(\lambda+\frac{q}{2})} (Q + i0)^{-\lambda-\frac{n}{2}}\right] \end{aligned} \tag{4}$$

(see [5], p. 284, formula (4)) and

$$\begin{aligned} F\left\{P_-^\lambda\right\} = {} & - \pi^{\frac{n}{2}-1} 2^{2\lambda+n} \Gamma(\lambda + \frac{n}{2}) \Gamma(\lambda+1).(2i)^{-1} \\ & \times \left[e^{-\frac{q\pi i}{2}} (Q - i0)^{-\lambda-\frac{n}{2}} - e^{\pi i\frac{q}{2}} (Q + i0)^{-\lambda-\frac{n}{2}}\right] \end{aligned} \tag{5}$$

(see [5], p. 284, formula (4′), where

$$F\{f(x)\} = \int \exp\left[-i(x_1 s_1 + ... + x_n s_n)\right] dx, \tag{6}$$

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm\lambda\pi i} P_-^\lambda \tag{7}$$

(see [5], p. 276, formulae (2) and (2′)),

$$P_+^\lambda = \begin{cases} P^\lambda, & \text{if } P \geq 0, \\ 0, & \text{if } P < 0, \end{cases} \tag{8}$$

and

$$P_-^\lambda = \begin{cases} (-P)^\lambda, & \text{if } P \leq 0, \\ 0, & \text{if } P > 0. \end{cases} \tag{9}$$

The funcionals

$$\left(P_+^\lambda, \varphi\right) \qquad \text{and} \qquad \left(P_-^\lambda, \varphi\right) \tag{10}$$

correspond to the functions P_+^λ and P_-^λ defined by(8) and(9) respectively.

Now we are going to study the Fourier transform of P_+^λ .

2. The Fourier Transform of P_+^λ

Let P_+^λ be defined by(8),the Fourier transform of P_+^λ is defined by the following formula

$$F\left\{P_+^\lambda(x)\right\} = \int\limits_{R^n} e^{-\langle x,y\rangle} P_+^\lambda(x)dx, \tag{11}$$

where

$$\begin{aligned} \langle x, y\rangle &= x_1y_1 + ... + x_py_p + x_{p+1}y_{p+1} + ... + x_{p+q}y_{p+q}, \\ dx &= dx_1...dx_pdx_{p+1}...dx_{p+q}, \end{aligned} \tag{12}$$

and $p + q = n$ dimension of the space.

From (11) and using (8) and (12), we have

$$\begin{aligned} F\left\{P_+^\lambda(x)\right\} &= \int\limits_{R^p} \left(\int\limits_{R^q} e^{-i\langle x,y\rangle} P_+^\lambda(x)\right) dx_1...dx_pdx_{p+1}...dx_{p+q} = \\ &= I_p\left(I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda\right), \end{aligned} \tag{13}$$

where

$$I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda = \int\limits_{R^q} e^{-i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})} \left(x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2\right)_+^\lambda dx_{p+1}...dx_{p+q}, \tag{14}$$

and

$$I_p\left(I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda\right) = \int\limits_{R^p} e^{-i(x_1y_1+...+x_py_p)} I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda dx_p...dx_1. \tag{15}$$

By calling

$$r^2 = x_1^2 + ... + x_p^2, \tag{16}$$

$$s^2 = x_{p+1}^2 + ... + x_{p+q}^2, \tag{17}$$

$$\bar{y_p} = (y_1.,,,y_p), \tag{18}$$

$$\bar{y_q} = (y_{p+1}.,,,y_{p+q}) \tag{19}$$

and without loss of generality we may assume that the component of y_q are given by $\bar{y_q} = (|\bar{y_q}|, 0, 0, ...0)$ so that the integration(14) becomes

$$I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda =$$

$$= \int_{R^q} e^{-i|\bar{y_q}|x_{p+1}} \left(r^2 - s^2\right)_+^\lambda dx_{p+1}...dx_{p+q}. \tag{20}$$

We shall perform the integration(20) by going to polar coordinates. After integration over angles $\varphi_{p+2}, \varphi_{p+3}, ...\varphi_{p+q-1}$ and using the fact that

$$\Omega_{q-1} = \frac{2\left(\sqrt[2]{\pi}\right)^{q-1}}{\Gamma(\frac{q-1}{2})} \tag{21}$$

area of unit sphere in R^{q-1} we arrive at

$$I_q((P(x_1, ...x_p, x_{p+1}, ...x_{p+q}))^\lambda =$$

$$= \frac{2\left(\sqrt[2]{\pi}\right)^{q-1}}{\Gamma(\frac{q-1}{2})} \int_0^\infty \int_0^\pi \left(r^2 - s^2\right)_+^\lambda e^{-i|\bar{y_q}|s\cos\varphi_{p+1}} sen^{q-2}(\varphi_{p+1}) s^{q-1} d\varphi_{p+1} ds. \tag{22}$$

Now using the integral representation of the Bessel function $J_\gamma(x)$:

$$J_\gamma(x) = \frac{1}{2^\gamma \sqrt[2]{\pi}\Gamma(\gamma + \frac{1}{2})} \int_0^\pi e^{\pm ix\cos\theta} x^\gamma sen^{2\gamma}\theta d\theta \tag{23}$$

([1].page 409 and [2], page953,formula7) where

$$J_\gamma(x) = \sum_{j\geq 0} (-1)^j \frac{(-1)^j \left(\frac{x}{2}\right)^{\gamma+2j}}{j!\Gamma(\gamma + j + 1)} \tag{24}$$

we have

$$I_q((P(x_1,...x_p,x_{p+1},...x_{p+q}))^\lambda = \frac{2\left(\sqrt[2]{\pi}\right)^{q-1}}{\Gamma(\frac{q-1}{2})}.$$

$$\int\limits_0^\infty s^{q-1}\left(r^2-s^2\right)_+^\lambda \frac{\sqrt[2]{\pi}2^{\frac{q-2}{2}}\Gamma(\frac{q-1}{2})J_{\frac{q-2}{2}}(s\left|\bar{y_q}\right|)}{(s\left|\bar{y_q}\right|)^{\frac{q-2}{2}}}ds = \tag{25}$$

$$= 2\left(\sqrt[2]{\pi}\right)^{q-1}\sqrt[2]{\pi}2^{\frac{q-2}{2}}\left(\left|\bar{y_q}\right|\right)^{1-\frac{q}{2}}\int\limits_0^\infty s^{\frac{q}{2}}\left(r^2-s^2\right)_+^\lambda J_{\frac{q-2}{2}}(s\left|\bar{y_q}\right|)ds.$$

Now taking into account the formula(8),from(25) we have

$$I_q((P(x_1,...x_p,x_{p+1},...x_{p+q}))^\lambda =$$

$$= 2^{\frac{q}{2}}\pi^{\frac{q-1}{2}}\left(\left|\bar{y_q}\right|\right)^{1-\frac{q}{2}}\int\limits_0^r s^{\frac{q}{2}}\left(r^2-s^2\right)^\lambda J_{\frac{q-2}{2}}(s\left|\bar{y_q}\right|)ds. \tag{26}$$

On the other hand, considering the formula

$$H_\nu\left\{f(x)\right\} = \int\limits_0^\infty f(x)J_\nu(xy)(xy)^{\frac{1}{2}}dx = \int\limits_0^a x^{\nu+\frac{1}{2}}\left(a^2-x^2\right)^\mu J_\nu(xy)(xy)^{\frac{1}{2}}dx =$$

$$= 2^\mu\Gamma(\mu+1)y^{-\mu-\frac{1}{2}}a^{\nu+\mu+1}J_{\nu+\mu+1}(ay) \tag{27}$$

([3],page26,formula33), for

$$f(x) = \begin{cases} x^{\nu+\frac{1}{2}}\left(a^2-x^2\right)^\mu \text{ if } 0<x<a \\ 0 \text{ if } a<x<\infty, \end{cases} \tag{28}$$

from(26)and using(27) we have

$$I_q((P(x_1,...x_p,x_{p+1},...x_{p+q}))^\lambda =$$

$$= 2^{\frac{q}{2}}\pi^{\frac{q-1}{2}}\sqrt[2]{\pi}\left(\left|\bar{y_q}\right|\right)^{1-\frac{q}{2}-\lambda-1}2^\lambda\Gamma(\lambda+1)r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\left|\bar{y_q}\right|). \tag{29}$$

On the other hand,from(15)and(29) we have

$$I_p\left(I_q((P(x_1,...x_p,x_{p+1},...x_{p+q}))^{\lambda}\right)=2^{\frac{q}{2}}\pi^{\frac{q-1}{2}}\sqrt[2]{\pi}\left(|\bar{y_q}|\right)^{1-\frac{q}{2}-\lambda-1}2^{\lambda}\Gamma(\lambda+1).$$

$$\int_{R^p} e^{-i(x_1y_1+...+x_py_p)}r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dx_p...dx_1. \tag{30}$$

Now we are going to study the integral

$$\int_{R^p} e^{-i(x_1y_1+...+x_py_p)}r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dx_p...dx_1 \tag{31}$$

and without loss of generality we may assume that the component of $\bar{y_p}$ are given by $\bar{y_p}=(|\bar{y_p}|\,,0,0,...0)$ so that the integration(31) becomes

$$\int_{R^p} e^{-i|\bar{y_p}|x_1}r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dx_p...dx_1. \tag{32}$$

We shall perform the integration(32) by going to polar coordinate. After integration over angles $\theta_2,...\theta_{p-1}$ and using the fact

$$\Omega_{p-1}=\frac{2\left(\sqrt[2]{\pi}\right)^{p-1}}{\Gamma(\frac{p-1}{2})} \tag{33}$$

we arrive at

$$\int_{R^p} e^{-i|\bar{y_p}|x_1}r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dx_p...dx_1=$$

$$=\frac{2\left(\sqrt[2]{\pi}\right)^{p-1}}{\Gamma(\frac{p-1}{2})}\int_0^{\infty}\left(\int_0^{\pi}e^{-i|\bar{y_p}|r\cos\theta_1}sen^{p-2}(\theta_1)r^{p-1}d\theta_1\right)r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dr. \tag{34}$$

From(34) and using the formula(23) we have

$$\int_{R^p} e^{-i|\bar{y_p}|x_1}r^{\lambda+\frac{q}{2}}J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dx_p...dx_1= \tag{35}$$

$$2^{\frac{p}{2}}\pi^{\frac{p}{2}}\left(|\bar{y_p}|\right)^{1-\frac{p}{2}}\int_0^{\infty}r^{\lambda+\frac{q}{2}}J_{\frac{p-2}{2}}(r\,|\bar{y_p}|)J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dr.$$

Now considering the formula

$$H_\nu\left\{x^{-\lambda-\frac{1}{2}}J_\mu(ax)\right\} = \frac{\Gamma(\frac{\mu+\nu-\lambda+1}{2})\ a^\mu}{2^\lambda y^{\mu-\lambda+\frac{1}{2}}\Gamma(\mu+1)\Gamma(\frac{\lambda+\nu-\mu+1}{2})}. \cdot {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\mu-\lambda-\nu+1}{2}; \mu+1; \frac{a^2}{y^2}\right) \tag{36}$$

([3],page48,formula9) where $H_\nu\{f(x)\}$ is the Hankel transform defined by

$$H_\nu\{f(x)\} = \int_0^\infty f(x)J_\nu(xy)(xy)^{\frac{1}{2}}dx, \tag{37}$$

$${}_2F_1(a,b;c;z) = F(a,b;c;z) = \sum_{j=0}^\infty \frac{(a)_j(b)_j}{(c)_j\ j!}z^j \tag{38}$$

and

$$(a)_j = a(a+1)...(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)} \tag{39}$$

we have

$$\int_0^\infty x^{\beta+\frac{n}{2}}J_{\beta+\frac{q}{2}}(|\bar{y_q}|\,x)J_{\frac{p}{2}-1}(|\bar{y_p}|\,x)dx$$

$$= \sum_{j\geq 0}\frac{\Gamma(\beta+\frac{n}{2}+j)\Gamma(\beta+\frac{q}{2}+j+1)}{\Gamma(\beta+\frac{n}{2})\ j!\Gamma(\beta+\frac{q}{2}+1)}.\frac{\Gamma(\beta+\frac{q}{2}+1)}{\Gamma(\beta+\frac{q}{2}+j+1)}.\left(\frac{|\bar{y_q}|^2}{|\bar{y_p}|^2}\right)$$

$$\times\frac{|\bar{y_q}|^{\beta+\frac{q}{2}}\,\Gamma(\beta+\frac{n}{2})}{2^{-\beta-\frac{n}{2}}\,(|\bar{y_p}|)^{2\beta+\frac{q}{2}+\frac{n}{2}+1}\,\Gamma(-\beta-\frac{q}{2})} \tag{40}$$

where $a = |\bar{y_q}|\,, y = |\bar{y_p}|$ and $|\bar{y_q}| < |\bar{y_p}|\,.$

By putting $x = r$, $\beta = \lambda$ in(40) and considering the formula

$$(1-z)^\gamma = \sum_{j=0}^\infty(-1)^j\frac{\Gamma(\gamma+1)}{j!\Gamma(\gamma+1-j)}z^j = \sum_{j=0}^\infty\frac{\Gamma(-\gamma+j)}{j!\Gamma(-\gamma)}z^j \tag{41}$$

for $|z| < 1$ and putting $\gamma = -\lambda-\frac{n}{2}$, we have

$$\int_0^\infty r^{\lambda+\frac{q}{2}}J_{\frac{p-2}{2}}(r\,|\bar{y_p}|)J_{\lambda+\frac{q}{2}}(r\,|\bar{y_q}|)dr = $$

$$= \frac{(|\bar{y_q}|)^{\lambda+\frac{q}{2}}\Gamma(\lambda+\frac{n}{2})\ 2^{\lambda+\frac{n}{2}}(|\bar{y_p}|)^{\frac{p}{2}-1}}{\Gamma(1+\lambda+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\left(|\bar{y_p}|^2-|\bar{y_q}|^2\right)^{-\lambda-\frac{n}{2}} \tag{42}$$

for $|\bar{y_q}| < |\bar{y_p}|$.

From(30)and using(42) we have

$$I_p\left(I_q((P(x_1,...x_p,x_{p+1},...x_{p+q}))^\lambda\right) =$$

$$\frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+\frac{n}{2})\Gamma(\lambda+1)}{\Gamma(1+\lambda+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\left(|\bar{y_p}|^2-|\bar{y_q}|^2\right)^{-\lambda-\frac{n}{2}} \tag{43}$$

$$\text{if } |\bar{y_q}|^2 < |\bar{y_p}|^2.$$

From(11),(13),using(10) and(43) we obtain the Fourier transform of $P_+^\lambda(x)$ by mean of the following formula

$$F\left\{P_+^\lambda(x)\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+\frac{n}{2})\Gamma(\lambda+1)}{\Gamma(1+\lambda+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}(Q(y))^{-\lambda-\frac{n}{2}} \tag{44}$$

if $Q(y) > 0$. Where

$$Q(y) = |\bar{y_p}|^2 - |\bar{y_q}|^2 = y_1^2 + ... + y_p^2 - y_{p+1}^2 - ... - y_{p+q}^2. \tag{45}$$

We observe that using the Bocher´s formula([6],page 187,formula15),

$$F\{f(x)\} = \int_{R^n} \exp\left[-i(x_1y_1+...+x_ny_n)\right]f(x)dx$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{|y|^{\frac{n-2}{2}}}\int_0^\infty f_0(r)r^{\frac{n}{2}}J_{\frac{n-2}{2}}(r\,|y|)dr \tag{46}$$

where $f(x) = f_0(|x|) = f_0(r)$,we obtain the same formula(44).

In fact, from(13), we have,

$$F\left\{P_+^\lambda(x)\right\} = \int_{R^p} f_p(|x|_p)e^{-i(x_1y_1+...+x_py_p)}dx_1...dx_p \tag{47}$$

where

$$f_p(|x|_p) = \int_{R^q}\left(|x|_p^2 - |x|_q^2\right)_+^\lambda e^{-i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})}dx_{p+1}...dx_{p+q}. \tag{48}$$

Now using(46)Bocner´s formula, we have

$$f_p(|x|_p) = 2\left(\sqrt[2]{\pi}\right)^{q-1}\sqrt[2]{\pi}2^{\frac{q-2}{2}}\left(|y|_q\right)^{1-\frac{q}{2}}\int_0^\infty s^{\frac{q}{2}}\left(r^2-s^2\right)_+^\lambda J_{\frac{q-2}{2}}(s\,|y|_q)ds \quad (49)$$

Now using the formula(8) and(24), we have

$$f_p(|x|_p) = 2\left(\sqrt[2]{\pi}\right)^{q-1}\sqrt[2]{\pi}2^{\frac{q-2}{2}}\sum_{j\geq 0}(-1)^j\frac{\left(\frac{|y|_q}{2}\right)^{\frac{q-2}{2}+2j}}{j!\Gamma(\frac{q-2}{2}+j+1)}\int_0^r s^{q+2j-1}\left(r^2-s^2\right)^\lambda ds \quad (50)$$

On the otherhand, using the formula

$$\int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad (51)$$

for Re(a) >0 and Re(b) > 0,([4],page8,formula(1),we have,

$$\int_0^r s^{q+2j-1}\left(r^2-s^2\right)^\lambda ds = \frac{1}{2}r^{2\lambda+q+2j}\frac{\Gamma(\frac{q}{2}+j)\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{q}{2}+j)} \quad (52)$$

From(50)and(52), we have

$$f_p(|x|_p) = 2\left(\sqrt[2]{\pi}\right)^{q-1}\sqrt[2]{\pi}2^{\frac{q-2}{2}}\Gamma(\lambda+1).$$

$$\sum_{j\geq 0}(-1)^j\frac{\left(\frac{|y|_q}{2}\right)^{\frac{q-2}{2}+2j}}{j!\Gamma(\frac{q-2}{2}+j+1)}\frac{1}{2}r^{2\lambda+q+2j}\frac{\Gamma(\frac{q}{2}+j)}{\Gamma(\lambda+1+\frac{q}{2}+j)} = \pi^{\frac{q}{2}}\Gamma(\lambda+1)\Gamma(\lambda+1).$$

$$\cdot\sum_{j\geq 0}(-1)^j\frac{\left(|y|_q\right)^{2j}}{j!2^{2j}\Gamma(\lambda+1+\frac{q}{2}+j)}r^{2\lambda+q+2j}. \quad (53)$$

From(47)and using(53), we have

$$F\left\{P_+^\lambda(x)\right\} = \int\limits_{R^p} f_p(|x|_p)e^{-i(x_1y_1+...+x_py_p)}dx_1...dx_p = \pi^{\frac{q}{2}}\Gamma(\lambda+1)\Gamma(\lambda+1).$$

$$\sum_{j\geq 0}(-1)^j \frac{\left(|y|_q\right)^{2j}}{j!2^{2j}\Gamma(\lambda+1+\frac{q}{2}+j)} \int\limits_{R^p} e^{-i(x_1y_1+...+x_py_p)}r^{2\lambda+q+2j}dx_1...dx_p \tag{54}$$

Now using that the Fourier Transform of r^λ is given by the following formulae

$$F\left\{r^\lambda(x)\right\} = \int\limits_{R^n} r^\lambda(x)e^{-i(x_1\sigma_1...+x_n\sigma_n)}dx_1...dx_n = \tag{55}$$

$$= \frac{\pi^{\frac{n}{2}}2^{\lambda+n}\Gamma(\frac{\lambda+n}{2})}{\Gamma(-\frac{\lambda}{2})}\rho^{-\lambda-n}$$

([5], page194, formula(2)), where

$$r^2 = x_1^2 + ... + x_n^2 \tag{56}$$

and

$$\rho^2 = \sigma_1^2 + ... + \sigma_n^2 \tag{57}$$

we have

$$\int\limits_{R^p} e^{-i(x_1y_1+...+x_py_p)}r^{2\lambda+q+2j}dx_1...dx_p = \tag{58}$$

$$= \frac{\pi^{\frac{p}{2}}2^{2\lambda+q+2j+p}\Gamma(\frac{2\lambda+q+2j+p}{2})}{\Gamma(-\lambda-\frac{q}{2}-j)}(|y|_p)^{-2\lambda-q-2j-p}.$$

From(54)and(58) we have

$$F\left\{P_+^\lambda(x)\right\} = \pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+1).$$

$$\sum_{j\geq 0}(-1)^j \frac{\left(|y|_q^2\right)^j}{j!\Gamma(\lambda+1+\frac{q}{2}+j)}\frac{\Gamma(\lambda+\frac{n}{2}+j)}{\Gamma(-\lambda-\frac{q}{2}-j)}(|y|_p^2)^{-\lambda-j-\frac{n}{2}}. \tag{59}$$

Using the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sen(z\pi)} \tag{60}$$

([4],page5,formula(6), the formula(59) can be written in the following form

$$F\left\{P_+^\lambda(x)\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\sum_{j\geq 0}\frac{\Gamma(\lambda+\frac{n}{2}+j)\left(|y|_q^2\right)^j}{j!}(|y|_p^2)^{-\lambda-j-\frac{n}{2}}. \tag{61}$$

On the other hand, using the formula(41) and the formula

$$\frac{\Gamma(z)}{\Gamma(z-m)} = \frac{(-1)^m\Gamma(-z+m+1)}{\Gamma(1-z)}, m = 1,2,3,.. \tag{62}$$

we have

$$\left(|y|_p^2 - |y|_q^2\right)^\gamma = \sum_{j=0}^{\infty}\frac{\Gamma(-\gamma+j)}{j!\Gamma(-\gamma)}\left(|y|_q^2\right)^j\left(|y|_p^2\right)^{\gamma-j} \tag{63}$$

if $|y|_p^2 - |y|_q^2 > 0$. From(61)and(63) we obtain the following formula

$$F\left\{P_+^\lambda(x)\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{q}{2})\Gamma(-\lambda-\frac{q}{2})}\left(|y|_p^2 - |y|_q^2\right)^{-\lambda-\frac{n}{2}}. \tag{64}$$

if $|y|_p^2 - |y|_q^2 > 0$.
The formula(64) coincide with the formula(44).

3. Relationship between the Formula (4) and the Formula (44)

From(9), we have

$$P_-^\lambda = 0 \text{ if } P > 0. \tag{65}$$

From(4), using the properties(9)and(65), we have

$$\begin{aligned} F\left\{P_+^\lambda\right\} = \; & \pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1).(2i)^{-1}. \\ & \left[e^{-\pi i(\lambda+\frac{q}{2})}(Q-i0)^{-\lambda-\frac{n}{2}} - e^{\pi i(\lambda+\frac{q}{2})}(Q+i0)^{-\lambda-\frac{n}{2}}\right] = \\ & = \pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1).sen(\pi(-\lambda-\tfrac{q}{2})Q_+{}^{-\lambda-\frac{n}{2}}. \end{aligned} \tag{66}$$

From(66)and using the formula(60), we have

$$F\left\{P_+^\lambda\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+\frac{n}{2})\Gamma(\lambda+1)}{\Gamma(-\lambda-\frac{q}{2})\Gamma(1+\lambda+\frac{q}{2}))}Q_+{}^{-\lambda-\frac{n}{2}} \tag{67}$$

The formula(67) coincide with the formula(44), therefore the formulae(4)and the formula(44) are equivalent.

4. The Fourier Transform of P_-^λ

Let P_-^λ be defined by(9),the Fourier transform of P_-^λ is defined by the following formula

$$F\left\{P_-^\lambda\right\} = \int\limits_{R^n} e^{-\langle x,y\rangle} P_-^\lambda(x)dx \tag{68}$$

where $\langle x, y\rangle$ and dx are defined by(12)and $p+q=n$ dimension of the space.

From(68),using(16)and(17),we have

$$F\left\{P_-^\lambda\right\} = \int\limits_{R^n} e^{-\langle x,y\rangle}\left(r^2-s^2\right)_-^\lambda dx =$$

$$\int\limits_{R^q}\left(\int\limits_{R^p} e^{-i\langle x_1y_1+...+x_py_p+x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q}\rangle}\left(r^2-s^2\right)_-^\lambda dx_1...dx_p\right. \tag{69}$$

$$\left. dx_{p+1}...dx_{p+q}\right)$$

Using the definition(9)

$$\left(r^2-s^2\right)_-^\lambda(x) = \begin{cases} (-\left(r^2-s^2\right)^\lambda \text{ if } r^2-s^2 \le 0 \\ \quad and \\ 0 \text{ if } r^2-s^2 > 0. \end{cases} \tag{70}$$

and using((46) Bochner´s formula) we have

$$f_q(|x|_q) = f_q(s) = \int\limits_{R^p} e^{-i\langle x_1y_1+...+x_py_p\rangle} \left(r^2 - s^2\right)_-^\lambda dx_1...dx_p =$$

$$= \int\limits_{R^p} e^{-i\langle x_1y_1+...+x_py_p\rangle} \left(s^2 - r^2\right)^\lambda dx_1...dx_p =$$

$$2\left(\sqrt[2]{\pi}\right)^{p-1} \sqrt[2]{\pi} 2^{\frac{p-2}{2}} \left(|y|_p\right)^{1-\frac{p}{2}} \int\limits_0^\infty r^{\frac{p}{2}} \left(s^2 - r^2\right)^\lambda J_{\frac{p-2}{2}}(r\,|y|_p) dr = \tag{71}$$

$$= \frac{(2\pi)^{\frac{p}{2}}}{(|y|_p)^{\frac{p}{2}-1}} \int\limits_0^\infty r^{\frac{p}{2}} \left(s^2 - r^2\right)^\lambda J_{\frac{p-2}{2}}(r\,|y|_p) dr.$$

Now using(24), we have

$$f_q(|x|_q) = \frac{(2\pi)^{\frac{p}{2}}}{(|y|_p)^{\frac{p}{2}-1}} \int\limits_0^\infty r^{\frac{p}{2}} \left(s^2 - r^2\right)^\lambda J_{\frac{p-2}{2}}(r\,|y|_p) dr =$$

$$= (2\pi)^{\frac{p}{2}} \sum_{j\geq 0} \frac{(-1)^j (|y|_p)^{2j}}{2^{\frac{p}{2}-1} 2^{2j} j! \Gamma(\frac{p}{2}+j)} \int\limits_0^\infty r^{p+2j-1} \left(s^2 - r^2\right)^\lambda dr \tag{72}$$

Using(70)and the formula(51), we have

On the other hand, using(66)and(70), we have

$$\int\limits_0^\infty r^{p+2j-1} \left(s^2 - r^2\right)^\lambda dr = \int\limits_0^s r^{p+2j-1} \left(s^2 - r^2\right)^\lambda dr =$$

$$= \tfrac{1}{2} s^{2\lambda+p+2j} \tfrac{\Gamma(\lambda+1)\Gamma(\frac{p}{2}+j)}{\Gamma(\lambda+1+\frac{p}{2}+j)}. \tag{73}$$

From(72))and using(73), we have

$$f_q(|x|_q) = f_q(s) = \pi^{\frac{p}{2}} \sum_{j\geq 0} \frac{(-1)^j (|y|_p)^{2j}}{2^{2j} j! \Gamma(\frac{p}{2}+j)} s^{2\lambda+p+2j} \frac{\Gamma(\lambda+1)\Gamma(\frac{p}{2}+j)}{\Gamma(\lambda+1+\frac{p}{2}+j)} \tag{74}$$

From(69)and using(74), we have

$$\int_{R^q}\left(\int_{R^p} e^{-i\langle x,y\rangle}\left(r^2-s^2\right)_-^{\lambda} dx_1...dx_p\right) dx_{p+1}...dx_{p+q} =$$

$$\int_{R^q} e^{i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})} f_q(|x|_q) dx_{p+1}...dx_{p+q} =$$

$$= \pi^{\frac{p}{2}} \sum_{j\geq 0}\left(\frac{(-1)^j (|y|_p)^{2j}}{2^{2j} j!} s^{2\lambda+p+2j} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{p}{2}+j)}\right. \tag{75}$$

$$\left.\int_{R^q} e^{i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})} s^{2\lambda+p+2j} dx_{p+1}...dx_{p+q}.\right) .$$

Now using(55), we have

$$\int_{R^q} e^{i(x_{p+1}y_{p+1}+...+x_{p+q}y_{p+q})} s^{2\lambda+p+2j} dx_{p+1}...dx_{p+q} =$$

$$= \frac{\pi^{\frac{q}{2}} 2^{2\lambda+p+2j+q}\Gamma(\frac{2\lambda+p+2j+q}{2})}{\Gamma(-\lambda-\frac{p}{2}-j)}\left(|y|_q\right)^{-2\lambda-2j-n}, \tag{76}$$

form(75)and(76), we have,

$$\int_{R^q}\left(\int_{R^p} e^{-i\langle x,y\rangle}\left(r^2-s^2\right)_-^{\lambda} dx_1...dx_p\right) dx_{p+1}...dx_{p+q} =$$

$$= \pi^{\frac{p}{2}} \sum_{j\geq 0} \frac{(-1)^j (|y|_p)^{2j}}{j!} s^{2\lambda+p+2j} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{p}{2}+j)} \frac{\pi^{\frac{q}{2}} 2^{2\lambda+p+q}\Gamma(\frac{2\lambda+p+2j+q}{2})}{\Gamma(-\lambda-\frac{p}{2}-j)}\left(|y|_q\right)^{-2\lambda-2j-n} =$$

$$= \pi^{\frac{n}{2}} 2^{2\lambda+n} \sum_{j\geq 0} \frac{(-1)^j (|y|_p)^{2j}}{j!} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{p}{2}+j)} \frac{\Gamma(\lambda+\frac{n}{2}+j)}{\Gamma(-\lambda-\frac{p}{2}-j)}\left(|y|_q\right)^{-2\lambda-2j-n}. \tag{77}$$

Using the formula(60), we have

$$\frac{1}{\Gamma(\lambda+1+\frac{p}{2}+j)\Gamma(-\lambda-\frac{p}{2}-j)} = \frac{sen(-\lambda-\frac{p}{2}-j)\pi}{\pi} = \frac{(-1)^j}{\Gamma(\lambda+1+\frac{p}{2})\Gamma(-\lambda-\frac{p}{2})}. \tag{78}$$

From(69),using(77)and(78), we have

$$F\left\{P_-^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{p}{2})}\frac{\pi^{\frac{n}{2}}2^{2\lambda+n}}{\Gamma(-\lambda-\frac{p}{2})}\sum_{j\geq 0}\frac{(-1)^j(-1)^j\Gamma(\lambda+\frac{n}{2}+j)}{j!}(|y|_p)^{2j}\left(|y|_q\right)^{-2\lambda-2j-n} =$$

$$= \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}}{\Gamma(\lambda+1+\frac{p}{2})}\frac{\Gamma(\lambda+1)}{\Gamma(-\lambda-\frac{p}{2})}\sum_{j\geq 0}\frac{\Gamma(\lambda+\frac{n}{2}+j)}{j!}(|y|_p)^{2j}\left(|y|_q\right)^{-2\lambda-2j-n}. \tag{79}$$

Taking into account the formula(63), the formula(79 can be written in the following form,

$$F\left\{P_-^\lambda\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}}{\Gamma(\lambda+1+\frac{p}{2})}\frac{\Gamma(\lambda+1)\Gamma(\lambda+\frac{n}{2})}{\Gamma(-\lambda-\frac{p}{2})}\left(|y|_q^2-|y|_p^2\right)^{-\lambda-\frac{n}{2}} \tag{80}$$

if $|y|_q^2 - |y|_p^2 > 0$. Using(45), the formula(80),can be rewritten in the following form

$$F\left\{P_-^\lambda\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}}{\Gamma(\lambda+1+\frac{p}{2})}\frac{\Gamma(\lambda+1)\Gamma(\lambda+\frac{n}{2})}{\Gamma(-\lambda-\frac{p}{2})}\left(-Q(y)\right)^{-\lambda-\frac{n}{2}} \tag{81}$$

if $Q(y) < 0$. Now using the formula(9), the formula(81), can be rewritten in the following form

$$F\left\{P_-^\lambda\right\} = \frac{\pi^{\frac{n}{2}}2^{2\lambda+n}}{\Gamma(\lambda+1+\frac{p}{2})}\frac{\Gamma(\lambda+1)\Gamma(\lambda+\frac{n}{2})}{\Gamma(-\lambda-\frac{p}{2})}\left(Q(y)\right)_-^{-\lambda-\frac{n}{2}}. \tag{82}$$

5. Relationship between the Formula (5) and the Formula (81)

From(8), we have

$$P_+^\lambda = 0 \text{ if } P < 0. \tag{83}$$

Now using(83), the formula(5),can be rewritten in the following form

$$F\left\{P_{-}^{\lambda}\right\} = -\,\pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1).(2i)^{-1}$$

$$\left[e^{-\frac{q\pi i}{2}}e^{\lambda+\frac{n}{2}}Q_{-}{}^{-\lambda-\frac{n}{2}} - e^{\pi i\frac{q}{2}}e^{-\lambda-\frac{n}{2}}Q_{-}{}^{-\lambda-\frac{n}{2}}\right] =$$

$$-\,\pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1).(2i)^{-1}\left[e^{(\lambda+\frac{p}{2})\pi i} - e^{-(\lambda+\frac{p}{2})\pi i}\right]Q_{-}{}^{-\lambda-\frac{n}{2}} =$$

$$-\,\pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1).(2i)^{-1}(2i)sen(\lambda+\tfrac{p}{2})\pi Q_{-}{}^{-\lambda-\frac{n}{2}}$$

$$= \pi^{\frac{n}{2}-1}2^{2\lambda+n}\Gamma(\lambda+\tfrac{n}{2})\Gamma(\lambda+1)sen(-\lambda-\tfrac{p}{2})\pi Q_{-}{}^{-\lambda-\frac{n}{2}}$$

$$\frac{\pi^{\frac{n}{2}}2^{2\lambda+n}\Gamma(\lambda+\frac{n}{2})\Gamma(\lambda+1)}{\Gamma(\lambda+1+\frac{p}{2})\Gamma(-\lambda-\frac{p}{2})}Q_{-}{}^{-\lambda-\frac{n}{2}}. \tag{84}$$

The formula(84) coincide with the formula(82), therefore the formulae(5)and the formula(82) are equivalent.

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