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ASYMPTOTIC AND OSCILLATORY BEHAVIOUR OF SOLUTIONS OF THIRD ORDER GENERALIZED MIXED DIFFERENCE EQUATION

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Abstract: In this paper, the authors discuss the asymptotic and oscillatory behaviour of solutions of the third order generalized mixed difference equation of the form

$$\Delta_\ell\Big(\Delta^2_{\alpha(\ell)}u(k)\Big) + p(k)u(k+\ell) = 0, \quad k \in [a,\infty), \tag{1}$$

where $p(k) \geq 0$ and α, ℓ are positive reals.

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [14]) have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \quad k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\}, \tag{2}$$

no significant progress took place on this line. But recently, E. Thandapani, M.M.S. Manuel, G.B.A. Xavier [6] considered the definition of Δ as given in (2) and developed the theory of difference equations in a different direction. For convenience, the operator Δ defined by (2) is labelled as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory were obtained. The results obtained using Δ_ℓ are found in ([6],[7],[12],[13]).

In 1984, Jerzy Popenda and B.Szmanda [4] defined Δ on $u(k)$ as

$$\Delta_\alpha u(k) = u(k+1) - \alpha u(k) \tag{3}$$

and based on this definition they ([4],[5]) studied the qualitative properties of a particular difference equation and no one else has handled this operator.

In [10] the authors extended the definition of Δ_α to $\Delta_{\alpha(\ell)}$ defined on $u(k)$ as $\Delta_{\alpha(\ell)} v(k) = v(k+\ell) - \alpha v(k)$, where $\alpha \neq 0$, $\ell > 0$ are fixed and $k \in [0, \infty)$ is a variable. By defining the inverse $\Delta_{\alpha(\ell)}^{-1}$, several interesting results on number theory were obtained ([8],[9],[10],[11]).

An equation involving both Δ and Δ_α is called a mixed difference equation. Oscillatory behaviour of solutions of certain types of mixed difference equations we already studied in [2, 3, 15, 16]. An equation involving Δ_ℓ and $\Delta_{\alpha(\ell)}$ is called as generalized mixed difference equation.

In this paper, we consider the third order mixed difference equation (1) and present the sufficient conditions for the existence of oscillatory solutions of the equation (1). Throughout this paper, we make use the following assumptions:

(i) $\lceil x \rceil$ and $[x]$ denotes upper integer and integer part of x respectively.

(ii) $0 \leq k_1 < k_2 < k$ such that $j = k - k_i - \left[\frac{k-k_i}{\ell}\right]\ell$.

2. Main Results

In this section, by using the Riccati transformation techniques, we establish some new conditions which are sufficient for all solutions of (1) to be oscillatory or tend to zero as $n \to \infty$.

Theorem 2.1. *Let $\ell > 0, 0 \leq k_1 < k_2 < k$. Assume that*

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\left[\frac{k_3+r\ell-\ell}{\ell}\right]} \sum_{t=0}^{\left[\frac{k_2+s\ell-\ell}{\ell}\right]} \alpha^{\left\lceil\frac{k_3+r\ell}{\ell}\right\rceil} \alpha^{\left\lceil\frac{k_3+s\ell}{\ell}\right\rceil} p(k_1+j+t\ell) = \infty \tag{4}$$

and there exists a positive function $f(k)$ such that

$$\limsup_{k\to\infty} \sum_{t=0}^{\frac{k-k_2-j}{\ell}} \Big[f(k_2+j+t\ell)p(k_2+j+t\ell) - \frac{(\Delta_{\alpha(\ell)} f(k_2+j+t\ell))^2}{4f(k_2+j+t\ell)(k_2+j+t\ell-k_1)}\Big] = \infty, \tag{5}$$

for $k_2 > k_1$. Then, every solution $u(k)$ of (1) oscillates or $\lim_{k\to\infty} u(k) = 0$.

Proof. Let $u(k)$ be a nonoscillatory solution of (1). Without loss of generality we may assume that $u(k) > 0$ for $k > k_1$ where k_1 is chosen so large. From (1) we have $\Delta_\ell\Big(\Delta^2_{\alpha(\ell)} u(k)\Big) \leq 0$ for $k > k_1$. Then $u(k), \Delta_{\alpha(\ell)} u(k)$ and $\Delta^2_{\alpha(\ell)} u(k)$ are monotone and eventually of one sign. We claim $\Delta^2_{\alpha(\ell)} u(k) > 0$. Suppose to the contrary that $\Delta^2_{\alpha(\ell)} u(k) \leq 0$ for $k > k_2 > k_1$. Since $\Delta^2_{\alpha(\ell)} u(k)$ is nonincreasing, there exists a constant $c < 0$ and $k_3 > k_2$ such that $\Delta^2_{\alpha(\ell)} u(k) \leq c$ for $k \geq k_3$. Summing from k_3 to $k - \ell$, we obtain

$$\Delta_{\alpha(\ell)} u(k) \leq \alpha^{\left[\frac{k}{\ell}\right]} \Delta_{\alpha(\ell)} u(k_3+j) + c(k-\ell-k_3-j).$$

Letting $k \to \infty$, then $\Delta_{\alpha(\ell)} u(k) \to -\infty$. Thus, there is an integer $k_4 \geq k_3$ such that for $k \geq k_4, \Delta_{\alpha(\ell)} u(k) \leq \Delta_{\alpha(\ell)} u(k_4+j) < 0$. Summing from k_4 to $k - \ell$ we obtain

$$u(k) - \alpha^{\left[\frac{k}{\ell}\right]} u(k_4+j) \leq c(k-\ell-k_4-j).$$

This implies that $u(k) \to -\infty$ as $n \to \infty$, which is a contradiction to the fact that $u(k)$ is positive. Then $\Delta^2_{\alpha(\ell)} u(k) > 0$. Therefore, there are only the following two cases for $k \geq k_1$ sufficiently large.

Case (i) $u(k) > 0, \Delta_{\alpha(\ell)} u(k) > 0, \Delta^2_{\alpha(\ell)} u(k) > 0$.

Case (ii) $u(k) > 0, \Delta_{\alpha(\ell)} u(k) < 0, \Delta^2_{\alpha(\ell)} u(k) > 0$.

First we consider the Case (i). Define $w(k)$ by the Riccati substitution

$$w(k) = f(k) \frac{\Delta^2_{\alpha(\ell)} u(k)}{u(k+\ell)}, k \geq k_1. \tag{6}$$

We have $w(k) > 0$ and

$$\Delta_\ell w(k) = \Delta^2_{\alpha(\ell)} u(k+\ell) \Delta_\ell \Big(\frac{f(k)}{u(k+\ell)} \Big) + \frac{f(k)\Delta_\ell \big(\Delta^2_{\alpha(\ell)} u(k)\big)}{u(k+\ell)}.$$

This together with equation (1) imply that

$$\Delta_\ell w(k) \leq -f(k)p(k) + \frac{\Delta_\ell f(k)}{f(k+\ell)} w(k+\ell) - \frac{f(k)\Delta^2_{\alpha(\ell)} u(k) \Delta_\ell u(k+\ell)}{u(k+\ell)u(k+2\ell)}. \quad (7)$$

From Case (i) we have $u(k+2\ell) \geq u(k+\ell)$, then from (7) we obtain

$$\Delta_{\alpha(\ell)} w(k) \leq -f(k)p(k) + \frac{\Delta_\ell f(k)}{f(k+\ell)} w(k+\ell) - \frac{f(k)\Delta^2_{\alpha(\ell)} u(k+\ell) \Delta_\ell u(k+\ell)}{u(k+2\ell)^2}. \quad (8)$$

Also from Case (i) and equation (1) we have $\Delta_\ell u(k) = \Delta_\ell u(k_1+j)$

$$+ \sum_{t=0}^{\frac{k-\ell-k_1-j}{\ell}} \Delta^2_\ell u(k_1+j+t\ell) \geq (k-\ell-k_1)\Delta^2_{\alpha(\ell)} u(k), k \geq k_1+\ell. \quad (9)$$

This implies that

$$\Delta_\ell u(k+\ell) \geq (k-k_1)\Delta^2_{\alpha(\ell)} u(k+\ell), k \geq k_2 = k_1+\ell. \quad (10)$$

Substituting (10) in (8), we obtain

$$\Delta_\ell w(k) \leq -f(k)p(k) + \frac{\Delta_\ell f(k)}{f(k+\ell)} w(k+\ell) - \frac{f(k)(k-k_1)(\Delta^2_{\alpha(\ell)} u(k+\ell))^2}{u(k+2\ell)^2}. \quad (11)$$

From (6) and (11) we obtain

$$\Delta_\ell w(k) \leq -f(k)p(k) + \frac{\Delta_\ell f(k)}{f(k+\ell)} w(k+\ell) - \frac{f(k)(k-k_1)w(k+\ell)^2}{f(k+\ell)^2}. \quad (12)$$

By completing the square, we have

$$\Delta_\ell w(k) \leq -f(k)p(k) + \frac{(\Delta_\ell f(k))^2}{4f(k)(k-k_1)}$$
$$- \left[\frac{\sqrt{f(k)(k-k_1)}}{f(k+\ell)} w(k+\ell) - \frac{\Delta_\ell f(k)}{2\sqrt{f(k)(k-k_1)}} \right]^2$$

$$< -\left[f(k)p(k) - \frac{(\Delta_\ell f(k))^2}{4f(k)(k-k_1)}\right],$$

which implies

$$\Delta_\ell w(k) < -\left[f(k)p(k) - \frac{(\Delta_\ell f(k))^2}{4f(k)(k-k_1)}\right]. \tag{13}$$

Summing (13) from k_2 to k, we obtain

$$-w(k_2+j) < w(k+\ell) - w(k_2+j)$$
$$< -\sum_{t=0}^{\frac{k-k_2-j}{\ell}} \left[f(k_2+j+t\ell)p(k_2+j+t\ell) - \frac{(\Delta_{\alpha(\ell)} f(k_2+j+t\ell))^2}{4f(k_2+j+t\ell)(k_2+j+t\ell-k_1)}\right],$$

which yields

$$\sum_{t=0}^{\frac{k-k_2-j}{\ell}} \left[f(k_2+j+t\ell)p(k_2+j+t\ell) - \frac{(\Delta_{\alpha(\ell)} f(k_2+j+t\ell))^2}{4f(k_2+j+t\ell)(k_2+j+t\ell-k_1)}\right] < c_1, \tag{14}$$

for all large k, and this is contrary to (5).

Next, we assume that the Case (ii) holds. Since $u(k)$ is positive and decreasing, it follows that $\lim_{n\to\infty} u(k) = b \geq 0$. Now we claim that $b = 0$. If not, then $u(k) \to b > 0$ as $k \to \infty$ and hence there exists $k_2 \geq k_1$ such that $u(k+\ell) \geq b$. Therefore from (1) we have

$$\Delta_\ell\Big(\Delta^2_{\alpha(\ell)} u(k)\Big) + p(k)b \leq 0, k \geq k_2. \tag{15}$$

Define the function $r(k) = \Delta^2_{\alpha(\ell)} u(k)$ for $k \geq k_2$. Then we have

$$\Delta_\ell r(k) \leq -p(k)b.$$

Summing the last inequality from k_2 to $k-\ell$, we have

$$r(k) \leq r(k_2+j) - b \sum_{t=0}^{\frac{k-\ell-k_2-j}{\ell}} p(k_2+j+t\ell). \tag{16}$$

From (5), by choosing $f(k) = 1$ we have $\sum_{t=0}^{\infty} p(k_0+j+t\ell) = \infty$, and then from (16) it is possible to choose an integer k_3 sufficiently large such that for all

$k \geq k_3$

$$r(k) \leq -\frac{b}{2} \sum_{t=0}^{\frac{k-\ell-k_2-j}{\ell}} p(k_2 + j + t\ell),$$

and hence

$$\Delta^2_{\alpha(\ell)} u(k) \leq -\frac{b}{2} \sum_{t=0}^{\frac{k-\ell-k_2-j}{\ell}} p(k_2 + j + t\ell).$$

Summing the last inequality from k_3 to $k - \ell$ we obtain

$$\Delta_{\alpha(\ell)} u(k) \leq \alpha^{\lceil \frac{k}{\ell} \rceil} \Delta_{\alpha(\ell)} u(k_3 + j) - \frac{b}{2} \sum_{s=0}^{\frac{k-\ell-k_3-j}{\ell}} \alpha^{-\lceil \frac{k_3+j+s\ell}{\ell} \rceil} \left(\sum_{t=0}^{\frac{s-\ell-k_2-j}{\ell}} p(k_2 + j + t\ell) \right).$$

Since $\Delta_{\alpha(\ell)} u(k) < 0$ for $k \geq k_0$, the last inequality implies that

$$\Delta_{\alpha(\ell)} u(k) \leq -\frac{b}{2} \sum_{s=0}^{\frac{k-\ell-k_3-j}{\ell}} \alpha^{-\lceil \frac{k_3+j+s\ell}{\ell} \rceil} \left(\sum_{t=0}^{\frac{s-\ell-k_2-j}{\ell}} p(k_2 + j + t\ell) \right).$$

Summing from k_3 to $k - \ell$, we have

$$u(k) \leq u(k_3 + j) - \frac{b}{2} \sum_{r=0}^{\frac{k-\ell-k_3-j}{\ell}} \alpha^{-\lceil \frac{k_3+j+r\ell}{\ell} \rceil} \left(\sum_{s=0}^{\frac{r-\ell-k_3-j}{\ell}} \alpha^{-\lceil \frac{k_3+j+s\ell}{\ell} \rceil} \sum_{t=0}^{\frac{s-\ell-k_2-j}{\ell}} p(k_2 + j + t\ell) \right).$$

Condition (1) implies that $u(k) \to -\infty$ as $k \to \infty$ which is a contradiction to the fact that $u(k)$ is positive. Hence $b = 0$ and this completes the proof. □

Example 2.2. *For the generalized mixed difference equation*

$$\Delta_\ell \Big(\Delta_{\alpha(\ell)} u(k) \Big) + 4(-\alpha) \frac{(k(\alpha+1) + \ell(2\alpha+1))}{k+\ell} u(k+\ell) = 0$$

and for $f = k^\lambda, \lambda \geq 1$, the conditions of Theorem 2.1 hold and hence every solution oscillates. In fact, $u(k) = (-\alpha)^{\lceil \frac{k}{\ell} \rceil} k$ is one such solution.

Theorem 2.3. *Assume that (1) holds. Let $f(k)$ be a positive function. Furthermore, we assume that there exists a function $h(k,m), k \geq m \geq 0$ such that:*

(i) $h(m,m) = 0$ for $m \geq 0$,

(ii) $h(k,m) > 0$ for $k > m \geq 0$,

(iii) $\Delta_{\alpha(\ell)} h(k,m) = h(k,m+\ell) - \alpha h(k,m) \leq 0$ for $k \geq m \geq 0$.

If

$$\limsup_{k\to\infty} \frac{1}{h(k,m_2)} \sum_{m=0}^{\frac{k-m_2-j}{\ell}} \Big[h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell) - \frac{(f(m_2+\ell+j+m\ell))^2}{4f(m_2+j+m\ell)(m_2+m\ell-m_1)}\Big(q(k,m_2+j+m\ell) - \frac{\Delta_{\alpha(\ell)} f(m_2+j+m\ell)}{f(m_2+j+\ell+m\ell)}\sqrt{h(k,m_2+j+m\ell)}\Big)^2\Big] = \infty \quad \text{for } k_2 > k_1, \tag{17}$$

where $q(k,m) = -\dfrac{\Delta_{\alpha(\ell)2} h(k,m)}{\sqrt{h(k,m)}}$, $k > m \geq 0$. Then, every solution of the equation (1) oscillates or $\lim\limits_{k\to\infty} u(k) = 0$.

Proof. Proceeding as in Theorem 2.1, we assume that (1) has a nonoscillatory solution, say $u(k) > 0$, for all $k \geq k_1$. From the proof of Theorem 2.1 there are two possible cases. First, we consider the case Case (i). Defining again $w(k)$ by (6), then from Theorem 2.1, we have $w(k) > 0$ and (12) holds. From (12) we have for $k \geq k_2$

$$f(k)p(k) \leq -\Delta_{\alpha(\ell)} w(k) + \frac{\Delta_{\alpha(\ell)} f(k)}{f(k+\ell)} w(k+\ell) - \frac{\bar{f}(k)}{(f(k+\ell))^2} w(k+\ell)^2. \tag{18}$$

Therefore, we have

$$\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell)$$

$$\leq -\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\Delta_{\alpha(\ell)} w(m_2+j+m\ell)$$

$$+\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}w(m_2+j+m\ell+\ell)$$
$$-\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\bar{f}(m_2+j+m\ell)}{(f(m_2+j+m\ell+\ell))^2}w(m_2+j+m\ell+\ell)^2, \quad (19)$$

which yields after summing by parts

$$\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell)$$
$$\leq h(k,m_2+j+m\ell)w(m_2+j)+\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} w(m_2+j+\ell+m\ell)\Delta_{\alpha(\ell)2}h(k,m_2+j+m\ell)$$
$$+\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}w(m_2+j+m\ell+\ell)$$
$$-\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\bar{f}(m_2+j+m\ell)}{(f(m_2+j+m\ell+\ell))^2}w(m_2+j+m\ell+\ell)^2.$$

Hence,

$$\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell)=h(k,m_2+j+m\ell)w(m_2+j)$$
$$-\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} q(k,m_2+j+m\ell)\sqrt{h(k,m_2+j+\ell+m\ell)}w(m_2+j+\ell+m\ell)$$
$$+\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}w(m_2+j+m\ell+\ell)$$
$$-\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}} h(k,m_2+j+m\ell)\frac{\bar{f}(m_2+j+m\ell)}{(f(m_2+j+m\ell+\ell))^2}w(m_2+j+m\ell+\ell)^2$$
$$=h(k,m_2+j+m\ell)w(m_2+j)$$
$$-\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}}\Bigg[\frac{\sqrt{h(k,m_2+j+\ell+m\ell)\bar{f}(m_2+j+m\ell)}}{f(m_2+j+m\ell+\ell)}w(m_2+j+m\ell)$$

$$+\frac{f(m_2+j+m\ell+\ell)}{2\sqrt{h(k,m_2+j+\ell+m\ell)\bar{f}(m_2+j+m\ell)}}\Big(q(k,m_2+j+m\ell)\sqrt{h(k,m_2+j+\ell+m\ell)}$$

$$-\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}h(k,m_2+j+m\ell)\Big)\Big]^2$$

$$+\frac{1}{4}\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}}\frac{(f(m_2+j+m\ell+\ell))^2}{\bar{f}(m_2+j+m\ell)}\Big(h(k,m_2+j+m\ell)$$

$$-\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}\sqrt{h(k,m_2+j+m\ell)}\Big)^2.$$

Then, $\displaystyle\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}}$ $[h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell)$

$$-\frac{(f(m_2+j+m\ell+\ell))^2}{4f(m_2+j+m\ell)}\Big(q(k,m_2+j+m\ell)-\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}\sqrt{h(k,m_2+j+m\ell)}\Big)^2\Big]$$

$$< h(k,m_2+j+m\ell)w(m_2+j).$$

which implies that

$$\lim_{k\to\infty}\sup\frac{1}{h(k,m_2)}\sum_{m=0}^{\frac{k-\ell-m_2-j}{\ell}}[h(k,m_2+j+m\ell)f(m_2+j+m\ell)p(m_2+j+m\ell)$$

$$-\frac{(f(m_2+j+m\ell+\ell))^2}{4\bar{f}(m_2+j+m\ell)}\Big(q(k,m_2+j+m\ell)-\frac{\Delta_{\alpha(\ell)}f(m_2+j+m\ell)}{f(m_2+j+m\ell+\ell)}\sqrt{h(k,m_2+j+m\ell)}\Big)^2\Big]$$

$$< w(m_2+j) < \infty,$$

which contradicts (17). If the Case (ii) holds, we are then back to the proof of the second case of Theorem 2.1 to prove that $\lim_{k\to\infty} u(k)=0$. The proof is now complete. □

Example 2.4. *For the generalized mixed difference equation*

$$\Delta_\ell\Big(\Delta_{\alpha(\ell)}u(k)\Big)+4(-\alpha)\frac{(k(\alpha+1)+\ell(2\alpha+1))}{k+\ell}u(k+\ell)=0$$

and for $f=(k-m)_\ell^\lambda, \lambda\geq 1, k\geq m\geq 0$, the conditions of Theorem 2.1 hold and hence every solution oscillates. Infact, $u(k)=(-\alpha)^{\lceil\frac{k}{\ell}\rceil}k$ is one such solution.

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