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SOLITARY WAVE SOLUTIONS FOR A CLASS OF DISPERSIVE EQUATIONS

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Abstract: The focus of the present work is the one-dimensional nonlinear equation

$$u_t - u_{xxt} + u_x + u_{xxx} + \alpha u u_x = \lambda(u u_{xxx} + 2u_x u_{xx}), \tag{1}$$

modeling the wave breaking phenomenon in the shallow water regime. When $\alpha, \lambda > 0$, using a variational approximation, we show that (1) admits solitary wave solutions which propagate in the negative $x-$direction.

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1. Introduction

The water waves problem has a long history which counts more than two hundred years. In the 1870's, J. Boussinesq derived some model equations to describe the propagation of small-amplitude, long wavelength, gravity waves on the surface of water in a canal. These equations possess special solutions called solitary wave solutions or solitons, and in one of his papers (Boussinesq, 1872) he also proposed what we would now call a Lyapunov function, which he argued

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was connected to the stability of these solitary waves. The Boussinesq's theory was the first scientific approximation of the phenomenon of solitary wave discovered by Scott-Russell and reported more than thirty years earlier, but a satisfactory theoretical confirmation had to wait until 1898 when two Dutch physicists, Diederik Johannes Korteweg and Gustav de Vries, presented their famous (KdV) equation. The original equation due to J. Boussinesq is not the only mathematical model for small-amplitude, planar, long waves on the surface of water. Different choices of the dependent variables, plus the possibility of modifying lower order terms by the use of the leading order relationships can lead to a whole range of equations, all of which have the same formal validity.

The models for dispersive and nonlinear long water waves with small amplitude in finite depth are derived from the full water waves problem through an approximation process, under the imposition of some restrictions on the parameters that affect the propagation of gravity water waves, as the nonlinearity (amplitude parameter) and the dispersion (long-wave parameter), and also by assuming that the free surface elevation, its derivatives, and the derivatives of the velocity potential are small quantities compared with the amplitude parameter and the long wave parameter. As it is well known, the study of water waves is reduced to determine the free surface elevation and the velocity potential on the free surface. So, if the vertical variable is eliminated from the equations by using Taylor expansion approximation about some hight with respect to the vertical spatial variable, then it is possible to obtain some water wave models in one or two spatial variables. A typical example is the Korteweg-de Vries (KdV) equation (see [5]) which models shallow water waves propagation:

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

In this equation, the steeping effect of the nonlinearity, represented by uu_x, and the effect of dispersion, represented by u_{xxx}, are in balance with each other. More recently, it has been noticed by Benjamin, Bona, Mahony (see [2]) that the (KdV) equation belongs to a wider class of equations which provide an approximation of the exact water wave equations of the same accuracy as the (KdV) equation:

$$u_t + u_x + u_{xxx} - u_{xxt} + uu_x = 0.$$

Since the (KdV) and (BBM) type equations does not model breaking waves, several model equations were proposed to capture this phenomenon. In particular we recall the Camassa-Holm (CH) equation (see [3]):

$$u_t + u_x - u_{xxt} + 3uu_x = uu_{xxx} + 2u_xu_{xx},$$

that arises as a model describing the evolution of the horizontal fluid velocity at a certain depth within the regime of shallow water waves. In terms of the two fundamental parameters μ (shallowness parameter) and ϵ (amplitude parameter), the shallow water regime of waves of small amplitude (proper to KdV and BBM) is characterized by $\mu \ll 1$ and $\epsilon = O(\mu)$, while the regime of shallow water waves of moderate amplitude (proper to CH) corresponds to

$$\mu \ll 1, \quad \epsilon = O(\sqrt{\mu}). \tag{2}$$

In a recent paper, A. Constantin and D. Lannes in [4] showed that the correct generalization of the (KdV), (BBM) and (CH) equations under the scaling (2) is provided by the class of equations (1) with $\alpha = \frac{3}{2}\epsilon$ and $\lambda = \epsilon\mu$. These equations capture stronger nonlinear effects than the classical nonlinear dispersive Benjamin-Bona-Mahony and Korteweg-de Vries equations. In particular, they accommodate wave breaking, a fundamental phenomenon in the theory of water waves.

For the nonlinear equations that model the evolution of water waves, it is very important to determine the existence and uniqueness of solution for the associated initial value problem, and the existence of special solutions as the solitary wave solutions. For instance, solitary waves are important in the study of dynamics of wave propagation in many applied models such as fluid dynamics, oceanography, and weather forecasting. An important application is the use of solitons (solitary waves of finite energy) as an efficient means of long-distance communication.

In this paper we investigate the existence of solitary wave solutions of (1) propagate in the direction of the $x-$axis. This is, solutions of the form

$$u(x,t) = v(x - ct),$$

where c denote the speed of the wave. By using the concentration-compactness principle, A. Montes in [6], showed the existence of solitary waves which propagate to the right at the positive velocity $c > 1$. In this paper, we establish the existence of solitary wave solutions with speed of wave $c < -1$, i.e., solitary waves which propagate to the left. Consequently, the existence of solitary waves with speed of wave $|c| > 1$ is obtained. We will get the result by using a variational approximation for which solitary waves corresponding a critical points of a suitable action functional.

The paper is organized as follows. In Section 2, we include some preliminaries. In Section 3, we find the natural finite-energy space for solitary wave solutions, and characterize solitary waves variationally as critical points of an action functional. Then we prove the existence of solitary waves for the equation (1) by using the mountain pass theorem. Throughout this work $L^p = L^p(\mathbb{R})$ denote the usual Lebesgue space; $H^1 = H^1(\mathbb{R})$ is the usual Sobolev space of order 1; $\|\cdot\|_X$ denotes norm in the Hilbert space X, $<,>_X$ is its inner product and X' represents the dual space.

2. Preliminaries

In this section we present some results that are used in this paper.

Theorem 1 (Mountain pass theorem). *Let X be a Hilbert space, $\varphi \in C^1(X,\mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\|_X > r$ and*

$$b = \inf_{\|u\|_X = r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

Then, given $n \in \mathbb{N}$, there is $u_n \in X$ such that

$$\varphi(u_n) \to d, \quad \text{and} \quad \varphi'(u_n) \to 0 \quad \text{in } X', \tag{PS}$$

where

$$d = \inf_{\pi\in\Pi} \max_{t\in[0,1]} \varphi(\pi(t)), \quad \Pi = \{\pi \in C([0,1], X) : \pi(0) = 0, \quad \pi(1) = e\}.$$

Proof. See [1] or [7]. □

Next, we establish an important result for our analysis, which is related with the characterization of "vanishing sequences" in $H^1(\mathbb{R})$. Hereafter, $B_r(\zeta)$ denotes the ball in $\mathbb{R}$ of center ζ and radius $r > 0$.

Lemma 2.1. *If $(v_n)_n$ is a bounded sequence in $H^1(\mathbb{R})$ and there is a positive constant $r > 0$ such that*

$$\lim_{n\to\infty} \sup_{\zeta\in\mathbb{R}} \|v_n\|_{L^\infty(B_r(\zeta))} = 0.$$

Then we have that

$$\lim_{n\to\infty} \int_{\mathbb{R}} v_n^3 \, dx = \lim_{n\to\infty} \int_{\mathbb{R}} v_n (v_n')^2 \, dx = 0. \tag{3}$$

Proof. Let $\zeta \in \mathbb{R}$ and $r > 0$. The Hölder inequality implies that

$$\int_{B_r(\zeta)} |v_n|^3 dx \leq \|v_n\|_{L^\infty(B_r(\zeta))} \int_{B_r(\zeta)} |v_n|^2 dx$$
$$\leq \|v_n\|_{L^\infty(B_r(\zeta))} \left(\int_{B_r(\zeta)} |v_n|^2 dx + \int_{B_r(\zeta)} |v_n'|^2 dx \right).$$

Now, covering $\mathbb{R}$ by balls of radius r in such a way that each point of $\mathbb{R}$ is contained in at most two balls, we find

$$\int_{\mathbb{R}} |v_n|^3 dx \leq 2 \sup_{\zeta \in \mathbb{R}} \|v_n\|_{L^\infty(B_r(\zeta))} \left(\int_{\mathbb{R}} |v_n|^2 dx + \int_{\mathbb{R}} |v_n'|^2 dx \right)$$
$$\leq 2 \sup_{\zeta \in \mathbb{R}} \|v_n\|_{L^\infty(B_r(\zeta))} \|v_n\|^2_{H^1(\mathbb{R})}.$$

Thus, under the assumptions of the lemma,

$$\lim_{n\to\infty} \int_{\mathbb{R}} |v_n|^3 \to 0.$$

In a similar fashion we obtain the other result. □

3. Existence of Solitary Waves

Remember that a solitary wave solution for the equation (1) is a solution of the form $u(x,t) = v(x - ct)$. Indeed, when this Ansatz is substituted into (1) we obtain the ordinary differential equation

$$(1-c)v' + (1+c)v''' + \alpha v v' - \lambda \left(v v''' + 2v'v'' \right) = 0,$$

which, upon integration, yields

$$(1-c)v + (1+c)v'' + \frac{\alpha}{2} v^2 - \frac{\lambda}{2} \left(\left(v'\right)^2 + 2vv'' \right) + C = 0, \tag{4}$$

where C is a constant of integration. Among all the solitary wave solutions of (1) we shall focus on solutions which have the additional property that the waves are localized and that v and its derivatives decay at infinity, that is,

$$v^{(n)}(y) \to 0 \quad \text{as} \quad |y| \to \infty, \quad n \in \mathbb{N}.$$

Under this decay assumption the constant of integration in (4) vanishes and then the solitary wave equation takes the form

$$(1-c)v + (1+c)v'' + \frac{\alpha}{2}v^2 - \frac{\lambda}{2}\left(\left(v'\right)^2 + 2vv''\right) = 0. \tag{5}$$

The existence result of solitary wave solutions is a consequence of a variational approach which uses a minimax type result, since solutions v of (5) are critical points of the functional J_c given by

$$J_c(v) = I_c(v) + G(v),$$

where the functionals I_c and G are defined on the space $H^1(\mathbb{R})$ by

$$\begin{aligned} I_c(v) &= \frac{1}{2}\int_{\mathbb{R}} \left((1-c)v^2 - (1+c)(v')^2\right) dx, \\ G(v) &= \frac{1}{2}\int_{\mathbb{R}} \left(\frac{\alpha}{3}v^3 + \lambda v(v')^2\right) dx. \end{aligned}$$

First we have that $I_c, G, J_c \in C^1(H^1(\mathbb{R}), \mathbb{R})$ and its derivatives in v in the direction of w are given by

$$\begin{aligned} \left\langle I_c'(v), w\right\rangle &= \int_{\mathbb{R}} \left((1-c)vw - (1+c)v'w'\right) dx \\ \left\langle G'(v), w\right\rangle &= \frac{1}{2}\int_{\mathbb{R}} \left(\alpha v^2 w + \lambda\left((v')^2 w + 2vv'w'\right)\right) dx. \end{aligned}$$

As a consequence of this, after integration by parts, we conclude that

$$J_c'(v) = (1-c)v + (1+c)v'' + \frac{\alpha}{2}v^2 - \frac{\lambda}{2}\left(\left(v'\right)^2 + 2vv''\right),$$

meaning that critical points of the functional J_c satisfy the solitary wave equation (5). In particular, we have that

$$\begin{aligned} \left\langle J_c'(v), v\right\rangle &= 2I_c(v) + 3G(v) \\ &= 2J_c(v) + G(v). \end{aligned} \tag{6}$$

Thus on any critical point $v \in H^1(\mathbb{R})$, we have that

$$\begin{aligned} J_c(v) &= \frac{1}{3}I_c(v), \\ J_c(v) &= -\frac{1}{2}G(v), \end{aligned}$$

$$I_c(v) = -\frac{3}{2}G(v).$$

We will say that weak solutions for (5) are critical points of the functional J_c. Next, it is easy to show the following results on properties of I_c and G.

Lemma 3.1. *The functional I_c is well-defined on $H^1(\mathbb{R})$. In addition; for $\alpha, \lambda > 0$ and $c < -1$, we have that $I_c(v) \geq 0$. Moreover, there are some positive constants $C_1(c) < C_2(c)$ such that*

$$C_1(c)\|v\|^2_{H^1(\mathbb{R})} \leq I_c(v) \leq C_2(c)\|v\|^2_{H^1(\mathbb{R})}. \tag{7}$$

Lemma 3.2. *The functional G is well-defined on $H^1(\mathbb{R})$. Moreover, there is a positive constant $C = C(\alpha, \lambda)$ such that*

$$|G(v)| \leq C\|v\|^3_{H^1}. \tag{8}$$

Proof. Since the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous we have that there is $C > 0$ such that

$$\int_{\mathbb{R}} |v||v'|^2 dx \leq \|v\|_{L^\infty}\|v'\|^2_{L^2} \leq C\|v\|^3_{H^1}.$$

In a similar fashion we see that

$$\int_{\mathbb{R}} |v|^3 dx \leq C\|v\|^3_{H^1}.$$

Then the result follows. □

Our approach to show the existence of a non trivial critical point for J_c is to use the mountain pass theorem without the Palais-Smale condition (see A. Ambrosetti *et al.* [1], M. Willem [7]). We will build a Palais-Smale sequence for J_c for a minimax value and use an embedding result to obtain a critical point for J_c as a weak limit of such Palais-Smale sequence. In fact, in the following theorem we want to verify the mountain pass theorem hypotheses given in Theorem 1 and to build a Palais-Smale sequence for J_c.

Theorem 2. *Let $\alpha, \lambda > 0$ and $c < -1$. Then*

1. *There exists $\rho > 0$ small enough such that*

$$b(c) := \inf_{\|z\|_{H^1(\mathbb{R})}=\rho} J_c(z) > 0.$$

2. *There is $e \in H^1(\mathbb{R})$ such that $\|e\|_{H^1} \geq \rho$ and $J_c(e) \leq 0$.*

3. If $d(c)$ is defined as

$$d(c) = \inf_{\pi\in\Pi} \max_{t\in[0,1]} J_c(\pi(t)),$$

$$\Pi = \{\pi \in C([0,1], H^1(\mathbb{R})) \ : \ \pi(0) = 0, \ \pi(1) = e\},$$

then $d(c) \geq b(c)$ and there is a sequence $(v_n)_n$ in $H^1(\mathbb{R})$ such that

$$J_c(v_n) \to d, \quad J_c'(v_n) \to \ 0 \ \textit{ in } \ \left(H^1(\mathbb{R})\right)'. \tag{9}$$

Proof. From inequalities (7)-(8), we have for any $v \in H^1(\mathbb{R})$ that

$$\begin{aligned} J_c(v) &\geq C_1(c)\|v\|_{H^1}^2 - C(\alpha,\lambda)\|v\|_{H^1}^3 \\ &\geq \left(C_1(c) - C(\alpha,\lambda)\|v\|_{H^1}\right)\|v\|_{H^1}^2. \end{aligned}$$

Then for $\rho > 0$ small enough such that

$$C_1 - \rho C > 0, \tag{10}$$

we conclude for $\|v\|_{H^1(\mathbb{R})} = \rho$ that

$$J_c(v) \geq (C_1 - \rho C)\,\rho^2 := \delta > 0.$$

In particular, we also have that

$$b(c) = \inf_{\|z\|_{H^1}=\rho} J_c(z) \geq \delta > 0. \tag{11}$$

Now, it is not hard to prove that there exist $v_0 \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} v_0^3 dx$ and $\int_{\mathbb{R}} v_0(v_0')^2 dx$ are negative quantities. For any $t \in \mathbb{R}$ we see that

$$J_c(tv_0) = t^2\left(I(v_0) + \frac{t}{2}\int_{\mathbb{R}}\left(\frac{\alpha}{3}v_0^3 + \lambda v_0(v_0')^2\right)dx\right).$$

Using the hypothesis we have that

$$\lim_{t\to\infty} J_c(tv_0) = -\infty,$$

because $0 \leq I(v_0) \leq C_2(c)\|v_0\|_{H^1}^2$. So that, there is $t_0 > 0$ such that $e = t_0 v_0 \in H^1(\mathbb{R})$ satisfies that

$$t_0\|v_0\|_{H^1} = \|e\|_{H^1} > \rho$$

and that $J_c(e) \leq J_c(0) = 0$. The third part follows by applying theorem 1. □

The following theorem is our main result.

Theorem 3. *For $\alpha, \lambda > 0$ and $c < -1$, the equation (1) admits nontrivial solitary wave solutions in $H^1(\mathbb{R})$.*

Proof. We will see that $d(c)$ is in fact a critical value of J_c. Let $(v_n)_n$ be the sequence in $H^1(\mathbb{R})$ given by previous lemma. First note from (11) that $d(c) \geq b(c) \geq \delta$. Using the definition of J_c and (6) we have that

$$I_c(v_n) = 3J_c(v_n) - \left\langle J_c'(v_n), v_n \right\rangle.$$

But from (7) we conclude for n large enough that

$$C_1(c)\|v_n\|_{H^1}^2 \leq I(v_n) \leq 3(d(c) + 1) + \|v_n\|_{H^1}.$$

Then we have shown that $(v_n)_n$ is a bounded sequence in $H^1(\mathbb{R})$. We claim that

$$\delta^* = \overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \|v_n\|_{L^\infty(B_1(\zeta))} > 0.$$

Suppose that

$$\overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \|v_n\|_{L^\infty(B_1(\zeta))} = 0.$$

Hence, from theorem 2.1 we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}} v_n^3\, dx = \lim_{n\to\infty} \int_{\mathbb{R}} v_n(v_n')^2\, dx = \lim_{n\to\infty} G(v_n) = 0.$$

Then, we have from (6) and (11) that

$$\begin{aligned} 0 < \delta \leq d(c) &= J_c(v_n) - \frac{1}{2}\left\langle J_c'(v_n), v_n \right\rangle + o(1) \\ &\leq -\frac{1}{2}G(v_n) + o(1) \\ &\leq o(1), \end{aligned}$$

but this is a contradiction. Thus, there is a subsequence of $(v_n)_n$, denoted the same, and a sequence $\zeta_n \in \mathbb{R}$ such that

$$\|v_n\|_{L^\infty(B_r(\zeta_n))} \geq \frac{\delta^*}{2}. \tag{12}$$

Now, we define the sequence $\tilde{v}_n(x) = v_n(x+\zeta_n)$. For this sequence we also have that

$$\|\tilde{v}_n\|_{H^1} = \|v_n\|_{H^1}, \quad J_c(\tilde{v}_n) \to d \quad \text{and} \quad J_c'(\tilde{v}_n) \to 0 \;\text{ in }\; \left(H^1(\mathbb{R})\right)'.$$

Then $(\tilde{v}_n)_n$ is a bounded sequence in $H^1(\mathbb{R})$. Thus, for some subsequence of $(\tilde{v}_n)_n$, denoted the same, and for some $v \in H^1(\mathbb{R})$ we have that

$$\tilde{v}_n \rightharpoonup v, \quad \text{as } n \to \infty \quad (\text{weakly in } H^1(\mathbb{R})).$$

Since the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty_{loc}(\mathbb{R})$ is locally compact we see that

$$\tilde{v}_n \to v \ \text{ in } L^\infty_{loc}(\mathbb{R}).$$

Note that $v \neq 0$ because using (12) we have that

$$\|v\|_{B_1(0)} = \lim_{n\to\infty} \|\tilde{v}_n\|_{B_1(0)} \geq \frac{\delta^*}{2}.$$

Moreover, if $W \in (C_0^\infty(\mathbb{R}))^2$, then for $K = supp\, W$ we have that

$$\begin{aligned}
\left\langle I'(v), W \right\rangle &= \int_{\mathbb{R}} \left((1-c)vW - (1+c)v'W'\right) dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}} \left((1-c)\tilde{v}_n W - (1+c)\left(\tilde{v}_n\right)' W'\right) dx \\
&= \lim_{n\to\infty} \left\langle I'(\tilde{v}_n), W \right\rangle .
\end{aligned}$$

Now (taking a subsequence, if necessary) noting that

$$(\tilde{v}_n)^2 \rightharpoonup v^2, \quad (\tilde{v}_n')^2 \rightharpoonup (v')^2, \quad \tilde{v}_n(\tilde{v}_n') \rightharpoonup vv' \quad \text{in} \quad L^2_{loc}(\mathbb{R}),$$

we have that

$$\int_K (\tilde{v}_n)^2 W\, dx \to \int_K v^2 W\, dx, \quad \int_K (\tilde{v}_n')^2 W\, dx \to \int_K (v')^2 W dx$$

and

$$\int_K \tilde{v}_n(\tilde{v}_n') W' dx \to \int_K vv'W'dx.$$

Then we conclude that

$$\left\langle G'(v), W \right\rangle = \lim_{n\to\infty} \left\langle G'(\tilde{v}_n), W \right\rangle,$$

and also that

$$\left\langle J_c'(v), W \right\rangle = \lim_{n\to\infty} \left\langle J_c'(\tilde{v}_n), W \right\rangle = 0.$$

If $W \in H^1(\mathbb{R})$, by using density, there is $W_k \in C_0^\infty(\mathbb{R})$ such that $W_k \to W$ in $H^1(\mathbb{R})$. Hence,

$$\left|\left\langle J_c'(v), W \right\rangle\right| \leq \left|\left\langle J_c'(v), W - W_k \right\rangle\right| + \left|\left\langle J_c'(v), W_k \right\rangle\right|$$

$$\leq \left\| J_c'(v) \right\|_{(H^1)'} \|W - W_k\|_{H^1} + \left| \left\langle J_c'(v), W_k \right\rangle \right| \to 0.$$

Thus, we have already established that $J_c'(v) = 0$. In other words, v is a nontrivial solution for problem (5). □

4. Conclusion

In this manuscript we prove the existence of solitary wave solutions for a one-dimensional nonlinear equation, which is a generalization of the so-called KdV, BBM and CH equations. The equation depends on two parameter $\alpha > 0$ and $\lambda > 0$. The solitary wave is searched under the form $u(t, x) = v(x - ct)$ where c is the speed of the wave. The case $c > 1$ is treated in [6] and the purpose of this manuscript is to investigate the case $c < -1$. The proof of the existence is based on the mountain pass theorem and uses classical variational methods.

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