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ANALYTICAL SOLUTIONS FOR VOLTERRA AND ABEL INTEGRAL EQUATIONS USING A GENERALIZED POWER SERIES METHOD

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Abstract: Even though they have a rather specialized structure, Volterra and Abel integral equations form an important class of integral equations in applications. This happens because completely independent problems lead to the solution of such equations. In this paper we consider the nonlinear Volterra integral equation of second kind and Abel integral equations of first kind. Authors have been proposed a new method for constructing solutions of Abel integral equations by a generalized power series.

AMS Subject Classification: 45D05
Key Words: generalized Abel integral equations, integral equation, power series

1. Introduction and Preliminary

The real world problems in scientific fields [1], [2] such as solid state physics, plasma physics, fluid mechanics, chemical kinetics and mathematical biology are nonlinear in general when formulated as partial differential equations or integral equations. In many situations of stochastic processes we are to deal

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with integral equations and some of them are non-linear, even some problems of stochastic processes can be formulated in terms of integral equations. One of the outstanding practical problems met in the solution of Volterra integral equations is the form of power series. Volterra integral equations has wide application in physics and other science. In this paper, we use the method of generalized power series, to solve linear Volterra integral equations and generalized Volterra integral equations of the first and second kind.This power series method are undetermined coefficients method, or a method based on the application of the Taylor series. The result obtained in the form of generalized power series solution further converted to the inversion formula of the integral equation. The main purpose of this paper is to introduce and study the relation between Volterra integral equations and generalized power series.

2. Nonlinear Volterra Integral Equations of Second Kind

Consider the nonlinear Volterra integral equation of second kind

$$v(x) = g(x) + \int_0^x \frac{K(x,t)}{(x-t)^\alpha} G\left[v(t)\right] dt, \quad x \in [0,1] \tag{1}$$

where $0 < \alpha < 1$, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued continuous functions, and $G(v(x,t)$ is a nonlinear function of $v(t)$such as $v^2(x)$, $\cos(x(x))$ and $e^{v(x)}$. The solution of this equation will be sought in the form of a sum of two generalized power series:

$$y(x) = x^{(1-\alpha)} \sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} b_n x^n \tag{2}$$

Suppose the solution of equation (1) be as

$$v(x) = a_0 x^{(1-\alpha)} + b_0, \tag{3}$$

where $b_0 = f(0) = v(0)$ and a_0, is a unknown parameter. By substituting Eq.(3) into Eq.(1) with simple calculations, we get

$$(a_0 - c_0) x^{1-\alpha} + Q(x^{1-\alpha}) = 0, \tag{4}$$

where $Q(x^{1-\alpha})$ is a polynomial of order greater than $1-\alpha$. By neglecting $Q(x^{1-\alpha})$, we have linear equation of a_0 in the form,

$$a_0 = c_0, \tag{5}$$

where the parameter a_0 of $x^{1-\alpha}$ in Eq. (3) is then obtained. In the next step, we assume that the solution of Eq.(1) is

$$v(x) = a_0 x^{(1-\alpha)} + a_1 x^{(2-\alpha)} x + b_0 + b_1 x, \tag{6}$$

where a_0 and b_0 both are known and a_1, b_1 is unknown parameter. By substituting Eq.(6) into Eq.(1), we get following two equation

$$(a_1 - c_1)x^{2-\alpha} + Q(x^{2-\alpha}) = 0, (b_1 - d_1)x^2 + P(x^2) = 0, \tag{7}$$

where $Q(x^{2-\alpha})$ is a polynomial of order greater than $2 - \alpha$ and $P(x^2)$ is a polynomial of order greater than two. By neglecting $Q(x^{2-\alpha})$ and $P(x^2)$, we have linear equations of a_1 and b_1 in the form,

$$a_1 = c_1, b_1 = d_1, \tag{8}$$

the unknown parameters a_1 of $x^{2-\alpha}$ and b_1 of x^2in Eq.(6) are then obtained. Having repeated the above procedure for n iterations, a power series of the following form is derived:

$$\begin{aligned} v(x) &= a_0 x^{1-\alpha} + a_1 x^{2-\alpha} + a_2 x^{3-\alpha} + ... \\ &... + a_n x^{n+1-\alpha} + b_0 + b_1 x + b_2 x^2 + ... + b_n x^n \end{aligned} \tag{9}$$

Eq.(9) is an approximation for the exact solution $v(x)$ of Eq.(1) in the interval [0, 1].

Theorem 1. *Let $v = v(x)$ be the exact solution of the following Volterra integral eqution*

$$v(x) = g(x) + \int_0^x \frac{K(x,t)}{(x-t)^\alpha} G\left[v(t)\right] dt, \quad x \in [0,1] \tag{10}$$

Then, the proposed method obtains the generalized Taylor expansion of $v = v(x)$ in the form 2.

If the exact solution to Eq.(10) be a polynomial, then the proposed method will obtain the real solution.

3. Abel Integral Equation of First Kind

Consider the generalized Abel integral equation of the first kind:

$$\int_a^x \frac{\varphi(t)dt}{(x-t)^{\alpha}} = f(x), \tag{11}$$

where $0 < \alpha < 1$ – arbitrary real constant, $f(x) : [a, b] \to R$ is given function. Let's assume that the function $f(x)$can be represented as follows:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ... \tag{12}$$

We shall seek for a solution equation in the form of the following generalized power series:

$$\varphi(t) = a_0(t-a)^{\gamma} + a_1(t-a)^{1+\gamma} + ... + a_n(t-a)^{n+\gamma} + ..., \tag{13}$$

where a_n – the unknown coefficients that must be determined. Substituting the power series (12), (13) into equation (11), we obtain:

$$\int_a^x \frac{\left(a_0(t-a)^{\gamma} + a_1(t-a)^{1+\gamma} + ... + a_n(t-a)^{n+\gamma} + ...\right) dt}{(x-t)^{\alpha}} =$$

$$= c_0 + c_1(x-a) + ... + c_n(x-a)^n + ...$$

$$\int_a^x \frac{\sum\limits_{m=0}^{+\infty} a_m(t-a)^{m+\gamma}}{(x-t)^{\alpha}} dt = \left| \begin{array}{l} t = x-(x-a)z \\ dt = -(x-a)dz \\ t-a = (x-a)(1-z) \\ x-t = (x-a)z \\ t = x \to z = 0 \\ t = a \to z = 1 \end{array} \right| =$$

$$= \sum_{m=0}^{+\infty} a_m(x-a)^{m+\gamma-\alpha+1} \int_0^1 (1-z)^{m+\gamma} z^{-\alpha} dz = \tag{14}$$

$$= \sum_{m=0}^{+\infty} a_m(x-a)^{m+\gamma-\alpha+1} B(-\alpha+1, m+\gamma+1) = \sum_{m=0}^{+\infty} c_m\,(x-a)^m.$$

Here $\mathrm{B}(a, b)$–Euler beta function, $\Gamma(a)$ – Euler gamma function. Let $\gamma = \alpha - 1$, then, equating the terms with the same power of x in (14) yields:

$$a_m B(-\alpha+1, m+\alpha) = c_m \Rightarrow a_m = \frac{c_m}{B(-\alpha+1, m+\alpha)}.$$

If we substitute obtained coefficients in (13), then by a simple calculations we obtain:

$$\varphi(x)=\sum_{k=0}^{+\infty}\frac{c_k(x-a)^{k+\alpha-1}}{B(-\alpha+1,k+\alpha)}=\sum_{k=0}^{+\infty}\frac{\Gamma(k+1)}{\Gamma(1-\alpha)\Gamma(k+\alpha)}c_k(x-a)^{k+\alpha-1}=$$

$$=\sum_{k=0}^{+\infty}\frac{k!}{\Gamma(1-\alpha)\Gamma(\alpha)(\alpha)_k}c_k(x-a)^{k+\alpha-1}=\frac{\sin\alpha\pi}{\pi}\sum_{k=0}^{+\infty}\frac{k!c_k(x-a)^{k+\alpha-1}}{(\alpha)_k}=$$

$$=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\sum_{k=0}^{+\infty}\frac{k!c_k(x-a)^{k+\alpha}}{(\alpha)_{k+1}}$$

$$=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\sum_{k=0}^{+\infty}c_k(x-a)^{k+\alpha}\frac{\Gamma(\alpha)\Gamma(k+1)}{\Gamma(\alpha+k+1)}=$$

$$=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\left[\sum_{k=0}^{+\infty}A_k(x-a)^{k+\alpha}\int_0^1(1-z)^m z^{\alpha-1}dz\right]=\left|\begin{array}{l} z=\frac{x-t}{x-a}\\ 0\to x,\\ 1\to a.\end{array}\right|=$$

$$=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x\frac{(c_0+c_1(t-a)+...+c_n(t-a)^n+...)\,dt}{(x-t)^{1-\alpha}}$$

$$=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x\frac{f(t)dt}{(x-t)^{1-\alpha}}$$

We are thus led to the solution:

$$\varphi(x)=\frac{\sin\alpha\pi}{\pi}\frac{\partial}{\partial x}\int_a^x\frac{f(t)dt}{(x-t)^{1-\alpha}}. \tag{15}$$

This solution identical to the solution obtained in [4].

Theorem 2. *If $f(t)\in C[a,b]$, then there exists a unique solution of equation Abel of first kind, which is expressed in the form* (15).

The theorem is proved by direct verification that the formula (15) is a solution of equation (11).

4. Conclusions

In this paper, solution is obtained by power series method. This may be used in more combinatorial way to obtain solution of higher degree non-linear integral equations.

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