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A PUT OPTION'S VALUE FOR A NONLINEAR BLACK-SCHOLES EQUATION

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Abstract: We study a nonlinear Black-Scholes partial differential equation for modelling illiquid markets with feedback effects. After reducing the equation into a second-order nonlinear partial differential equation, we find that the assumption of a traveling wave profile to the second-order equation reduces it further to ordinary differential equations. Solutions to all these transformed equations facilitate an analytic solution to the nonlinear Black-Scholes equation. Use of the put-call parity gave rise to the put option's current value. These solutions can be used for pricing a European *call* and *put* options respectively at $t \geq 0$ and when $c \neq r$.

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1. Introduction

Two primary assumptions are used in formulating classical arbitrage pricing theory: *frictionless* and *competitive* markets. There are no transaction costs and restrictions on trade in a frictionless market. In a competitive market, any quantity of a security can be traded without changing its price.

The notion of liquidity risk is introduced on relaxing the assumptions above.

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The purpose of this paper is to obtain analytic solutions of the nonlinear Black-Scholes equation arising from transaction costs in the absence of a price slippage impact by Bakstein and Howison in [1]. This is done by substitutions and transformations, which give a second-order nonlinear partial differential equation (PDE). Assuming a traveling wave solution to the second-order PDE reduces it further to ordinary differential equations (ODEs). All these transformed equations are solved to obtain an analytic solution to the nonlinear Black-Scholes PDE which is used for pricing a *call* option at $t \geq 0$. Use of the put-call parity yielded the *put* option's value. The solutions can be used for pricing these options when $c \neq r$

This paper is organized as follows. Section 2 describes the nonlinear Black-Scholes PDE used for modelling illiquid markets with a price slippage impact. Smooth solutions to the nonlinear Black-Scholes equation without price slippage are presented in Section 3. Section 4 concludes the paper.

2. Bakstein and Howison (2003) Equation

In this section, we will consider the continuous-time feedback effects equation for illiquid markets by Bakstein and Howison in [1]. Two assets are used in the model: a bond (or a risk-free money market account with spot rate of interest $r \geq 0$), and a stock. The stock is assumed to be risky and illiquid while the bond is assumed to be riskless and liquid. This equation (see Theorem 3.1 of [1]) is given by

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + \tfrac{1}{2}\rho^2(1-\alpha)^2\sigma^2 S^4 u_{SS}^3 + rSu_S - ru = 0, \quad (2.1)$$

where t is current time, S is the price of the stock, $\rho \geq 0$ is a measure of the liquidity of the market, σ is volatility, $u(S,t)$ is the option price and α is a measure of the *price slippage impact* of a trade felt by all participants of a market (see Theorem 3.1 of [1]).

The terminal condition for a European call option is given by

$$h(S_T) = u(S,T) = \max\{S - K, 0\} \quad \text{for} \quad S \geq 0,$$

where T is expiry time, $K > 0$ is the striking price and $h(S_T)$ is a terminal claim whose hedge cost, $u(S_t, t)$, is the solution to (2.1). The boundary conditions for the option are as follows:

$$u(0,t) = 0 \quad \text{for} \quad 0 \leq t \leq T,$$

$$u(S,t) \sim S - Ke^{-r(T-t)} \quad \text{as} \quad S \to \infty.$$

We take the last condition to mean that

$$\lim_{S\to\infty} \frac{u(S,t)}{S-Ke^{-r(T-t)}} = 1$$

uniformly for $0 \le t \le T$ with the constraint $u(S,t) \ge 0$.

The payoff profile for the put option is given by

$$h(S_T) = u^P(S,T) = \max\{K - S, 0\} \quad \text{for} \quad S \ge 0.$$

Liquidity in (2.1) has been defined through a combination of transaction cost and a price slippage impact. Due to ρ, bid-ask spreads dominate the price elasticity effect. When $\alpha = 1$, this corresponds to no slippage and (2.1) reduces to

$$u_t + \tfrac{1}{2}\sigma^2 S^2 u_{SS}(1 + 2\rho S u_{SS}) + rSu_S - ru = 0. \tag{2.2}$$

The solution to (2.2) is found in Theorem 3.0.2 of [3] for $r > 0$, and in Theorem 3.2 of [4] and Theorem 4.1 of [5] for $r = 0$. However, the expiry time, T, and the transformation $x = \ln\left(\frac{K}{S}\right)$ were not considered in all these solutions and this can hinder the study of put-call-parity as it relies on time to expiry $T - t$.

The magnitude of the market impact is determined by ρS. Large ρ implies a big market impact of hedging. If $\rho = 0$, the asset's price in (2.1) follows the standard Black-Scholes model in [2] with constant volatility σ.

In the remainder of the work, we will ignore price slippage.

3. Smooth Solution to the Bakstein and Howison (2003) Equation

3.1. Call Option's Current Value

Lemma 3.1. *If $\nu(\xi)$ is a twice continuously differentiable function, and x and τ are the spatial and time variables respectively, then there exists a traveling wave solution to the equation,*

$$V_\tau - \tfrac{\sigma^2}{2}(VV_x - \tfrac{1}{2}V^2)_x + rV_x = 0 \tag{3.1}$$

in $\mathbb{R} \times [0,\infty)$ of the form

$$V(x,\tau) = \nu(\xi) \quad \text{where} \quad \xi = x - c\tau, \quad \xi \in \mathbb{R} \tag{3.2}$$

for $r, \sigma > 0$, $\tau \ge 0$ and $x \in \mathbb{R}$ such that $V(x,\tau)$ is a traveling wave of permanent form which translates to the right with constant speed $c > 0$.

Proof. Applying the chain rule to (3.2) gives

$$V_\tau = -c\nu'(\xi), \quad V_x = \nu'(\xi), \quad \text{and} \quad V_{xx} = \nu''(\xi),$$

where the prime denotes $\frac{d}{d\xi}$. Substituting these expressions into (3.1) and rearranging, we conclude that $\nu(\xi)$ must satisfy the nonlinear second order ODE

$$c\nu' + \frac{\sigma^2}{2}\Big(\nu\nu'' + \big(\nu'\big)^2 - \nu\nu'\Big) - r\nu' = 0 \tag{3.3}$$

in $\mathbb{R}$ and hence $V(x,\tau)$ solves (3.1).

Equation (3.3) can now be solved in a closed-form by first writing it as

$$\frac{d\Big(c\nu + \frac{\sigma^2}{2}(\nu\nu' - \frac{1}{2}\nu^2) - r\nu\Big)}{d\xi} = 0 \quad \text{in} \quad \mathbb{R}.$$

The equation above is solved to get

$$c\nu + \frac{\sigma^2}{2}\big(\nu\nu' - \tfrac{1}{2}\nu^2\big) - r\nu = \kappa_1 \quad \text{in} \quad \mathbb{R},$$

where κ_1 is a constant of integration.

Applying localizing boundary conditions reduces the equation above to the variable separable standard form (see [6])

$$\nu' = \tfrac{1}{2}\nu + \tfrac{2}{\sigma^2}(r-c),$$

since $\kappa_1 = 0$ from these boundary conditions. This is the first order linear autonomous and separable ODE whose solution upon integration is given by

$$\nu(\xi) = \frac{4}{\sigma^2}(r-c)\left(e^{\frac{\xi}{2}} - 1\right)$$

for $\xi \in \mathbb{R}$, $r, c, \sigma > 0$. Substituting into (3.2) gives

$$V(x,\tau) = \frac{4}{\sigma^2}(r-c)\left(e^{\frac{1}{2}(x-c\tau)} - 1\right), \tag{3.4}$$

for $x \in \mathbb{R}$, $r, c, \sigma > 0$, and $\tau \geq 0$. □

Theorem 3.2. *If $V(x,\tau)$ is any positive solution to the nonlinear equation*

$$V_\tau - \tfrac{\sigma^2}{2}(VV_x - \tfrac{1}{2}V^2)_x + rV_x = 0 \tag{3.5}$$

in $\mathbb{R} \times [0,\infty)$ then

$$u^C = \tfrac{1}{\rho}\Big[-\tfrac{4}{\sigma^2}(r-c)\sqrt{KS}e^{-\frac{c}{2}(T-t)} + \big(\tfrac{r-c}{\sigma^2} + \tfrac{1}{4}\big)(1 - ln\ S)S \tag{3.6}$$

$$+\left(\tfrac{\sigma^2+4r}{16}+\tfrac{c(r-c)}{\sigma^2}\right)St+\tfrac{r-c}{\sigma^2}K\left(e^{-c(T-t)}-(r-c)e^{-r(T-t)}\right)\Big]$$

in $\mathbb{R}_+\times[0,\infty)$ *solves the nonlinear Black-Scholes equation*

$$u_t+\tfrac{1}{2}\sigma^2S^2u_{SS}(1+2\rho Su_{SS})+rSu_S-ru=0 \tag{3.7}$$

for $c,r,K,\sigma,\rho>0$, $T\geq 0$.

Proof. To solve (3.7), differentiate it twice with respect to S. Then set $w=u_{SS}$ to get

$$\begin{aligned}w_t+\tfrac{\sigma^2S^2}{2}(1+4\rho Sw)w_{SS}+2\rho\sigma^2S^3w_S^2+2\sigma^2S(1+6\rho Sw)w_S\\ +rSw_S+\sigma^2(1+6\rho Sw)w+rw=0.\end{aligned} \tag{3.8}$$

The transformations $w=\frac{v}{\rho S}$, $x=\ln\left(\frac{K}{S}\right)$ and $\tau=T-t$ yield

$$w_t=-\tfrac{v_\tau}{\rho S},\quad w_S=-\tfrac{(v_x+v)}{\rho S^2},\quad w_{SS}=\tfrac{v_{xx}+3v_x+2v}{\rho S^3}.$$

The justification of the transformation $x=\ln\left(\frac{K}{S}\right)$ is that since this can be re-written as $S=Ke^{-x}$, it brings the notion of discounting.

Substituting these expressions into (3.8) and simplifying gives

$$v_\tau-\tfrac{\sigma^2}{2}(1+4v)v_{xx}-2\sigma^2v_x^2+\tfrac{\sigma^2}{2}(1+4v)v_x+rv_x=0. \tag{3.9}$$

If we let $v=\frac{V-1}{4}$, equation (3.9) reduces to (3.5). Hence, applying the transformation above into (3.4) gives

$$v(x,\tau)=\tfrac{r-c}{\sigma^2}\left(e^{\frac{1}{2}(x-c\tau)}-1\right)-\tfrac{1}{4}\quad\text{in}\quad\mathbb{R}\times[0,\infty)$$

for $r,c,\sigma>0$ and thus

$$u_{SS}=\tfrac{1}{\rho S}\left(\tfrac{r-c}{\sigma^2}\left(\sqrt{\tfrac{K}{S}}e^{-\frac{c}{2}(T-t)}-1\right)-\tfrac{1}{4}\right)$$

for $\rho,r,c,\sigma,S,K>0,\quad T,t\geq 0$. Integrating the equation above twice with respect to S yields (3.6) after a little algebra. □

3.2. Put-Call Parity

The put-call parity relationship is given by

$$u^P = u^C - S + Ke^{-r(T-t)}, \tag{3.10}$$

where $u^P(S,t)$ is the put option's current value.

Corollary 1. *Substituting (3.6) into (3.10) and rearranging gives the put option's value as*

$$\begin{aligned} u^P = {} & \frac{1}{\rho}\Big[-\frac{4}{\sigma^2}(r-c)\sqrt{KS}e^{-\frac{c}{2}(T-t)} + \left(\frac{r-c}{\sigma^2}+\frac{1}{4}\right)(1-\ln S)S \\ & + \left(\frac{\sigma^2+4r}{16}+\frac{c(r-c)}{\sigma^2}\right)St + \frac{r-c}{\sigma^2}Ke^{-c(T-t)}\Big] - S \\ & + Ke^{-r(T-t)}\left(1-\frac{(r-c)^2}{\rho\sigma^2}\right) \end{aligned}$$

in $\mathbb{R}_+ \times [0,\infty)$ *for* $c, r, K, \sigma, \rho > 0$, $T \geq 0$.

4. Conclusion

We have studied the hedging of derivatives in the presence of feedback effects but with no price slippage impact. Assuming the solution of a forward wave, classical solutions for a nonlinear Black-Scholes equation were found. The solutions obtained can be used for pricing European *call* and *put* options at time $t \geq 0$ and when $c \neq r$. The put-call parity was used to arrive at the put option's value.

In conclusion, future work will involve analysis of straddles and strangles. Another part of our work will be to use put-call parity to study the exposure from writing a covered call and the exposure from writing a naked put. We will also consider solving the Bakstein and Howison (2003) equation in the presence of slippage using the transformations used in this work.

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