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GROUP THEORETICAL JUSTIFICATION OF A "SIMPLE" COSMOLOGICAL MODEL

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Abstract: We have found a group theoretical justification of a simple cosmological model that describes a spherically symmetrical expansion of the Universe in the case of equality of the initial and critical densities of matter.

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1. Introduction

Many exact analytical solutions of linear and nonlinear equations in mathemat-

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ical physics are invariant under certain groups of transformations of dependent and independent variables allowed by the mathematical model [1, 2]. Knowledge of the full symmetry group of the mathematical model (this group is algorithmically computed [1, 2]) allows us to formulate a program for SUBMODELS, with the aim of obtaining a "full set" of invariant and partially invariant solutions of the original "big model" [3]. Among invariant solutions, "simple" solutions play a crucial role. To find "simple" solutions, you need to solve algebraic equations instead of differential ones [4]. In [5], the program SUBMODELS was formulated to model Newtonian cosmology. In the study, we consider a "simple" solution for a spherically symmetrical submodel within the domain of Newtonian cosmology, which would be of interest to astrophysics.

2. Model and it's "simple" invariant solution

The system of differential equations describing the model of spherically symmetrical motion of an ideal nonrelativistic self-gravitating fluid with zero pressure is as follows:

$$\begin{aligned} &U_t + UU_r + \Phi_r = 0, \\ &\rho_t + U\rho_r + \rho\left(\frac{2U}{r} + U_r\right) = 0, \\ &\frac{2\Phi_r}{r} + \Phi_{rr} = \rho, \end{aligned} \tag{1}$$

where Φ is the gravitational potential, ρ is the density, U is the radial velocity, r is the polar radius, t is time. All variables are dimensionless [5].

This system admits an infinite-dimensional Lie algebra with a three-dimensional subalgebra L_3 with basic generators

$$\begin{aligned} &\hat{X}_1 = 2\Phi\partial_\Phi + U\partial_U + r\partial_r, \\ &\hat{X}_2 = -2\rho\partial_\rho + r\partial_r + t\partial_t, \\ &\hat{X}_3 = \partial_t, \end{aligned} \tag{2}$$

and an infinite-dimensional subalgebra with generator $\hat{X}_\infty = F_1(t)\partial_\Phi$, where $F_1(t)$ is an arbitrary function of time t. To obtain simple solutions to system (1), it is necessary to know the optimal system of two-dimensional subalgebras of the algebra L_3. In [5], it was established that the optimal system consists of subalgebras $\langle X_1, X_2\rangle$, $\langle X_1, X_3\rangle$, $\langle aX_1 + X_2, X_3\rangle$, $a \geq 0$.

Consider the subalgebra $\langle X_1, X_2\rangle$; invariants are to be found from the conditions $X_1\psi = 0, X_2\psi = 0$ [1]. In other words, the invariant of this two-dimensional subalgebra is the function that satisfies the overdetermined system of equations:

$$\begin{aligned} 2\Phi\psi_\Phi + U\psi_U + r\psi_r &= 0, \\ -2\rho\psi_\rho + r\psi_r + t\psi_t &= 0. \end{aligned} \tag{3}$$

The system of characteristic equations for the first equation of the system (2) has the form

$$\frac{d\Phi}{2\Phi} = \frac{dU}{U} = \frac{dr}{r}.$$

Hence, we find the basic invariants $J_1 = \Phi/r^2$ and $J_2 = U/r$, t, ρ. Consequently, $\psi = \psi(J_1, J_2, t, \rho)$. Substituting ψ in the second equation of the system (3) gives

$$2\rho\psi_\rho + 2J_1\psi_{J_1} + J_2\psi_{J_2} - t\psi_t = 0. \tag{4}$$

The system of characteristic equations for equation (4) has the form

$$\frac{d\rho}{2\rho} = \frac{dJ_1}{2J_1} = \frac{dJ_2}{J_2} = \frac{dt}{-t}.$$

Hence, we find the following list of basic invariants of subalgebra $\langle X_1, X_2\rangle$: ρt^2, $J_1 t^2$, $J_2 t$. Finally, we have

$$\begin{aligned} \rho &= \frac{C_1}{t^2}, \\ \Phi &= C_2\left(\frac{r}{t}\right)^2, \\ U &= C_3\frac{r}{t}, \end{aligned} \tag{5}$$

where C_1, C_2, and C_3 are undefined constants. Substituting (5) in (1), we obtain the system for determining the constants:

$$\begin{aligned} &-C_3 + C_3^2 + 2C_2 = 0, \\ &C_1(-2 + 3C_3) = 0, \\ &C_1 = 6C_2. \end{aligned}$$

This system has three solutions:

(a) $C_1 = C_2 = C_3 = 0$;

(b) $C_1 = C_2 = 0$, $C_3 = 1$;

(c) $C_1 = 2/3 = C_3$, $C_2 = 1/9$.

Case (a) leads to the trivial solution $U = 0$, $\rho = 0$, $\Phi = 0$.
In case (b), we obtain a solution with no physical meaning.
In case (c), we obtain a nontrivial solution:

$$\rho = \frac{2}{3}\frac{1}{t^2},$$
$$U = \frac{2}{3}\frac{r}{t},$$
$$\Phi = \frac{1}{9}\left(\frac{r}{t}\right)^2.$$

This solution describes (in terms of Newtonian approximation) the spherically symmetrical expansion of the Universe in the case of equality of the initial and critical densities of matter [6]. The group theoretical nature of this solution has not been previously addressed in physical literature. The study also shows that the solutions, which are invariant under the subalgebras $\langle X_1, X_3\rangle$ and $\langle aX_1 + X_2, X_3\rangle$, are either trivial solutions $U = 0$, $\rho = 0$, $\Phi = 0$ or reduced to trivial solutions by an appropriate choice of gravitational potential calibrations.

3. Conclusion

In the framework of the SUBMODELS program, which was originally formulated for modeling Newtonian cosmology, we found “simple” solutions of spherically symmetrical submodels. In this assessment, we demonstrated that a unique nontrivial simple invariant solution coincides with the solution itself, which is derived from physical plane considerations. The search for simple solutions of the original “big model” is a subject of future research.

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