On Diophantine equations $X^6 + 6Z^3 = Y^6 \pm 6W^3$

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Abstract: We show that the Diophantine equations $X^6 + 6Z^3 = Y^6 \pm 6W^3$ has infinite non-trivial primitive integer solutions by rationally transferring these equation to elliptic curves with Mordel-Weil rank 1.

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1. Introduction

Symmetric Diaphantine equations has been studied by many authors. In [2] Choudhry uses certain properties of rational binary forms to solve several Diophantine equations of the type $f(x, y) = f(u, v)$. One of the symmetric Diaphantine equations is sums of like powers

$$\sum_{i=1}^m a_i^k = \sum_{j=1}^m b_j^k,$$

which has been studied in [5, 7] by computer search. One special case is the equal sums of sixth powers [1, 4]. Following this special type, S. K. Jena [6] solved parametrically the equations $2A^6 + B^6 = 2C^6 \pm D^3$.

In this article we prove the following main result.

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Theorem 1. *The Diophantine equations*

$$X^6 + 6Z^3 = Y^6 \pm 6W^3 \quad (1)$$

has infinite non-trivial primitive integer solutions.

To do this, using some rational transformations we first change these equations to some elliptic curves. Then show that these elliptic curves have positive ranks.

2. Preliminaries

Consider the equations of the form

$$v^2 = au^4 + bu^3 + cu^2 + du + e, \quad a \neq 0, \quad (2)$$

defined over the field K , i.e., $a, b, c, d, e, f \in K$. There are curves of the form (2) that do not have points with coordinates in K . Suppose we have a curve defined by an equation (2) and suppose we have a point (p, q) lying on the curve. By changing u to $u + p$, we may assume $p = 0$, so the point has the form $(0, q)$. First, suppose $q = 0$. If $d = 0$, then the curve has a singularity at $(u, v) = (0, 0)$. Therefore, assume $d \neq 0$. Then

$$\left(\frac{v}{u^2}\right)^2 = d\left(\frac{1}{u}\right)^3 + c\left(\frac{1}{u}\right)^2 + b\left(\frac{1}{u}\right) + a.$$

This can be easily transformed into a Weierstrass equation in d/u and dv/u^2 . The harder case is when $q \neq 0$. We have the following result [8].

Theorem 2. *Let K be a field of characteristic not 2. Consider the quartic equation*

$$v^2 = au^4 + bu^3 + cu^2 + du + q^2 \quad (3)$$

with $a, b, c, d, q \in K$. Let

$$x = \frac{2q(v + q) + du}{u^2}, \quad y = \frac{4q^2(v + q) + 2q(du + cu^2) - (d^2u^2/2q)}{u^3}.$$

Define

$$a_1 = d/q, \quad a_2 = c - (d^2/4q^2), \quad a_3 = 2qb, \quad a_4 = -4q^2a, \quad a_6 = a_2a_4.$$

Then

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The inverse transformation is

$$u = \frac{2q(x+c) - (d^2/2q)}{y}, \quad v = -q + \frac{u(ux-d)}{2q}. \quad (4)$$

The point $(u, v) = (0, q)$ corresponds to the point $(x, y) = \infty$ and $(u, v) = (0, -q)$ corresponds to $(x, y) = (-a_2, a_1a_2 - a_3)$.

3. Proof of the Main Result

Proof of the main theorem 1. Through this proof we assume the primitive case $X \neq Y$ and $Z \neq W$. Consider first the positive-sign equation

$$X^6 + 6Z^3 = Y^6 + 6W^3. \quad (5)$$

Take the rational transformation

$$X = -u + h, \quad Y = u + h, \quad Z = u + v/3, \quad W = -u + v/3. \quad (6)$$

Assuming the non-trivial case $u \neq 0$, the equation (5) turns to

$$v^2 = 3hu^4 + (10h^3 - 3)u^2 + 3h^5.$$

For different values h we have different quartic equations of the form (3). However, we should take such an h that leads to an elliptic curve with positive rank. For example for $h = 3$ we get

$$v^2 = 9u^4 + 267u^2 + 729.$$

Now use Theorem 2 for $a = 9$, $b = 0$, $c = 267$, $d = 0$, and $q = 27$. With the inverse transformation

$$u = \frac{54(x+267)}{y}, \quad v = -27 + \frac{u^2x}{54}, \quad (7)$$

we get the elliptic curve

$$E_1 : y^2 = x^3 + 267x^2 - 26244x - 7007148.$$

An easy calculation with Cremona's MWrank program [3] shows that E_1 has Mordel-Weil rank 1 with the generator $[201 : 2574 : 1]$.

Note that each point on the elliptic curve can be represented in the form $(r/s^2, t/s^3)$. Applying the transformations (7) for this point now and then reversing the equations (6) gives the solutions

$$\begin{aligned} X &= -54s(rs + 267s^3) + 3ts, \\ Y &= 54s(rs + 267s^3) + 3ts, \\ Z &= 54ts^2(rs + 267s^3) - 9t^2s^2 + 18(rs + 267s^3)^2r, \\ W &= -54ts^2(rs + 267s^3) - 9t^2s^2 + 18(rs + 267s^3)^2r. \end{aligned}$$

Now consider the minus-sign equation

$$X^6 + 6Z^3 = Y^6 - 6W^3. \quad (8)$$

In this case, the rational transformation

$$X = -u + 2h, \quad Y = u + 2h, \quad Z = 8u + v/6, \quad W = 8u - v/6, \quad (9)$$

in the non-trivial case $u \neq 0$, turns (8) to

$$v^2 = 3hu^4 + (40h^3 - 768)u^2 + 48h^5.$$

Here, again, for each value h we get a various quartic equation of the form (3) but, as above, we should take h in such a way that leads to an elliptic curve with positive rank. Taking, say, $h = 3$ we get

$$v^2 = 9u^4 + 312u^2 + 11664.$$

Using Theorem 2 now for $a = 9$, $b = 0$, $c = 312$, $d = 0$, and $q = 108$ with the inverse transformation

$$u = \frac{216(x + 312)}{y}, \quad v = -108 + \frac{u^2x}{216}, \quad (10)$$

gives the elliptic curve

$$E_2: \quad y^2 = x^3 + 312x^2 - 219904x - 131010048.$$

Similarly, calculation with MWrank gives the same rank 1 for E_2 and, this time, with the generator $[-632 : 2560 : 1]$.

Finally, applying the transformations (10) for the point $(r/s^2, t/s^3)$ and reversing the equations (9) gives this time the solutions

$$X = -216s^2(r + 312s^2) + 6ts,$$

$$\begin{aligned}
Y &= 216s^2(r + 312s^2) + 6ts, \\
Z &= 1728ts^3(r + 312s^2) - 18t^2s^2 + 36(r + 312s^2)^2s^2r, \\
W &= 1728ts^3(r + 312s^3) - 9t^2s^2 + 36(r + 267s^2)^2s^2r.
\end{aligned}$$

Of course we used the fact that if (X, Y, Z, W) is a solution for either of (1) then, for any $\mu \neq 0$, $(\mu X, \mu Y, \mu^2 Z, \mu^2 W)$ is so. This fact also shows that the equations (1) has infinite integer solutions. This completes the proof of Theorem 1. $\square$

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