ON THE DIOPHANTINE EQUATION $p^a + (p + 1)^b = z^2$

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Abstract: In this paper, we shall prove some results related to the equation $p^a + (p + 1)^b = z^2$. In particular, we shall prove that there is no solution when $p > 3$, $b \geq 2$ and z even.

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1. Introduction

Recently, many papers have been devoted to the study of the Diophantine equation $x^a + y^b = z^2$ where x, y are positive integers and a, b, z are non-negative integers. In a series of papers (see [7, 8, 10, 11, 12, 9, 13, 14, 15, 16]) Sroysang solved this equation for some pairs (x, y) of integers (e.g., $(3, 5)$, $(3, 17)$, $(3, 85)$, $(3, 45)$, $(143, 145)$). He also solved the equation for the pairs $(7, 8)$ and $(31, 32)$ (see [5, 6]) and posed a conjecture about the equation $p^a + (p + 1)^b = z^2$. In 2013, Chotchaisthit [1] found all solutions of the previous equation where p is the Mersenne prime and a special forms are solved in [4, 2].

In this paper, we continue this program by studying the Diophantine equation $p^a + (p + 1)^b = z^2$ and some of its variants. More precisely, our main result is the following

Theorem 1. *Let p be a prime number. We have that*

(i) *If $p > 3$, then the equation*

$$p^a + (p+1)^b = z^2 \quad (1)$$

does not have solution in a, b, z with $b \geq 2$ and z even.

(i) *If $p > 2$, then the equation*

$$p^a + (p+1)^b = z^4 \quad (2)$$

does not have solution in a, b, z if b has at least one prime factor ≥ 7 .

2. The Proof

2.1. Proof of Item (i)

Suppose that $p \equiv 1 \pmod{3}$, then by considering equation (1) $\pmod{3}$ we have that $1 + 2^b \equiv z^2 \equiv 0, 1 \pmod{3}$. Thus b is even, say $2b'$. In this case, we can factor $p^a + (p+1)^b = z^2$ as

$$p^a = (z - (p+1)^{b'})(z + (p+1)^{b'}).$$

Therefore $z - (p+1)^{b'} = p^r$ and $z + (p+1)^{b'} = p^s$, where $r+s = a$. By subtracting these equations, we obtain $2(p+1)^{b'} = p^s - p^r$ (note that $s > r$). Since p does not divide $2(p+1)$, then $r = 0$ and the equation becomes $2(p+1)^{b'} = p^a - 1$. By looking this relation $\pmod{p}$, we obtain (using also the Binomial theorem) that $3 \equiv 0 \pmod{p}$ and so $p = 3$ contradicting the assumption of $p > 3$.

Now, suppose that $p \equiv 2 \pmod{3}$, then by looking (1) $\pmod{3}$ we have that $2^a \equiv z^2 \equiv 0, 1 \pmod{3}$. Thus a is even, say $2a'$. In this case, we can factor $p^a + (p+1)^b = z^2$ as

$$(p+1)^b = (z - p^{a'})(z + p^{a'}). \quad (3)$$

We use that the $\gcd(m+n, m-n)$ divides $2\gcd(m, n)$ to get that $\gcd(z - p^{a'}, z + p^{a'})$ divides $2\gcd(z, p^{a'})$. However, (1) implies that $p \nmid z$ and so $\gcd(z - p^{a'}, z + p^{a'}) = 1$ or 2 . Since z is even and p is odd, we conclude that this gcd is equal to 1. Thus, from (3), we obtain $z - p^{a'} = x^b$ and $z + p^{a'} = y^b$, where $xy = p+1$.

By subtracting these equations, we obtain $2p^{a'} = y^b - x^b$ and then y and x have the same parity. On the other hand, $xy = p+1$ is even (since p is odd) and the x and y must be both even numbers. In this case, the 2-adic order of $y^b - x^b$ is at least $b \geq 2$, while the 2-adic order of $2p^{a'}$ is obviously 1 contradicting the equality $2p^{a'} = y^b - x^b$. Therefore, there is no solution for this equation.

2.2. Proof of Item (ii)

The proof for item (ii) is quite similar, i.e., by following the same steps as in the previous proof we obtain that (2) does not have solution for $p \equiv 1 \pmod{3}$ and for $p \equiv 2 \pmod{3}$ when z is even. Then the novelty here is the case z odd. For this case, we have the relation

$$(p+1)^b = (z^2 - p^{a'})(z^2 + p^{a'}). \quad (4)$$

Since $\gcd(z^2 - p^{a'}, z^2 + p^{a'}) = 2$, we have two possibilities, namely

$$\begin{cases} z^2 - p^{a'} = 2x^b; \\ 2(z^2 + p^{a'}) = y^b \end{cases} \quad \text{or} \quad \begin{cases} 2(z^2 - p^{a'}) = x^b; \\ z^2 + p^{a'} = 2y^b, \end{cases}$$

where $\gcd(x, y) = 1$ and $xy = p+1$. In order to avoid unnecessary repetitions, let us work only on the first case, i.e., $z^2 - p^{a'} = 2x^b$ and $2(z^2 + p^{a'}) = y^b$.

First, let us show that x, y, z are pairwise coprime. In fact, $\gcd(x, y) = 1$. Suppose that q is a prime number dividing x and z , then by the relation $z^2 - p^{a'} = 2x^b$, we obtain that $q = p$ (since $q \mid p$). This leads to the following contradiction $p = q \mid x \mid p+1$. Now, suppose that q is a prime number dividing y and z , then by the relation $2(z^2 + p^{a'}) = y^b$, we obtain that $q \mid 2p^{a'}$. If $q \mid p$, then we arrive at the same contradiction as before. In the case, of $q \mid 2$ we obtain $2 = q \mid z$ which is a contradiction since z is odd. This leads to the following contradiction $p = q \mid x \mid p+1$.

By combining the relations $z^2 - p^{a'} = 2x^b$ and $2(z^2 + p^{a'}) = y^b$, we get $4z^2 = y^b + 4x^b$ and then $y = 2w$ for some positive integer w . Then, the previous relation can be rewritten as

$$z^2 = x^b + 2^{b-2}w^b$$

where b has a prime factor $t \geq 7$ (and so b is itself at least 7). Writing $b = tm$ we have

$$z^2 = x_1^t + 2^{mt-2}w_1^t,$$

where $x_1 = x^m$ and $w_1 = w^m$. Since $mt - 2 \geq t - 2 \geq 5$ this equation does not have solution by the following result (its proof depends on the Modular approach to prove the Fermat Last Theorem):

Theorem 2 (Siksek [3]). *The Diophantine equation*

$$z^2 = x^p + 2^k y^p$$

does not have solution in pairwise coprime integers x, y, z when $k \geq 4$ and $p \geq 7$ is prime.

This completes our proof. □

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