ON THE SMOOTHNESS OF FUNCTION OF REACHABILITY FOR AUTONOMOUS DIFFERENTIAL EQUATIONS

K.G. Dishlieva

Department of Differential Equations
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
1756 Sofia, 8 Kl. Ohridski Blvd, BULGARIA

Abstract: A function of reachability Θ^+ is investigated for a wide class of autonomous systems of differential equations and relatively wide class reachable sets. The sufficient conditions for differentiability of the function Θ^+ are found.

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1. Introduction

Assume that:

- The domain $G \subset R^n$ is a phase space of autonomous system of differential equations;
- The initial point $x_0 \in G$;
- The set $\Phi \neq \emptyset$;
- The curve γ , $\gamma \subset G$, is a trajectory of the autonomous system which:
 - at the moment $t_0 = 0$, passes through the initial point x_0 ;
 - crosses the set Φ , i.e. $\gamma \cap \Phi \neq \emptyset$.

We say that Θ^+ is a function of reachability if this function assigning to the initial point x_0 the moment θ at which the trajectory γ crosses Φ for the first time. The articles [7], [8] and [14] are devoted on the qualities of function of reachability. The sufficient conditions under which the function is continuous and bounded are found there. In this paper, the studies of function qualities are continued. The sufficient conditions for differentiability of Θ^+ in respect to the initial point x_0 are given.

The function of reachability has a fundamental role in the studies on the existence of periodic solutions of the systems of differential equations with variable moments of the impulsive effects. (see [6] and [13]). In the definitions of impulsive equations, the set of reachability is a set of impulses too. The impulsive effects take place when the trajectory of impulsive system reaches the set Φ . The trajectory instantaneously "bounces" at the moment of these impacts. Thus, the study of function of reachability is equivalent to the investigation of impulsive moments of the systems of differential equations with impulsive effects.

The dynamic processes that have interrupted nature can be modeled by the impulsive systems of differential equations (see [1]-[5] and [10]-[11]). The processes in which the impulsive effects are not predefined, and they are performed when the trajectory reaches the above-mentioned impulsive sets are particularly important. In these cases, the function of reachability is an important feature of such processes and their corresponding models.

The considerations above justify the research presented in this paper.

2. Preliminary Remarks

The scalar product and Euclidean norm in R^n are denoted by $\langle x, y \rangle$ and $\|x\|$, respectively. For the points $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in R^n , we have

$$\begin{aligned}\langle x, y \rangle &= x_1 y_1 + x_2 y_2 + \dots + x_n y_n, \\ \|x\| &= (\langle x, x \rangle)^{1/2} = ((x_1)^2 + (x_2)^2 + \dots + (x_n)^2)^{\frac{1}{2}}.\end{aligned}$$

We will use the unit vectors:

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \dots, \quad e_n(0, 0, \dots, 1) \in R^n.$$

The closure and contour of the set X are denoted by $\overline{X}$ and ∂X , correspondingly. Euclidean distance between the sets X and Y will denote by $\rho(X, Y)$.

The main object of investigation is the following initial value problem

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0, \quad (1)$$

where: the function $f : G \rightarrow R^n$; set $G \subset R^n$, $G \neq \emptyset$ and G is a domain (an open and connected set); the initial point $x_0 \in G$. The solution of problem (1) will be denoted by $x(t; x_0)$. The trajectory of system (1), closed between the points $x(0; x_0) = x_0$ and $x(\theta; x_0)$, where $\theta \in R$ will be denoted by $\gamma(\theta, x_0)$. We have

$$\gamma(\theta, x_0) = \begin{cases} \{x = x(t; x_0); 0 \leq t < \theta\}, & 0 < \theta; \\ \{x = x(t; x_0); \theta < t \leq 0\}, & \theta < 0. \end{cases}$$

In particular:

$$\gamma(\infty, x_0) = \{x = x(t; x_0); 0 \leq t < \infty\}$$

and

$$\gamma(-\infty, x_0) = \{x = x(t; x_0); -\infty < t \leq 0\}.$$

Definition 1. [8] Assume that:

1. The sets $X_0^+ \subset G$, $\Phi \subset G$, $X_0^+ \neq \emptyset$ and $\Phi \neq \emptyset$;
2. For any initial point $x_0 \in X_0^+$, the solution of problem (1) is defined and unique in the infinite interval $[0, \infty)$;
3. There exists a function $\Theta^+ : X_0^+ \rightarrow R^+$, which assigns to any point $x_0 \in X_0^+$ a positive constant $\theta = \Theta^+(x_0)$ for which it is satisfied:
 - $x(\theta; x_0) \in \Phi$;
 - $x(t; x_0) \notin \Phi$ for $0 \leq t < \theta$.

Then, we say that:

1. Φ is a **positive reachable set** from the set X_0^+ through the system (1);
2. X_0^+ is a **positive initial set** for Φ through the system (1);
3. Any initial point $x_0 \in X_0^+$ is named **positive initial point** for Φ through the system (1);
4. If $X_0^+ = G$, then Φ is a **totally positive reachable set** through the system (1);

5. The function $\Theta^+ : X_0^+ \rightarrow R^+$ is named **a positive function of reachability**.

The following concepts are defined in a similar manner:

1. Negative reachable set;
2. Negative initial set X_0^- ;
3. Negative initial point;
4. Totally negative reachable set;
5. Negative function of reachability $\Theta^- : X_0^- \rightarrow R^-$.

Let the reachable set has the form

$$\Phi = \{x \in G; \varphi(x) = 0\},$$

where the function $\varphi : G \rightarrow R$.

We introduce the following conditions:

- H1. There exists a constant $C_f > 0$ such that

$$(\forall x \in G) \Rightarrow \|f(x)\| \leq C_f.$$

- H2. There exists a constant $C_{Lip} > 0$ such that

$$(\forall x', x'' \in G) \Rightarrow \|f(x') - f(x'')\| \leq C_{Lip} \|x' - x''\|.$$

- H3. For any point $x_0 \in G$, the solution of problem (1) exists and is unique in R .

- H4. The function $\varphi \in C^1[G, R]$. There exists a constant $C_{\langle grad\varphi, f \rangle} > 0$ such that

$$(\forall x \in \Phi) \Rightarrow \langle grad\varphi(x), f(x) \rangle \geq C_{\langle grad\varphi, f \rangle}.$$

- H5. The set Φ is connected.

- H6. It is fulfilled $\overline{\Phi} \setminus \Phi \subset \partial G$.

- H7. There exists a constant $C_\varphi > 0$ such that

$$(\forall x \in G) \Rightarrow |\varphi(x)| \leq C_\varphi \rho(x, \Phi).$$

Theorem 1. [7], [8]. *Let the conditions H1-H7 be valid.*

Then:

1. *If the point $x_0 \in X_0^-$, then the trajectory $\gamma(\Theta^-(x_0); x_0) \subset X_0^-$;*

2. If the point $x_0 \in X_0^+$, then the trajectory $\gamma(\Theta^+(x_0); x_0) \subset X_0^+$;
3. If the point $x_0 \in X_0^-$, then the trajectory $\gamma(\infty; x_0) \subset X_0^-$;
4. If the point $x_0 \in X_0^+$, then the trajectory $\gamma(-\infty; x_0) \subset X_0^+$;
5. The sets $X_0^- \neq \emptyset$ and $X_0^+ \neq \emptyset$;
6. The set $\Phi \subset X_0^-$ and $\Phi \subset X_0^+$;
7. The sets X_0^- and X_0^+ are open;
8. The set $X_0 = X_0^- \cup X_0^+ \cup \Phi$ is open;
9. The set X_0 is connected;
10. The functions $\Theta^- \in C[X_0^-, R^-]$ and $\Theta^+ \in C[X_0^+, R^+]$.

Let us consider the particular case when the reachable set Φ is a part of one hyperplane, i.e.

$$(\exists a = (a_1, a_2, \dots, a_n) \in R^n, a \neq 0) (\exists a_0 \in R) : \Phi = \{x \in G, \langle a, x \rangle + a_0 = 0\}.$$

In other words, the function $\varphi(x) = \langle a, x \rangle + a_0, x \in G$.

In connection with this particular case, we introduce the following condition.

H8. There exists a constant $C_{\langle a, f \rangle} > 0$ such that

$$(\forall x \in \Phi \Leftrightarrow \langle a, x \rangle + a_0 = 0, x \in G) \Rightarrow \langle a, f(x) \rangle \geq C_{\langle a, f \rangle}.$$

As a consequence of the previous theorem (in the case where the reachable set is a part of the hyperplane), we formulate the following statement.

Corollary 1. *Let:*

1. The function $\varphi(x) = \langle a, x \rangle + a_0, x \in G$;
2. The conditions H1-H3 and H8 hold.

Then the assertions of Theorem 1 are satisfied.

Proof. We will check whether the conditions of Theorem 1 are valid, which is the evidence of this corollary. It is clear that the function $(\varphi(x) = \langle a, x \rangle + a_0) \in C^1[G, R]$. Furthermore, by using the condition H8, we find

$$(\forall x \in \Phi) \Rightarrow \langle \text{grad} \varphi(x), f(x) \rangle = \langle a, f(x) \rangle \geq C_{\langle a, f \rangle} = C_{\langle \text{grad} \varphi, f \rangle},$$

i.e. the condition H4 is fulfilled. The conditions H5 and H6 are checked trivial. In this case, we have

$$\begin{aligned} (\forall x \in G) \Rightarrow |\varphi(x)| &= |\langle a, x \rangle + a_0| = \frac{|\langle a, x \rangle + a_0|}{\|a\|} \cdot \|a\| \\ &= \|a\| \cdot \rho(x, \Phi) = C_{\varphi} \cdot \rho(x, \Phi), \end{aligned}$$

by which the condition H7 ($C_\varphi = \|a\|$) is established. $\square$

Remark 1. Let the initial point of problem (1) be fixed and $x_0 = (x_{01}, x_{02}, \dots, x_{0n}) \in X_0^+$. If we assume that the private derivatives

$$\frac{\partial}{\partial x_{0i}} \Theta^+(x_0), \quad i = 1, 2, \dots, n,$$

exist, then we will show the equations that they satisfy.

Firstly, we will examine the general case where the function $\varphi \in C^1[G, R]$. For any point $x_0 = (x_{01}, x_{02}, \dots, x_{0n}) \in X_0^+$, we have

$$\begin{aligned} \varphi(x(\Theta^+(x_0); x_0)) = 0 \Leftrightarrow & \quad \varphi\left(x_1(\Theta^+(x_{01}, x_{02}, \dots, x_{0n}); (x_{01}, x_{02}, \dots, x_{0n})), \right. \\ & \quad x_2(\Theta^+(x_{01}, x_{02}, \dots, x_{0n}); (x_{01}, x_{02}, \dots, x_{0n})), \dots, \\ & \quad \left. x_n(\Theta^+(x_{01}, x_{02}, \dots, x_{0n}); (x_{01}, x_{02}, \dots, x_{0n}))\right). \end{aligned}$$

Let us formally differentiate the equation above in respect to x_{0i} , where $i \in \{1, 2, \dots, n\}$. Consistently, we get:

$$\begin{aligned} & \frac{\partial}{\partial x_1} \varphi(x(\Theta^+(x_0); x_0)) \cdot \frac{\partial}{\partial x_{0i}} x_1(\Theta^+(x_0); x_0) \\ & + \frac{\partial}{\partial x_2} \varphi(x(\Theta^+(x_0); x_0)) \cdot \frac{\partial}{\partial x_{0i}} x_2(\Theta^+(x_0); x_0) \\ & + \dots + \\ & + \frac{\partial}{\partial x_n} \varphi(x(\Theta^+(x_0); x_0)) \cdot \frac{\partial}{\partial x_{0i}} x_n(\Theta^+(x_0); x_0) \\ \Leftrightarrow & \quad \left\langle \text{grad} \varphi(x(\Theta^+(x_0); x_0)), \frac{\partial}{\partial x_{0i}} x(\Theta^+(x_0); x_0) \right\rangle = 0 \\ \Leftrightarrow & \quad \left\langle \text{grad} \varphi(x(\Theta^+(x_0); x_0)), \frac{\partial}{\partial x_{0i}} \left(x_0 + \int_0^{\Theta^+(x_0)} f(x(\tau; x_0)) d\tau \right) \right\rangle = 0 \\ \Leftrightarrow & \quad \left\langle \text{grad} \varphi(x(\Theta^+(x_0); x_0)), \right. \\ & \quad \left. \left(e_i + \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x_{0i}} f(x(\tau; x_0)) d\tau + \frac{\partial}{\partial x_{0i}} \Theta^+(x_0) \cdot f(x(\Theta^+(x_0); x_0)) \right) \right\rangle = 0 \\ \Leftrightarrow & \quad \left\langle \text{grad} \varphi(x(\Theta^+(x_0); x_0)), \frac{\partial}{\partial x_{0i}} \Theta^+(x_0) \cdot f(x(\Theta^+(x_0); x_0)) \right\rangle \\ & \quad = - \left\langle \text{grad} \varphi(x(\Theta^+(x_0); x_0)), e_i \right\rangle \end{aligned}$$

$$-\left\langle \text{grad}\varphi(x(\Theta^+(x_0); x_0)), \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x_{0i}} f(x(\tau; x_0)) d\tau \right\rangle,$$

where $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$. From the equation above, after trivial transformations, we obtain

$$\begin{aligned} & \frac{\partial}{\partial x_{0i}} \Theta^+(x_0) \\ &= - \frac{\frac{\partial}{\partial x_i} \varphi(x(\Theta^+(x_0); x_0))}{\left\langle \text{grad}\varphi(x(\Theta^+(x_0); x_0)), f(x(\Theta^+(x_0); x_0)) \right\rangle} \\ & \quad - \frac{\left\langle \text{grad}\varphi(x(\Theta^+(x_0); x_0)), \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x} f(x(\tau; x_0)) \cdot \frac{\partial}{\partial x_{0i}} x(\tau; x_0) d\tau \right\rangle}{\left\langle \text{grad}\varphi(x(\Theta^+(x_0); x_0)), f(x(\Theta^+(x_0); x_0)) \right\rangle}. \end{aligned}$$

In the particular case, when the reachable set is a part of hyperplane, we find that

$$\frac{\partial}{\partial x_{0i}} \Theta^+(x_0) = - \frac{\langle a, e_i \rangle + \left\langle a, \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x} f(x(\tau; x_0)) \cdot \frac{\partial}{\partial x_{0i}} x(\tau; x_0) d\tau \right\rangle}{\left\langle a, f(x(\Theta^+(x_0); x_0)) \right\rangle} \quad (2)$$

The above formal equality allows us to formulate the main results in the present study.

3. Main Results

Theorem 2. *Let:*

1. *The function $\varphi(x) = \langle a, x \rangle + a_0$, $x \in G$.*
2. *The conditions H1-H3 and H8 hold.*

Then the function $\Theta^+ : X_0^+ \rightarrow R^+$ is differentiable in respect to $x_0 = (x_{01}, x_{02}, \dots, x_{0n}) \in X_0^+$ and the private derivatives

$$\frac{\partial}{\partial x_{0i}} \Theta^+(x_0), \quad i = 1, 2, \dots, n,$$

satisfy the corresponding equations (2).

Proof. Let $i \in \{1, 2, \dots, n\}$, i.e. i is a fixed number. Since the positive initial set $X_0^+ \neq \emptyset$, then there exists at least one initial point $x_0 = (x_{01}, x_{02}, \dots, x_{0n}) \in$

X_0^+ . As X_0^+ is an open set, then

$$\begin{aligned} & (\exists \Delta_i = \text{const} > 0) : (\forall \Delta x_i \in R, 0 \leq |\Delta x_i| < \Delta_i) \\ & \Rightarrow x_0^* = (x_{01}, \dots, x_{0i-1}, x_{0i} + \Delta x_i, x_{0i+1}, \dots, x_{0n}) \in X_0^+. \end{aligned}$$

According to the definition, the next inequalities are fulfilled:

$$\begin{aligned} \varphi\left(x(\Theta^+(x_0^*); x_0^*)\right) &= \langle a, x(\Theta^+(x_0^*); x_0^*) \rangle + a_0 = 0; \\ \varphi\left(x(\Theta^+(x_0); x_0)\right) &= \langle a, x(\Theta^+(x_0); x_0) \rangle + a_0 = 0, \end{aligned}$$

from where, for any $\Delta x_i \in R, 0 < |\Delta x_i| < \Delta_i$, we obtain

$$\begin{aligned} & \frac{1}{\Delta x_i} \left(\langle a, x(\Theta^+(x_0^*); x_0^*) \rangle + a_0 - \langle a, x(\Theta^+(x_0); x_0) \rangle - a_0 \right) = 0 \\ \Leftrightarrow & \left\langle a, \frac{1}{\Delta x_i} \left(x(\Theta^+(x_0^*); x_0^*) - x(\Theta^+(x_0); x_0) \right) \right\rangle = 0 \\ \Leftrightarrow & \left\langle a, \frac{1}{\Delta x_i} \left(x_0^* + \int_0^{\Theta^+(x_0^*)} f(x(\tau; x_0^*)) d\tau - x_0 - \int_0^{\Theta^+(x_0)} f(x(\tau; x_0)) d\tau \right) \right\rangle = 0 \\ \Leftrightarrow & \langle a, e_i \rangle + \left\langle a, \frac{1}{\Delta x_i} \left(\int_0^{\Theta^+(x_0^*)} f(x(\tau; x_0^*)) d\tau - \int_0^{\Theta^+(x_0)} f(x(\tau; x_0)) d\tau \right) \right\rangle = 0. \end{aligned}$$

Further, for convenience, we will assume that $\Theta^+(x_0) \leq \Theta^+(x_0^*)$. Then, from the last equality it follows:

$$\begin{aligned} & \langle a, e_i \rangle + \left\langle a, \frac{1}{\Delta x_i} \left(\int_0^{\Theta^+(x_0^*)} \left(f(x(\tau; x_0^*)) - f(x(\tau; x_0)) \right) d\tau \right. \right. \\ & \quad \left. \left. + \int_{\Theta^+(x_0)}^{\Theta^+(x_0^*)} f(x(\tau; x_0^*)) d\tau \right) \right\rangle = 0 \\ \Leftrightarrow & \langle a, e_i \rangle + \left\langle a, \int_0^{\Theta^+(x_0^*)} \frac{\partial}{\partial x} f(x(\tau; x_0 + \mu \cdot \Delta_i \cdot e_i)) \cdot \frac{1}{\Delta x_i} (x(\tau; x_0^*) - x(\tau; x_0)) d\tau \right\rangle \\ & \quad + \left\langle a, \frac{1}{\Delta x_i} (\Theta^+(x_0^*) - \Theta^+(x_0)) \cdot f(x(\theta; x_0^*)) \right\rangle = 0, \end{aligned}$$

where $0 \leq \mu \leq 1$ and $\Theta^+(x_0) \leq \theta \leq \Theta^+(x_0^*)$. Finally, we derive

$$\frac{1}{\Delta x_i} (\Theta^+(x_0^*) - \Theta^+(x_0)) \quad (3)$$

$$= - \frac{\langle a, e_i \rangle}{\left\langle a, f(x(\theta; x_0^*)) \right\rangle} - \frac{\left\langle a, \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x} f(x(\tau; x_0 + \mu \Delta_i \cdot e_i)) \cdot \frac{1}{\Delta x_i} (x(\tau; x_0^*) - x(\tau; x_0)) d\tau \right\rangle}{\left\langle a, f(x(\theta; x_0^*)) \right\rangle}.$$

Bearing in mind the way in which the constants θ and μ are obtained, we deduce that if

$$\Delta x_i \rightarrow 0 \quad \Rightarrow \quad \theta \rightarrow \Theta^+(x_0), \quad \mu \Delta x_i \rightarrow 0.$$

Then:

$$\begin{aligned} \lim_{\Delta x_i \rightarrow 0} f(x(\theta; x_0^*)) &= \lim_{\theta \rightarrow \Theta^+(x_0)} \lim_{\Delta x_i \rightarrow 0} f(x(\theta; x_0 + \Delta x_i \cdot e_i)) \\ &= f(x(\Theta^+(x_0); x_0)); \\ \lim_{\Delta x_i \rightarrow 0} \frac{\partial}{\partial x} f(x(\tau; x_0 + \mu \Delta_i \cdot e_i)) &= \lim_{\mu \Delta x_i \rightarrow 0} \frac{\partial}{\partial x} f(x(\tau; x_0 + \mu \Delta_i \cdot e_i)) \\ &= \frac{\partial}{\partial x} f(x(\tau; x_0)); \\ \lim_{\Delta x_i \rightarrow 0} \frac{1}{\Delta x_i} (x(\tau; x_0^*) - x(\tau; x_0)) &= \lim_{\Delta x_i \rightarrow 0} \frac{1}{\Delta x_i} (x(\tau; x_0 + \Delta x_i \cdot e_i) - x(\tau; x_0)) \\ &= \frac{\partial}{\partial x_{0i}} x(\tau, x_0). \end{aligned}$$

The last equality follows from the classical theorem for differentiating of the solution of differential equation on the initial condition (for example, see Theorem 1.16 [9]). Then from the equalities above and (3), we get

$$\begin{aligned} &\lim_{\Delta x_i \rightarrow 0} \frac{1}{\Delta x_i} (\Theta^+(x_0^*) - \Theta^+(x_0)) \\ = &- \lim_{\Delta x_i \rightarrow 0} \frac{\langle a, e_i \rangle}{\left\langle a, f(x(\theta; x_0^*)) \right\rangle} \\ &- \lim_{\Delta x_i \rightarrow 0} \frac{\left\langle a, \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x} f(x(\tau; x_0 + \mu \Delta_i \cdot e_i)) \cdot \frac{1}{\Delta x_i} (x(\tau; x_0^*) - x(\tau; x_0)) d\tau \right\rangle}{\left\langle a, f(x(\theta; x_0^*)) \right\rangle} \\ = &- \frac{\langle a, e_i \rangle + \left\langle a, \int_0^{\Theta^+(x_0)} \frac{\partial}{\partial x} f(x(\tau; x_0)) \cdot \frac{\partial}{\partial x_{0i}} x(\tau; x_0) d\tau \right\rangle}{\left\langle a, f(x(\Theta^+(x_0); x_0)) \right\rangle}. \end{aligned}$$

Thus, we have shown that there exists

$$\frac{\partial}{\partial x_{0i}} \Theta^+(x_0), \quad i = 1, 2, \dots, n,$$

and the conditions (2) are satisfied.

The theorem is proved. $\square$

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