ON NODES IN TRELLISES

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Abstract: In this paper, the concept of a node in a lattice introduced by J.Duda has been extended to trellises, a non- associative generalization of a lattice. It is proved that set of all nodes of a trellis forms a distributive lattice. Also we have proved that node of a trellis is standard if and only if it is not contained in a non trivial cycle.

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1. Introduction

A reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A is called a *pseudo-order* on A and $\langle A, \trianglelefteq \rangle$ is called a *pseudo-ordered set* or a *poset*. For $a, b \in A$ if $a \trianglelefteq b$ and $a \neq b$, then we write $a \triangleleft b$. For a subset B of A , the notions of a lower bound, an upper bound, the greatest lower bound (GLB or meet, denoted by $\wedge B$), the least upper bound (LUB or join, denoted by $\vee B$) are defined analogous to the corresponding notions in a poset (we refer to [3]).

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Define a relation $\sqsubseteq_B$ on a subset B of a poset $\langle A, \leq \rangle$ by setting $b \sqsubseteq_B b^1$ for two elements b and b^1 of B if and only if there is a directed path in B from b to b^1 say $b = b_0 \leq b_1 \leq \dots \leq b_n = b^1$ for some $n \geq 0$. The relation $\sqsupseteq_B$ is defined dually. If for each pair of elements b and b^1 of B at least one of the relations $b \sqsubseteq_B b^1$ or $b^1 \sqsubseteq_B b$ holds, then B will be called a *pseudo chain* or a *p-chain*. If for each pair of elements b and b^1 of B both the relations $b \sqsubseteq_B b^1$ and $b^1 \sqsubseteq_B b$ hold, then B will be called a *cycle*. The empty set and a single element set in a poset are cycles. A non-trivial cycle contains at least three elements. A poset is said to be *acyclic* if it does not contain any non-trivial cycle.

A poset $\langle A, \leq \rangle$ in which every pair of elements has a least upper bound and a greatest lower bound is called a *trellis*. In other words, a trellis is an algebra $\langle L, \wedge, \vee \rangle$ where the binary operations $\wedge$ and $\vee$ satisfy the following properties:

- (i) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for all $a, b \in L$;
- (ii) $a \vee (b \wedge a) = a = a \wedge (b \vee a)$ for all $a, b \in L$;
- (iii) $a \vee ((a \wedge b) \vee (a \wedge c)) = a = a \wedge ((a \vee b) \wedge (a \vee c))$ for all $a, b, c \in L$.

The notion of a trellis as a non-associative generalisation of a lattice is due to E.Fried [2] and H.L.Skala [5].

2. Definitions and Results

Definition 2.1. For elements x, y, d of a trellis L , d is said to be:

- i. *right transitive* if $d \leq x \leq y$ imply $d \leq y$;
- ii. *left transitive* if $x \leq y \leq d$ imply $x \leq d$;
- iii. *middle transitive* if $x \leq d \leq y$ imply $x \leq y$;
- iv. *transitive* if d is right, left and middle transitive.

Definition 2.2. An element d of a trellis L is said to be:

- (i) $\vee$ - *associative* if $d \vee (x \vee y) = (d \vee x) \vee y$ for all $x, y \in L$;
- (ii) $\wedge$ - *associative* if $d \wedge (x \wedge y) = (d \wedge x) \wedge y$ for all $x, y \in L$;
- (iii) *associative* if d is both $\vee$ - associative and $\wedge$ - associative.

Definition 2.3. A trellis L is said to be *modular* if for $x, y, z \in L$, $x \leq z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$.

J. Duda [1] has defined an element d of a poset A as a *node* if d is comparable with every element of A . We modify this definition in pseudo-ordered set as follows.

Definition 2.4. An element d of a psoset A is called a *node* if d is middle transitive and comparable with every element of A .

Proposition 2.5. A node d of a psoset is transitive if and only if it is not contained in a non-trivial cycle.

Proof. Let a node d of a psoset A be transitive. If d is contained in a non-trivial cycle C , then for any $x \in C$, we have $x \sqsubseteq_C d$ and $d \sqsubseteq_C x$ which in turn imply $x \leq d$ by left transitivity of d and $d \leq x$ by right transitivity of d so that $x = d$. Therefore C is the one element cycle $\{d\}$, a contradiction.

Conversely, let a node d of a psoset A be not contained in a non-trivial cycle. Let $x \leq y \leq d$ for $x, y, d \in L$. If either $x = y$ or $y = d$, then $x \leq d$ holds. Therefore, let us assume that $x \triangleleft y \triangleleft d$. Then $x \triangleleft d$. For, if $x \triangleright d$, then $x \triangleleft y \triangleleft d \triangleleft x$ so that $\{x, d, y\}$ is a non-trivial cycle containing d , a contradiction. Thus d is left transitive. Similarly the right transitivity of d can be proved. By the definition of a node, d is middle transitive. Therefore d is transitive. $\square$

Following definitions are due to Shashirekha [4].

Definition 2.6. An element d of a trellis L is said to be *standard* if:

- (i) d is $\vee$ - associative and;
- (ii) $x \wedge (d \vee y) = (x \wedge d) \vee (x \wedge y)$ for all $x, y \in L$.

A dually standard element is defined dually.

Definition 2.7. An element d of a trellis L is said to be *distributive* if:

- (i) d is $\vee$ - associative and;
- (ii) $d \vee (x \vee y) = (d \vee x) \vee (d \vee y)$ for all $x, y \in L$.

Proposition 2.8. Every standard element d of a trellis L is transitive.

Proof. By the definition of a standard element, d is $\vee$ - associative and therefore d is transitive. For if $d \leq x \leq y$ then $d \vee y = d \vee (x \vee y) = (d \vee x) \vee y = x \vee y = y$ imply d is right transitive. Similarly we can verify that d is left transitive and middle transitive. Thus d is transitive. $\square$

Theorem 2.9. Following statements are equivalent for a node d in a trellis L .

1. d is standard (dually standard) element;

2. d is not contained in a non-trivial cycle.

Proof. Direct part follows from Proposition 2.5 and Proposition 2.8.

Conversly, suppose that the node d is not contained in a non-trivial cycle. Then by Proposition 2.5 d is transitive. We have the following possibilities for elements d, x, y of L .

Case (i): $d \leq x, d \leq y$.

Since d is right transitive and $d \leq x \leq x \vee y$, we have $d \leq x \vee y$. Therefore $d \vee (x \vee y) = x \vee y = (d \vee x) \vee y$. Also $x \wedge (d \vee y) = x \wedge y = d \vee (x \wedge y) = (x \wedge d) \vee (x \wedge y)$.

Case (ii): $d \geq x, d \geq y$.

Then $d \vee (x \vee y) = d = (d \vee x) \vee y$. Also $x \wedge (d \vee y) = x \wedge d = x = x \vee (x \wedge y) = (x \wedge d) \vee (x \wedge y)$.

Case (iii): $d \geq x, d \leq y$.

Since d is middle transitive and $x \leq d \leq y$, we have $x \leq y$. Therefore $d \vee (x \vee y) = d \vee y = (d \vee x) \vee y$ and $x \wedge (d \vee y) = x \wedge y = x = x \vee (x \wedge y) = (x \wedge d) \vee (x \wedge y)$.

Case (iv): $d \leq x, d \geq y$.

Since d is middle transitive and $y \leq d \leq x$, we have $y \leq x$. Therefore $d \vee (x \vee y) = d \vee x = x = x \vee y = (d \vee x) \vee y$ and $x \wedge (d \vee y) = x \wedge d = d = d \vee y = (x \wedge d) \vee (x \wedge y)$.

Thus d is standard. □

Definition 2.10. A trellis L is said to be *normal* if $x \wedge y \leq x \vee y$ for every $x, y \in L$.

Proposition 2.11. If d is a node in an acyclic normal trellis L , then d is distributive.

Proof. Let d be a node in an acyclic normal trellis L . Then by Proposition 2.5, d is transitive which implies d is associative. It remains to prove that $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$. We have the following possibilities for elements d, x, y of L .

Case (i): $d \geq x, d \geq y$.

Then $d \geq x \vee y$. As L is normal $x \wedge y \leq x \vee y \leq d$. By the left transitivity of d $x \wedge y \leq d$. Therefore $d \vee (x \wedge y) = d$. Also $(d \vee x) \wedge (d \vee y) = d \vee d = d$.

Case (ii): $d \leq x, d \leq y$.

Then $d \leq x \wedge y$. Also $d \vee (x \wedge y) = x \wedge y$ and $(d \vee x) \wedge (d \vee y) = x \wedge y$.

Case (iii): $d \leq x, d \geq y$.

Then $d \vee x = x$ and $d \vee y = d$ and $y \leq d \leq x$ imply $y \leq x$ as d is middle transitive. Therefore $d \vee (x \wedge y) = d \vee y = d$ and $(d \vee x) \wedge (d \vee y) = x \wedge d = d$.

Thus d is distributive. $\square$

Theorem 2.12. *The set of all nodes of a trellis L forms a distributive lattice.*

Proof. Let A the set of all nodes of a trellis L and $\leq$ be the restriction of $\leq$ to A . Then $\leq$ is a partial order. For, reflexivity and antisymmetry are obvious. Transitivity of $\leq$ follows from the middle transitivity of a node.

Define $\wedge$ and $\vee$ in A by $a \wedge b = a$ if $a \leq b$ and $a \vee b = a$ if $b \leq a$. Then for any $a, b \in A$ either $a \leq b$ or $b \leq a$ as both are nodes. Therefore $a \leq b$ or $b \leq a$. Hence $\wedge$ and $\vee$ exist for every pair of elements of A proving $\langle A, \leq \rangle$ is a lattice. One can easily verify that this is a distributive lattice. $\square$

Theorem 2.13. *If $f : L \rightarrow L$ is a homomorphism of an acyclic trellis L onto itself and if a is a node in L then $f(a)$ is also a node.*

Proof. Let $a \in L$ be a node, then $f(a) \in L$. Let $l \leq f(a) \leq m$. Since $f : L \rightarrow L$ is onto there exists $b, c \in L$ such that $f(b) = l$ and $f(c) = m$ so that $f(b) \leq f(a) \leq f(c)$. Consider $f(b) \wedge f(c) = [f(b) \wedge f(a)] \wedge f(c) = f(b \wedge a) \wedge f(c) = f((b \wedge a) \wedge c) = f((a \wedge c) \wedge b)$ (since the node a is middle transitive, it is $\wedge$ associative) $= [f(a) \wedge f(c)] \wedge f(b) = f(a) \wedge f(b) = f(b)$. Therefore $f(b) \leq f(c)$ which imply $f(a)$ is middle transitive. Since $a \in L$ is a node, either $a \leq x$ or $a \geq x$ for all $x \in L$ which imply $f(a) \leq f(x)$ or $f(a) \geq f(x)$ for all $f(x) \in L$ which proves that $f(a)$ is a node. $\square$

Theorem 2.14. *The set of all nodes of a trellis L with least element 0 forms a commutative monoid.*

Proof. Let A denote the set of all nodes of a trellis L . Define a binary relation $\oplus$ on A by for $a, b \in A$, $a \oplus b = a \vee b$. Clearly $\oplus$ satisfies the closure property. If $a, b, c \in A$ such that $a \leq b$ and $b \leq c$. Then $a \leq b$ and $b \leq c$ in L so that $a \leq c$ as b is middle transitive. Now $a \oplus (b \oplus c) = a \vee (b \vee c) = a \vee c = c$ and $(a \oplus b) \oplus c = (a \vee b) \vee c = b \vee c = c$. Other cases can be proved dually. Therefore $\oplus$ is associative. Also the element 0 of the trellis is the identity element since $a \oplus 0 = a \vee 0 = a$ and $0 \oplus a = 0 \vee a = a$. $\oplus$ is commutative as $a \oplus b = a \vee b = b \vee a = b \oplus a$. Therefore $\langle A, \oplus \rangle$ forms a commutative monoid. $\square$

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