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ON PARAMETER ESTIMATION OF SEMI-VARYING COEFFICIENT MODELS WITH CORRELATED RANDOM ERRORS

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Abstract: This paper deals with estimation of semi-varying coefficient models with correlated random errors. The estimations of the function coefficients are are given by the use of generalized weighted least squares. The theorem of Gauss-Markov was very well known, but moreover we present an estimation of parameter σ^2.

AMS Subject Classification: 62G05, 62G08
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1. Introduction

Fan et al (see [1]) put forward the the semi-varying coefficient models when he studied whether the function coefficient of the varying coefficient really change,

$$Y = X\alpha(U) + Z\beta + \varepsilon. \tag{1.1}$$

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Suppose that we have a response variable Y and regressors $X = (X_1, X_2, \ldots X_p)$, $Z = (Z_1, Z_2, \ldots, Z_q)$ as well as another variable U. ε is a random error with $E(\varepsilon) = 0$ and it is independent of (U, Z, X); $\beta = (\beta_1, \beta_2, \ldots, \beta_q)^T$ is a q-dimensional vector of an unknown parameters and $\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \ldots, \alpha_p(\cdot))^T$ is a p-dimensional vector of an unknown coefficient function.

Most researches have focused on the case in which random error are independently and identically distributed in a model.

In practice, sometimes the random errors are correlated, so it is been necessary to discuss the question.

In this paper, we discuss the semi-varying coefficient models with correlated random errors, which can be defined as follows:

$$\begin{cases} Y = X\alpha(U) + Z\beta + e, \\ e \sim N(0, \sigma^2\Sigma), \quad \Sigma = diag(\omega_1, \omega_2, \ldots, \omega_n). \end{cases} \tag{1.2}$$

2. Estimation

Suppose that we have a random sample of size n. $\{Y_r, X_r, Z_r, U_r\}(r = 1, 2, \ldots, n)$ is a sample from the model (1.2), $e \sim N(0, \sigma^2\Sigma)$, $e = (e_1, e_2, \ldots, e_n)^T$, Σ is a given positive matrices, (1.2) can be written as

$$\begin{cases} Y^* = X\alpha(U) + e, \\ e \sim N(0, \sigma^2\Sigma). \end{cases} \tag{2.1}$$

where $Y^* = Y - Z\beta$,and β known,

$$X = (X_1^T, X_2^T, \ldots, X_n^T)^T, \quad X_r^T = (X_{r1}, X_{r2}, \ldots, X_{rp}),$$

$$Z = (Z_1^T, Z_2^T, \ldots, Z_n^T)^T, \quad Z_r^T = (Z_{r1}, Z_{r2}, \ldots, Z_{rq}).$$

As Σ is a given positive definite matrix,there must exist a orthogonal matrix. Hence, $\Sigma = P^T\Lambda P$, $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_r > 0$ $(r = 1, 2, \ldots, n)$ are characteristic roots of Σ.Use P^T left multiplied by (2.1).

So

$$\begin{cases} \widetilde{Y} = \widetilde{X}\alpha(U) + \varepsilon, \\ e \sim N(0, \sigma^2 I_n). \end{cases} \tag{2.2}$$

So (2.2) is the common varying coefficient model.

Let

$$\widetilde{Y} = (\widetilde{Y}_1, \widetilde{Y}_2, \dots, \widetilde{Y}_n)^T = \Sigma^{\frac{-1}{2}} Y^*, \quad \overline{Y} = \Sigma^{\frac{-1}{2}} Y,$$

$$\overline{Z} = \Sigma^{\frac{-1}{2}} Z, \quad \widetilde{X} = (\widetilde{X}_1^T, \widetilde{X}_2^T, \dots, \widetilde{X}_n^T)^T = \Sigma^{\frac{-1}{2}} X, \quad \widetilde{X}_r^T = (\widetilde{X}_{r1}, \widetilde{X}_{r2}, \dots, \widetilde{X}_{rp}),$$

$$\overline{Z} = (\overline{Z}_1^T, \overline{Z}_2^T, \dots, \overline{Z}_n^T)^T, \quad \overline{Z}_r^T = (\overline{Z}_{r1}, \overline{Z}_{r2}, \dots, \overline{Z}_{rq}),$$

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)^T = \Sigma^{\frac{-1}{2}} e, \quad W = diag(W_{11}, W_{22}, \dots, W_{nn})$$

$(r = 1, 2, \dots, n)$. The coefficient function $\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \dots, \alpha_p(\cdot))^T$ is estimated by minimizing

$$Q(\alpha(u)) = (\widetilde{Y} - \widetilde{X}\alpha(U))^T W (\widetilde{Y} - \widetilde{X}\alpha(U)). \tag{2.3}$$

According to the principle of weighted least squares, if we assume that the inverse of the matrix $\widetilde{X}^T W \widetilde{X}$ exists for any u, then the estimated coefficient function at u can be expressed as

$$\begin{aligned}\widehat{\alpha}(u) &= (\widehat{\alpha}_1(u), \widehat{\alpha}_2(u), \dots, \widehat{\alpha}_p(u))^T \\ &= [X^T \Sigma^{\frac{-1}{2}} W \Sigma^{\frac{-1}{2}} X]^{-1} X^T \Sigma^{\frac{-1}{2}} W \Sigma^{\frac{-1}{2}} Y^*. \end{aligned} \tag{2.4}$$

Then (2.2) can be written as

$$\overline{Y} - \overline{Z}\beta = M + \varepsilon. \tag{2.5}$$

The estimator for M is then

$$\widehat{M} = \widetilde{X}[\widetilde{X}^T W \widetilde{X}]^{-1} \widetilde{X}^T W (\overline{Y} - \overline{Z}\beta) = S(\overline{Y} - \overline{Z}\beta).$$

The matrix S is a smoothing matrix and dependent only on the observations (U_r, X_r^T), $r = 1, 2, \dots, n$.Substituting $\widehat{M}$ into (2.5),we obtain

$$(I - S)\bar{Y} = (I - S)\bar{Z}\beta + \varepsilon. \tag{2.6}$$

Applying least squares to the linear model (2.6), we obtain

$$\widehat{\beta} = \{\overline{Z}^T (I - S)^T (I - S)\overline{Z}\}^{-1} \overline{Z}^T (I - S)^T (I - S)\overline{Y}.$$

Moreover

$$\widehat{M} = S(\overline{Y} - \overline{Z}\widehat{\beta}). \tag{2.7}$$

On the basis of $\widehat{\alpha}(u)$, we inferred from $\widehat{\beta}$, hence $C^T\beta$ is two-step estimation method of $C^T\alpha(U)$, due to the complexity of local fitting, so it's difficult to analyze the statistical inference properties.thus, we mainly focus on some properties of $\widehat{\alpha}(u)$.

Suppose that $EY^* = X\alpha(u)$, then:

$$E[\widehat{\alpha}(u)] = [X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X]^{-1}X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X\alpha(u),$$
$$Var[\widehat{\alpha}(u)] = \sigma^2[X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X]^{-1}[X^T\Sigma^{\frac{-1}{2}}W^2\Sigma^{\frac{-1}{2}}X][X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X].$$

Theorem 1. *Consider model (2.2), $\widetilde{\alpha}(u)$ is the generalized weighted least squares estimation*

$$\widetilde{\alpha}(u) = [X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X]^{-1}X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}Y^*.$$

Suppose that $EY^* = X\alpha(u)$. Then:

(1) When X is a matrix with full column rank and the inverse of the matrix $X^T\Sigma^{\frac{-1}{2}}\ W\Sigma^{\frac{-1}{2}}X$ exist, then $\widehat{\alpha}(u)$ must be unique.

$$E[\widehat{\alpha}(u)] = \alpha(u).$$

Hence $\widehat{\alpha}(u)$ is an estimation of $\alpha(u)$, and we called $\widehat{\alpha}(u)$ the weighted linear unbiased estimate of $\alpha(u)$.

(2) When X is a matrix with full column rank and W is idempotent matrix,then

$$E[\widehat{\alpha}(u)] = \alpha(u).$$
$$Var[\widehat{\alpha}(u)] = \sigma^2[X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X]^{-1}.$$

(3) When X is a matrix with full column rank, W is idempotent matrix and $\lim\limits_{n\to\infty}(X^T\Sigma^{\frac{-1}{2}}\ W\Sigma^{\frac{-1}{2}}X)^{-1} = 0$. hence $\widehat{\alpha}(u)$ is the consistent estimate of $\alpha(u)$.

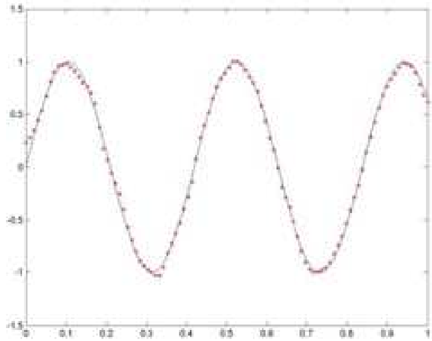
3. Simulation

In this section, we use Matlab conduct some research to estimator.

Case 1. $Y = sin(15U)X_1 + 4U(1-U)X_2 + 8Z_1 + \varepsilon$.

Case 2. $Y = cos(5piU)X_1 + exp(4U)X_2 + 8Z_1 + \varepsilon$.

In these cases,where $U \sim N[0.1]$, $X_1 \sim N[1,3]$, $X_2 \sim N[5,10^{-5}]$,$Z_1 \sim N[0,10^{-5}]$, $\varepsilon \sim N(0,A)$, A is a random number in $[0,1.1]$. Using *Epanechnikov kernels* window width $h_n = \frac{1}{40}$, sample data randomly generated by $Matlab$. The sample sizes is 1000. In Case 1 and Case 2 we use $*$ indicate estimated value, and use solid-line curve indicate actual value. Then, we found estimated results is fine.



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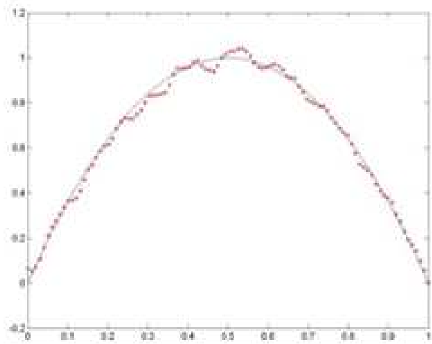
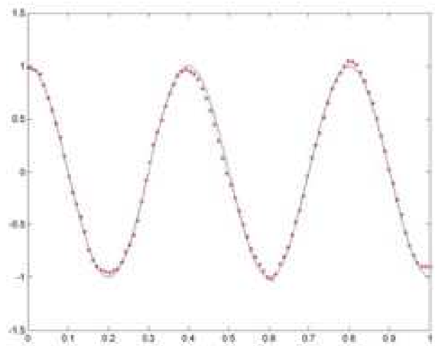
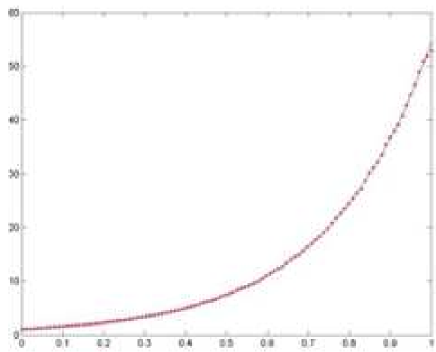


The estimate of $Sin(15U)$ The estimate of $4U(1-U)$.





The estimate of $Cos(5piU)$ The estimate of $exp(4U)$.

4. The Estimation of the Linear Estimable Function and Optimality

Theorem 2. (The Generalized Weighted Gauss-Markov Theorem) *In model* (2.2), $\widehat{\alpha}(u)$ *is a generalized weighted least squares estimate , Suppose that* $EY^* = X\alpha(u)$. *Then:*

(1) $\widehat{\alpha}(u)$ *is the unique weighted linear unbiased estimation of* $\alpha(u)$.

(2) when W is an idempotent matrix, then $\widehat{\alpha}(u)$ is the unique best weighted linear unbiased estimation of $\alpha(u)$.

(3) when X is a matrix with full row rank,W is an idempotent matrix and $\lim\limits_{n\to\infty} c^T(X^T\Sigma^{\frac{-1}{2}} W\Sigma^{\frac{-1}{2}}X)^{-1}c = 0$. Then $c^T\widehat{\alpha}(u)$ is the consistent estimate of $c^T\alpha(u)$.

In order to prove Theorem 2, we give some definitions and lemmas are defined in this paper.

Definition 1. (see [2]) In model (2.2) , $c \in R^p$, linear function $\alpha(u)$ is weighting estimation. If we assume that $b^TW\widetilde{Y}$ is the weighted linear unbiased estimation of $c^T\alpha(u)$ and $b^TW\widetilde{Y}$ exists , $b \in R^n$,then

$$E(b^TW\widetilde{Y}) = c^T\alpha(u),$$

for any $\alpha(u)$.

Lemma 1. (see [2]) *$c^T\alpha(u)$ is the weighting estimation $\Leftrightarrow c^T \in R(\widetilde{X}^TW\widetilde{X}) = R(\widetilde{X}^TW) = R(\widetilde{X}^T)$.*

Lemma 2. (see [2]) *If $c^T\alpha(u)$ is the weighting estimation ,then $c^T\widehat{\alpha}(u)$ must be unique. since $b \in R_n$, we have*

$$c^T\widehat{\alpha}(u) = b^T\widetilde{X}^TW\widetilde{Y}.$$

Definition 2. (see [2]) Considering model(2.2) and $c \in R^p$,$b \in R^n$,so $b^TW\widetilde{Y}$ is the best weighted linear unbiased estimate.

If:

(1) $b^TW\widetilde{Y}$ is the weighted linear unbiased estimate of $c^T\alpha(u)$.

(2) Any $g^TW\widetilde{Y}$ is the weighted linear unbiased estimate of $c^T\alpha(u)$, we have

$$Var(b^TW\widetilde{Y}) \le Var(g^TW\widetilde{Y}).$$

Proof of Theorem 2. (1) $c^T\alpha(u)$ is the weighting estimation.By Lemma 1, $c \in R(\widetilde{X}^T)$ and $b \in R^n$.

We have

$$c = \widetilde{X}^Tb = X^T\Sigma^{\frac{-1}{2}}b.$$

then $c^T\widehat{\alpha}(u) = b^T\widetilde{X}(\widetilde{X}^TW\widetilde{X})^{-1}\widetilde{X}^T\widetilde{Y}$. so

$$E(c^T\widehat{\alpha}(u)) = b^T\Sigma^{\frac{-1}{2}}X\alpha(u).$$

By Lemma 2, we know $c^T\alpha(u)$ must be unique. So $\widehat{\alpha}(u)$ is the unique weighted linear unbiased estimate of $\alpha(u)$.

(2) W is an idempotent matrix, any $g^T W\widetilde{Y}$ is the weighted linear unbiased estimate of $c^T\alpha(u)$. So

$$c^T\alpha(u) = E(g^T W\widetilde{Y}) = g^T W\widetilde{X}\alpha(u),$$

for any $\alpha(u)$.

We have $c^T = g^T W\widetilde{X}$, then $c = \widetilde{X}^T W g$,

$$\begin{aligned} &Var(g^T W\widetilde{Y}) - Var(c^T\alpha(u)) \\ &\qquad = \sigma^2 g^T W^{\frac{1}{2}}[I_n - W^{\frac{1}{2}}\Sigma^{\frac{-1}{2}}X(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}X^T\Sigma^{\frac{-1}{2}}W^{\frac{1}{2}}]W^{\frac{1}{2}}g, \end{aligned}$$

where $P_{W_{\frac{1}{2}}} = W^{\frac{1}{2}}\Sigma^{\frac{-1}{2}}X(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}X^T\Sigma^{\frac{-1}{2}}W^{\frac{1}{2}}$ is an Projection matrix.

Therefore

$$Var(g^T W\widetilde{Y}) - Var(c^T\widehat{\alpha}(u)) = \sigma^2 g^T W^{\frac{1}{2}}[I_n - P_{W_{\frac{1}{2}}}]W^{\frac{1}{2}}g \geq 0. \tag{3.1}$$

From Definition 2, we have that $\widehat{\alpha}(u)$ is the weighted linear unbiased estimate of $\alpha(u)$. The conditions of the holding water of sign of equality in (3.1) is $(I_n - P_{W_{\frac{1}{2}}})g = 0$, then

$$g^T W\widetilde{Y} = g^T W\widetilde{X}(\widetilde{X}^T W\widetilde{X})^{-1}\widetilde{X}^T W\widetilde{Y} = c^T\widehat{\alpha}(u).$$

Hence $\widehat{\alpha}(u)$ is the unique best weighted linear unbiased estimate of $\alpha(u)$.

(3) We have $E(c^T\widehat{\alpha}(u)) = c^T\alpha(u)$, X is a matrix with full column rank, W is idempotent matrix and $\lim_{n\to\infty} c^T(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}c = 0$.

By Theorem 1, we have

$$Var(c^T\widehat{\alpha}(u)) = c^T Var(\widehat{\alpha}(u))c = \sigma^2(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}c \to 0 \quad (n\to\infty).$$

Therefore, $c^T\widehat{\alpha}(u)$ is the consistent estimate of $c^T\alpha(u)$. □

5. The Estimation of σ^2 and Optimality

Let us set:

$$Rss_e^2 = (Y^* - X\alpha(u))^T(Y^* - X\alpha(u)) = \|Y^* - X\alpha(u)\|^2.$$

$$e = Y^* - X\alpha(u) = (I_n - X(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}})Y^*.$$

If

$$P = X(X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}X)^{-1}X^T\Sigma^{\frac{-1}{2}}W\Sigma^{\frac{-1}{2}}$$

and

$$P^2 = P,$$

then

$$e = (I_n - P)Y^*.$$

Setting $M = I_n - P$, $MX = X - PX = 0$, we obtain $Rss_e^2 = (Y^*)^T M^T MY^*$.

As before, suppose that $E(Y^*) = X\alpha(u)$, then

$$E(Rss_e^2) = E((Y^*)^T M^T MY^*) = tr(M^T Mcov(Y^*)) = \sigma^2 tr(\Sigma^{\frac{-1}{2}}M^T M\Sigma^{\frac{-1}{2}}).$$

Theorem 3. *Considering model (2.2) $c \in R^p$,suppose that$EY^* = X\alpha(u)$, then*

$$\widehat{\sigma}^2 = tr[(\Sigma^{\frac{-1}{2}})MM^T\Sigma^{\frac{-1}{2}}]^{-1}(Y^*)^T M^T MY^*.$$

is the unbiased estimation of σ^2.

In order to determine the probability distribution of σ^2, we have to modify the construction of σ^2.

As before, suppose that $E(Y^*) = X\alpha(u)$ $|M| \neq 0$ and $e \sim N(0, \Sigma)$, then $\Sigma^{\frac{-1}{2}}e \sim N(0, I_n)$;

$$\begin{aligned}
Rss_{e^*}^2 =& [(M^T\Sigma M)^{\frac{-1}{2}}(Y^* - X\alpha(u))]^T[(M^T\Sigma M)^{\frac{-1}{2}}(Y^* - X\alpha(u))] \\
=& (Y^*)^T M^T(M\Sigma M^T)^{-1}MY^*, \\
E[Rss_{e^*}^2] =& E[Y^*)^T M^T(M\Sigma M^T)^{-1}MY^*] \\
=& E[e^T M^T(M\Sigma M^T)^{-1}Me] \\
=& E[(\Sigma^{\frac{-1}{2}}e)^T\Sigma^{\frac{1}{2}}M^T(M\Sigma M^T)^{-1}M\Sigma^{\frac{1}{2}}(\Sigma^{\frac{-1}{2}}e)] \\
=& \sigma^2 tr[(M\Sigma M^T)^{-1}(M\Sigma M^T)] \\
=& \sigma^2 tr(I_n) \\
=& n\sigma^2.
\end{aligned}$$

Then $\widehat{\sigma}^2 = \frac{1}{n}(Y^*)^T M^T(M\Sigma M^T)^{-1}MY^*$ is the unbiased estimation of σ^2.

But $\Sigma^{\frac{-1}{2}}e \sim N(0, I_n)$ and $\Sigma^{\frac{1}{2}}M^T(M\Sigma M^T)^{-1}M\Sigma^{\frac{1}{2}}$ is an idempotent and symmetric matrix, so

$$tr[\Sigma^{\frac{1}{2}}M^T(M\Sigma M^T)^{-1}M\Sigma^{\frac{1}{2}}] = rk[\Sigma^{\frac{1}{2}}M^T(M\Sigma M^T)^{-1}M\Sigma^{\frac{1}{2}}] = n.$$

Hence

$$\frac{\Sigma^{\frac{-1}{2}}}{\sigma}e \sim N(0, I_n).$$

Therefore

$$\begin{aligned}\frac{Rss^2_{e^*}}{\sigma^2} =& \frac{1}{\sigma^2}(Y^*)^T M^T (M\Sigma M^T)^{-1} MY^* \\ =& \frac{1}{\sigma^2}(e^T M^T (M\Sigma M^T)^{-1} Me \\ =& \frac{1}{\sigma^2}(\Sigma^{\frac{-1}{2}}e)^T \Sigma^{\frac{1}{2}} M^T (M\Sigma M^T)^{-1} M\Sigma^{\frac{1}{2}}(\Sigma^{\frac{-1}{2}}e).\end{aligned}$$

$\frac{Rss^2_{e^*}}{\sigma^2}$ follows the χ^2 distribution with n degree of freedom and $Var(Rss^2_{e^*}) = 2n\sigma^2$.

Theorem 4. *Consider model (2.1). Suppose that* $EY^* = X\alpha(u)$,

$$\widehat{\sigma}^2 = \frac{1}{n}(Y^*)^T M^T (M\Sigma M^T)^{-1} MY^*$$

is the unbiased estimation of σ^2. *If* $\frac{\Sigma^{\frac{-1}{2}}}{\sigma}e \sim N(0, I_n)$, *then we have*

$$\frac{Rss^2_{e^*}}{\sigma^2} = \frac{1}{\sigma^2}(Y^*)^T M^T (M\Sigma M^T)^{-1} MY^*$$

which follows the χ^2 *distribution with* n *degree of freedom and* $Var(\widehat{\sigma}^2) = \frac{2\sigma^4}{n}$.

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