ON PARAMETER ESTIMATION OF SEMI-VARYING COEFFICIENT MODELS WITH CORRELATED RANDOM ERRORS

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Abstract: This paper deals with estimation of semi-varying coefficient models with correlated random errors. The estimations of the function coefficients are given by the use of generalized weighted least squares. The theorem of Gauss-Markov was very well known, but moreover we present an estimation of parameter $\sigma^2$.

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1. Introduction

Fan et al (see [1]) put forward the semi-varying coefficient models when he studied whether the function coefficient of the varying coefficient really change,

\[ Y = X\alpha(U) + Z\beta + \varepsilon. \] (1.1)
Suppose that we have a response variable $Y$ and regressors $X = (X_1, X_2, \ldots, X_p)$, $Z = (Z_1, Z_2, \ldots, Z_q)$ as well as another variable $U$. $\varepsilon$ is a random error with $E(\varepsilon) = 0$ and it is independent of $(U, Z, X)$; $\beta = (\beta_1, \beta_2, \ldots, \beta_q)^T$ is a q-dimensional vector of an unknown parameters and $\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \ldots, \alpha_p(\cdot))^T$ is a p-dimensional vector of an unknown coefficient function.

Most researches have focused on the case in which random error are independently and identically distributed in a model.

In practice, sometimes the random errors are correlated, so it is been necessary to discuss the question.

In this paper, we discuss the semi-varying coefficient models with correlated random errors, which can be defined as follows:

$$
\begin{align*}
Y &= X \alpha(U) + Z \beta + \varepsilon, \\
e &\sim N(0, \sigma^2 \Sigma), \quad \Sigma = \text{diag}(\omega_1, \omega_2, \ldots, \omega_n).
\end{align*}
$$

(1.2)

2. Estimation

Suppose that we have a random sample of size $n$. $\{Y_r, X_r, Z_r, U_r\} (r = 1, 2, \ldots, n)$ is a sample from the model (1.2), $e \sim N(0, \sigma^2 \Sigma)$, $e = (e_1, e_2, \ldots, e_n)^T$, $\Sigma$ is a given positive matrices, (1.2) can be written as

$$
\begin{align*}
Y^* &= X \alpha(U) + e, \\
e &\sim N(0, \sigma^2 \Sigma).
\end{align*}
$$

(2.1)

where $Y^* = Y - Z \beta$, and $\beta$ known,

$$
X = (X_1^T, X_2^T, \ldots, X_p^T)^T, \quad X_r^T = (X_{r1}, X_{r2}, \ldots, X_{rp}),
$$

$$
Z = (Z_1^T, Z_2^T, \ldots, Z_q^T)^T, \quad Z_r^T = (Z_{r1}, Z_{r2}, \ldots, Z_{rq}).
$$

As $\Sigma$ is a given positive definite matrix, there must exist a orthogonal matrix. Hence, $\Sigma = P^T \Lambda P$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_r > 0$ ($r = 1, 2, \ldots, n$) are characteristic roots of $\Sigma$. Use $P^T$ left multiplied by (2.1).

So

$$
\begin{align*}
\tilde{Y} &= \tilde{X} \alpha(U) + \varepsilon, \\
e &\sim N(0, \sigma^2 I_n).
\end{align*}
$$

(2.2)

So (2.2) is the common varying coefficient model.
Let
\[ \tilde{Y} = (\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_n)^T = \Sigma^{-\frac{1}{2}} Y^*, \quad \overline{Y} = \Sigma^{-\frac{1}{2}} Y, \]
\[ \overline{Z} = \Sigma^{-\frac{1}{2}} Z, \quad \tilde{X} = (\tilde{X}_1^T, \tilde{X}_2^T, \ldots, \tilde{X}_n^T)^T = \Sigma^{-\frac{1}{2}} X, \quad \tilde{X}_r^T = (\tilde{X}_r, \tilde{X}_r, \ldots, \tilde{X}_r), \]
\[ \tilde{Z} = (\tilde{Z}_1^T, \tilde{Z}_2^T, \ldots, \tilde{Z}_n^T)^T, \quad \overline{Z}_r^T = (\overline{Z}_r, \overline{Z}_r, \ldots, \overline{Z}_r), \]
\[ \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^T = \Sigma^{-\frac{1}{2}} e, \quad W = \text{diag}(W_{11}, W_{22}, \ldots, W_{nn}) \]
\((r = 1, 2, \ldots, n)\). The coefficient function \(\alpha(\cdot) = (\alpha_1(\cdot), \alpha_2(\cdot), \ldots, \alpha_p(\cdot))^T\) is estimated by minimizing
\[ Q(\alpha(u)) = (\tilde{Y} - \tilde{X}_r\alpha(U))^T W(\tilde{Y} - \tilde{X}_r\alpha(U)). \quad \text{(2.3)} \]

According to the principle of weighted least squares, if we assume that the inverse of the matrix \(\tilde{X}_r^T W \tilde{X}_r\) exists for any \(u\), then the estimated coefficient function at \(u\) can be expressed as
\[ \hat{\alpha}(u) = (\hat{\alpha}_1(u), \hat{\alpha}_2(u), \ldots, \hat{\alpha}_p(u))^T \]
\[ = [X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X]^{-1} X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} Y^*. \quad \text{(2.4)} \]

Then (2.2) can be written as
\[ \overline{Y} - \overline{Z}\beta = M + \varepsilon. \quad \text{(2.5)} \]

The estimator for \(M\) is then
\[ \hat{M} = \tilde{X}[\tilde{X}_r^T W \tilde{X}_r]^{-1} \tilde{X}_r^T W (\overline{Y} - \overline{Z}\beta) = S(\overline{Y} - \overline{Z}\beta). \]

The matrix \(S\) is a smoothing matrix and dependent only on the observations \((U_r, X_r^T), \ r = 1, 2, \ldots, n\).

Substituting \(\hat{M}\) into (2.5), we obtain
\[ (I - S)\tilde{Y} = (I - S)\tilde{Z}\beta + \varepsilon. \quad \text{(2.6)} \]

Applying least squares to the linear model (2.6), we obtain
\[ \hat{\beta} = \{\overline{Z}^T (I - S)^T (I - S)\overline{Z}\}^{-1} \overline{Z}^T (I - S)^T (I - S)\overline{Y}. \]

Moreover
\[ \hat{M} = S(\overline{Y} - \overline{Z}\hat{\beta}). \quad \text{(2.7)} \]

On the basis of \(\hat{\alpha}(u)\), we inferred from \(\hat{\beta}\), hence \(C^T \beta\) is two-step estimation method of \(C^T \alpha(U)\), due to the complexity of local fitting, so it’s difficult to
analyze the statistical inference properties. Thus, we mainly focus on some properties of \( \hat{\alpha}(u) \).

Suppose that \( EY^* = X\alpha(u) \), then:

\[
E[\hat{\alpha}(u)] = [X^T\Sigma^{-1/2}W\Sigma^{-1/2}X]^{-1}X^T\Sigma^{-1/2}W\Sigma^{-1/2}X\alpha(u),
\]

\[
\text{Var}[\hat{\alpha}(u)] = \sigma^2[X^T\Sigma^{-1/2}W\Sigma^{-1/2}X]^{-1}[X^T\Sigma^{-1/2}W^2\Sigma^{-1/2}X][X^T\Sigma^{-1/2}W\Sigma^{-1/2}X].
\]

**Theorem 1.** Consider model (2.2), \( \tilde{\alpha}(u) \) is the generalized weighted least squares estimation

\[
\hat{\alpha}(u) = [X^T\Sigma^{-1/2}W\Sigma^{-1/2}X]^{-1}X^T\Sigma^{-1/2}W\Sigma^{-1/2}Y^*.
\]

Suppose that \( EY^* = X\alpha(u) \). Then:

(1) When \( X \) is a matrix with full column rank and the inverse of the matrix \( X^T\Sigma^{-1/2}W\Sigma^{-1/2}X \) exist, then \( \hat{\alpha}(u) \) must be unique.

\[
E[\hat{\alpha}(u)] = \alpha(u).
\]

Hence \( \hat{\alpha}(u) \) is an estimation of \( \alpha(u) \), and we called \( \hat{\alpha}(u) \) the weighted linear unbiased estimate of \( \alpha(u) \).

(2) When \( X \) is a matrix with full column rank and \( W \) is idempotent matrix, then

\[
E[\hat{\alpha}(u)] = \alpha(u).
\]

\[
\text{Var}[\hat{\alpha}(u)] = \sigma^2[X^T\Sigma^{-1/2}W\Sigma^{-1/2}X]^{-1}.
\]

(3) When \( X \) is a matrix with full column rank, \( W \) is idempotent matrix and \( \lim_{n \to \infty} (X^T\Sigma^{-1/2}W\Sigma^{-1/2}X)^{-1} = 0 \). Hence \( \hat{\alpha}(u) \) is the consistent estimate of \( \alpha(u) \).

### 3. Simulation

In this section, we use Matlab conduct some research to estimator.

Case 1. \( Y = \sin(15U)X_1 + 4U(1 - U)X_2 + 8Z_1 + \varepsilon \).

Case 2. \( Y = \cos(5\pi U)X_1 + \exp(4U)X_2 + 8Z_1 + \varepsilon \).
In these cases, where $U \sim N[0, 1]$, $X_1 \sim N[1, 3]$, $X_2 \sim N[5, 10^{-5}], Z_1 \sim N[0, 10^{-5}], \varepsilon \sim N(0, A)$, $A$ is a random number in $[0, 1.1]$. Using Epanechnikov kernels window width $h_n = \frac{1}{40}$, sample data randomly generated by Matlab. The sample sizes is 1000. In Case 1 and Case 2 we use * indicate estimated value, and use solid-line curve indicate actual value. Then, we found estimated results is fine.

The estimate of $\sin(15U)$

The estimate of $4U(1 - U)$.

The estimate of $\cos(5\pi U)$

The estimate of $\exp(4U)$.

4. The Estimation of the Linear Estimable Function and Optimality

**Theorem 2.** (The Generalized Weighted Gauss-Markov Theorem) In model (2.2), $\hat{\alpha}(u)$ is a generalized weighted least squares estimate, Suppose that $EY^* = X\alpha(u)$. Then:

1. $\hat{\alpha}(u)$ is the unique weighted linear unbiased estimation of $\alpha(u)$. 
(2) When $W$ is an idempotent matrix, then $\hat{\alpha}(u)$ is the unique best weighted linear unbiased estimation of $\alpha(u)$.

(3) When $X$ is a matrix with full row rank, $W$ is an idempotent matrix and
\[ \lim_{n \to \infty} c^T \left( X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X \right)^{-1} c = 0. \] Then $c^T \hat{\alpha}(u)$ is the consistent estimate of $c^T \alpha(u)$.

In order to prove Theorem 2, we give some definitions and lemmas are defined in this paper.

**Definition 1.** (see [2]) In model (2.2), $c \in \mathbb{R}^p$, linear function $\alpha(u)$ is weighting estimation. If we assume that $b^T W \tilde{Y}$ is the weighted linear unbiased estimation of $c^T \alpha(u)$ and $b^T W \tilde{Y}$ exists, $b \in \mathbb{R}^n$, then
\[ E(b^T W \tilde{Y}) = c^T \alpha(u), \]
for any $\alpha(u)$.

**Lemma 1.** (see [2]) $c^T \alpha(u)$ is the weighting estimation $\iff c^T \in R(\tilde{X}^TW \tilde{X}) = R(\tilde{X}^TW) = R(\tilde{X}^T)$.

**Lemma 2.** (see [2]) If $c^T \alpha(u)$ is the weighting estimation, then $c^T \hat{\alpha}(u)$ must be unique. Since $b \in \mathbb{R}_n$, we have
\[ c^T \hat{\alpha}(u) = b^T \tilde{X}^T W \tilde{Y}. \]

**Definition 2.** (see [2]) Considering model(2.2) and $c \in \mathbb{R}^p, b \in \mathbb{R}^n$, so $b^T W \tilde{Y}$ is the best weighted linear unbiased estimate.

If:

1. $b^T W \tilde{Y}$ is the weighted linear unbiased estimate of $c^T \alpha(u)$.
2. Any $g^T W \tilde{Y}$ is the weighted linear unbiased estimate of $c^T \alpha(u)$, we have
\[ V ar(b^T W \tilde{Y}) \leq V ar(g^T W \tilde{Y}). \]

**Proof of Theorem 2.** (1) $c^T \alpha(u)$ is the weighting estimation. By Lemma 1, $c \in R(\tilde{X}^T)$ and $b \in \mathbb{R}^n$.

We have
\[ c = \tilde{X}^T b = X^T \Sigma^{-\frac{1}{2}} b. \]
then $c^T \hat{\alpha}(u) = b^T \tilde{X} (\tilde{X}^T W \tilde{X})^{-1} \tilde{X}^T \tilde{Y}$. so
\[ E(c^T \hat{\alpha}(u)) = b^T \Sigma^{-\frac{1}{2}} X \alpha(u). \]

By Lemma 2, we know $c^T \alpha(u)$ must be unique. So $\hat{\alpha}(u)$ is the unique weighted linear unbiased estimate of $\alpha(u)$. 
(2) \( W \) is an idempotent matrix, any \( g^T W \tilde{Y} \) is the weighted linear unbiased estimate of \( c^T \alpha(u) \). So
\[
c^T \alpha(u) = E(g^T W \tilde{Y}) = g^T W \tilde{X} \alpha(u),
\]
for any \( \alpha(u) \).
We have \( c^T = g^T W \tilde{X} \), then \( c = \tilde{X}^T W g \),
\[
\text{Var}(g^T W \tilde{Y}) - \text{Var}(c^T \alpha(u)) = \sigma^2 g^T W^{\frac{1}{2}} [I_n - W^{\frac{1}{2}} \Sigma^{-\frac{1}{2}} X (X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} X^T \Sigma^{-\frac{1}{2}} W^{\frac{1}{2}}] W^{\frac{1}{2}} g,
\]
where \( P_{W^{\frac{1}{2}}} = W^{\frac{1}{2}} \Sigma^{-\frac{1}{2}} X (X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} X^T \Sigma^{-\frac{1}{2}} W^{\frac{1}{2}} \) is an Projection matrix.
Therefore
\[
\text{Var}(g^T W \tilde{Y}) - \text{Var}(c^T \alpha(u)) = \sigma^2 g^T W^{\frac{1}{2}} [I_n - P_{W^{\frac{1}{2}}}] W^{\frac{1}{2}} g \geq 0. \tag{3.1}
\]
From Definition 2, we have that \( \tilde{\alpha}(u) \) is the weighted linear unbiased estimate of \( \alpha(u) \). The conditions of the holding water of sign of equality in (3.1) is \( (I_n - P_{W^{\frac{1}{2}}}) g = 0 \), then
\[
g^T W \tilde{Y} = g^T W \tilde{X} (\tilde{X}^T W \tilde{X})^{-1} \tilde{X}^T W \tilde{Y} = c^T \tilde{\alpha}(u).
\]
Hence \( \tilde{\alpha}(u) \) is the unique best weighted linear unbiased estimate of \( \alpha(u) \).

(3) We have \( E(c^T \tilde{\alpha}(u)) = c^T \alpha(u), \) \( X \) is a matrix with full column rank, \( W \) is idempotent matrix and \( \lim_{n \to \infty} c^T (X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} c = 0 \).
By Theorem 1, we have
\[
\text{Var}(c^T \tilde{\alpha}(u)) = c^T \text{Var}(\tilde{\alpha}(u)) c = \sigma^2 (X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} c \to 0 \quad (n \to \infty).
\]
Therefore, \( c^T \tilde{\alpha}(u) \) is the consistent estimate of \( c^T \alpha(u) \).

5. The Estimation of \( \sigma^2 \) and Optimality

Let us set:
\[
\text{Rss}_e^2 = (Y^* - X \alpha(u))^T (Y^* - X \alpha(u)) = \|Y^* - X \alpha(u)\|^2.
\]
\[
e = Y^* - X \alpha(u) = (I_n - X (X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} ) Y^*.
\]
If 
\[ P = X(X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} X)^{-1} X^T \Sigma^{-\frac{1}{2}} W \Sigma^{-\frac{1}{2}} \]
and 
\[ P^2 = P, \]
then
\[ e = (I_n - P)Y*. \]

Setting \( M = I_n - P \), \( MX = X - PX = 0 \), we obtain \( \text{Rss}_e^2 = (Y^*)^T M^T MY*. \)
As before, suppose that \( E(Y^*) = X\alpha(u) \), then
\[ E(\text{Rss}_e^2) = E((Y^*)^T M^T MY^*) = tr(M^T M \text{cov}(Y^*)) = \sigma^2 tr(\Sigma^{-\frac{1}{2}} M^T \Sigma \Sigma^{-\frac{1}{2}}). \]

**Theorem 3.** Considering model (2.2) \( c \in R^p \), suppose that \( EY^* = X\alpha(u) \), then
\[ \hat{\sigma}^2 = tr[(\Sigma^{-\frac{1}{2}}) M M^T \Sigma^{-\frac{1}{2}}]^{-1} (Y^*)^T M^T MY^*. \]
is the unbiased estimation of \( \sigma^2 \).

In order to determine the probability distribution of \( \sigma^2 \), we have to modify the construction of \( \sigma^2 \).
As before, suppose that \( E(Y^*) = X\alpha(u) \mid M \neq 0 \) and \( e \sim N(0, \Sigma) \), then \( \Sigma^{-\frac{1}{2}} e \sim N(0, I_n) \);
\[ \text{Rss}_e^2 = [((M^T \Sigma M)^{-\frac{1}{2}} (Y^* - X\alpha(u)))^T [(M^T \Sigma M)^{-\frac{1}{2}} (Y^* - X\alpha(u))] \]
\[ = (Y^*)^T M^T (M \Sigma M^T)^{-1} MY^*; \]
\[ E[\text{Rss}_e^2] = E[(Y^*)^T M^T (M \Sigma M^T)^{-1} MY^*] \]
\[ = E[e^T M^T (M \Sigma M^T)^{-1} Me] \]
\[ = E[(\Sigma^{-\frac{1}{2}} e)^T \Sigma^\frac{1}{2} M^T (M \Sigma M^T)^{-1} M \Sigma^\frac{1}{2} (\Sigma^{-\frac{1}{2}} e) ] \]
\[ = \sigma^2 tr[(M \Sigma M^T)^{-1} (M \Sigma M^T) ] \]
\[ = \sigma^2 tr(I_n) \]
\[ = n\sigma^2. \]

Then \( \hat{\sigma}^2 = \frac{1}{n}(Y^*)^T M^T (M \Sigma M^T)^{-1} MY^* \) is the unbiased estimation of \( \sigma^2 \).
But \( \Sigma^{-\frac{1}{2}} e \sim N(0, I_n) \) and \( \Sigma^\frac{1}{2} M^T (M \Sigma M^T)^{-1} M \Sigma^\frac{1}{2} \) is an idempotent and symmetric matrix, so
\[ tr[\Sigma^\frac{1}{2} M^T (M \Sigma M^T)^{-1} M \Sigma^\frac{1}{2}] = rk[\Sigma^\frac{1}{2} M^T (M \Sigma M^T)^{-1} M \Sigma^\frac{1}{2}] = n. \]
Hence
\[ \frac{\Sigma^{-\frac{1}{2}}}{\sigma} e \sim N(0, I_n). \]

Therefore
\[
\frac{Rss^{2}_e}{\sigma^2} = \frac{1}{\sigma^2} (Y^*)^T M^T (M \Sigma M^T)^{-1} M e = \frac{1}{\sigma^2} (\Sigma^{-\frac{1}{2}} e)^T \Sigma^{- \frac{1}{2}} M^T (M \Sigma M^T)^{-1} M \Sigma^{- \frac{1}{2}} (\Sigma^{-\frac{1}{2}} e).
\]

\( \frac{Rss^{2}_e}{\sigma^2} \) follows the \( \chi^2 \) distribution with \( n \) degree of freedom and \( \text{Var}(\frac{Rss^{2}_e}{\sigma^2}) = 2\sigma^4 n \).

**Theorem 4.** Consider model (2.1). Suppose that \( EY^* = X\alpha(u) \),
\[
\hat{\sigma}^2 = \frac{1}{n} (Y^*)^T M^T (M \Sigma M^T)^{-1} M Y^*
\]
is the unbiased estimation of \( \sigma^2 \). If \( \frac{\Sigma^{-\frac{1}{2}}}{\sigma} e \sim N(0, I_n) \), then we have
\[
\frac{Rss^{2}_e}{\sigma^2} = \frac{1}{\sigma^2} (Y^*)^T M^T (M \Sigma M^T)^{-1} M Y^*
\]
which follows the \( \chi^2 \) distribution with \( n \) degree of freedom and \( \text{Var}(\hat{\sigma}^2) = \frac{2\sigma^4}{n} \).

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**References**


