MONOIDS OF S-H FUZZY PARTITIONS OF A SET

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Abstract: This research note constructs S-H fuzzy collections and S-H fuzzy partitions of a finite set. The main objective of the paper lies in defining two different operations on a class of S-H fuzzy partitions of a set and, in turn, proving that these give rise to a monoid.

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1. Introduction

Zadeh [1] is known to be the pioneering work on extending the concepts of hard (i.e non-fuzzy) partitions and equivalence relations defined for finite sets to fuzzy partitions and similarity relations. Concurrently as well as subsequently, this endeavour of Zadeh attracted serious attention from a number of

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researchers, particularly [2], which also contains most extant related references. The most intriguing problem has been to justify \textit{transitivity} fragment of similarity relation in the context of fuzzy sets. In the recent years, some works such as [3, 4] have appeared which describe fuzzy equivalence relations and partitions at par with their hard counterparts, and name it \textit{S-H fuzzy partition}.

It is known that some important applications of the concept of \textit{equivalence}, \textit{similarity}, \textit{ordering}, etc., defined for fuzzy sets, have been found in both pure and applied mathematics [1]. In particular, in view of the applications of certain \textit{monoids} of partitions of a set in the areas of computer arithmetic, formal languages, and sequential machine [5], the objective of this paper centres at describing certain monoids of S-H fuzzy partitions of a set.

\section*{2. Preliminaries}

\textbf{Definition 1.} (Fuzzy set [6]) A fuzzy set $\tilde{A}$ in a nonempty universe set $X$ is a function from $X$ into $[0,1]$. Let $\tilde{A}(x)$ denote the degree of $x$ in $\tilde{A}$. The fuzzy set $\tilde{A}$ is said to be contained in a fuzzy set $\tilde{B}$ if and only if $\tilde{A}(x) \leq \tilde{B}(x), \forall x \in X$. The union of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \cup \tilde{B}$, is defined by $(\tilde{A} \cup \tilde{B})(x) = \max [\tilde{A}(x), \tilde{B}(x)], \forall x \in X$. The intersection of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \cap \tilde{B}$, is defined by $(\tilde{A} \cap \tilde{B})(x) = \min [\tilde{A}(x), \tilde{B}(x)], \forall x \in X$. The compliment of $\tilde{A}$, denoted $\tilde{A}'$, defined by $\tilde{A}'(x) = 1 - \tilde{A}(x), \forall x \in X$.

\textbf{Definition 2.} (weak-separated fuzzy subsets [7]) Let $\tilde{F}$ be a collection of fuzzy subset of a nonempty set $X$. $\tilde{A}, \tilde{B} \in \tilde{F}$ with $\tilde{A} \neq \tilde{B}$. If $\mu_{\tilde{A} \cap \tilde{B}}(x) < 0.5, \forall x \in X$, then $\tilde{A}$ and $\tilde{B}$ are called weak-separated fuzzy subsets.

\textbf{Definition 3.} (Fuzzy Partition [3]) Let $X$ be a nonempty set. A fuzzy partition $\tilde{T}$ of $X$ is a set of nonempty fuzzy subsets of $X$ such that

(i) If $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$, and

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$.

\textbf{Definition 4.} (Weakly Empty Fuzzy Subset, [3]) Let $X$ be a nonempty set and $\tilde{A}$ be a fuzzy subset of $X$. $\tilde{A}$ is called a weakly empty fuzzy subset of $X$ if $\mu_{\tilde{A}}(x) < 0.5, \forall x \in X$.

\textbf{Definition 5.} (Non-weakly empty fuzzy subset) A fuzzy subset of $X$ is called \textit{non-weakly empty fuzzy subset} if it is not weakly empty.
Definition 6. (S-H Fuzzy Collection [4]) Let $\tilde{F}$ be a collection of fuzzy subsets of a nonempty set $X$. $\tilde{F}$ is called a S-H collection if and only if $\tilde{B} \cap \tilde{A} \leq \tilde{B}(a)$, whenever $\tilde{A}, \tilde{B} \in \tilde{F}$ such that $\tilde{A}(a) = 1$.

Example 7. Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ and let $\tilde{F} = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\}$ be a collection of fuzzy subsets of $X$ where

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.9), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$
$$\tilde{A}_2 = \{(x_0, 0.9), (x_1, 1.0), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$
$$\tilde{A}_3 = \{(x_0, 0.8), (x_1, 0.8), (x_2, 1.0), (x_3, 0.7), (x_4, 0.1)\},$$
$$\tilde{A}_4 = \{(x_0, 0.7), (x_1, 0.7), (x_2, 0.7), (x_3, 1.0), (x_4, 0.1)\},$$
and
$$\tilde{A}_5 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 0.1), (x_4, 0.1)\}.$$

It is immediate to see that $\tilde{A}_i \cap \tilde{A}_j(x) \leq \tilde{A}_j(x)$, where $\tilde{A}_i(x) = 1$, $i = \overline{1, 4}$ and hence, $\tilde{F}$ is a S-H collection. Moreover, $\tilde{A}_5$ is a weak empty fuzzy set, and

$$\bigcup \tilde{A}_i = \{(x_0, 1), (x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0.1) \neq X.$$  

Also $\mu_{\tilde{A}_1 \cap \tilde{A}_2}(x_0) = 0.9 > 0.5$, and similarly for other such combinations.

It is to be noted that S-H collections may contain both weakly empty and non-weakly empty fuzzy subsets of a given set.

Definition 8. (S-H Fuzzy Partition [4]) Let $X$ be a nonempty set. An S-H fuzzy partition $\tilde{T}$ of $X$ is defined as a set of non-weakly empty fuzzy subsets of $X$ such that

(i) if $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$ i.e., $\tilde{A}, \tilde{B}$ are weak separated fuzzy subsets,

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$, and

(iii) $\tilde{T}$ is S-H collection.
3. Monoids of S-H Fuzzy Partitions of a Set

Let $\tilde{T}$ be a S-H fuzzy partition of a set $X$ and, let $\tilde{A}_i (i = 1, n)$ denote the blocks of the partition $\tilde{T}$. Let $\prod(X)$ denote the collection of all S-H fuzzy partitions of $X$.

We define a binary operation $\ast$ on $\prod(X)$ as follows:

Given $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, let $\tilde{T}_1 \ast \tilde{T}_2$ be the fuzzy set consisting of all nonempty intersections of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, viz.,

$$\tilde{T}_1 \ast \tilde{T}_2 = \{(\tilde{A}_i \cap \tilde{B}_j), i, j = 1, n\}$$

where $\tilde{T}_1 = \{\tilde{A}_i\}, \tilde{T}_2 = \{\tilde{B}_j\}, \tilde{X} = \{\tilde{x}_i\}$ and $\tilde{A}_i, \tilde{B}_j, i, j = 1, n$, are given by $\{(x_i, \mu_{\tilde{A}_i}(x_i))\}$ and $\{(x_i, \mu_{\tilde{B}_j}(x_i))\}$, $\forall x_i \in X$, respectively.

It is immediate to see that the operation $\ast$ on $\prod(X)$ is both associative and commutative, since the operation $\cap$ on fuzzy sets is both associative and commutative and also every element $\tilde{T}_k \in \prod(X)$ is idempotent with respect to $\ast$ i.e., $\tilde{T}_k \ast \tilde{T}_k = \tilde{T}_k$. Moreover, the identity element with respect to $\ast$ is the S-H fuzzy partition consisting of a single block. Thus $(\prod(X), \ast)$ is a commutative, idempotent monoid.

Similarly, if we define a binary operation $\circ$ on $\prod(X)$ such that every resulting fuzzy set consists of all non-empty union of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, for all $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, then $(\prod(X), \circ)$ also gives rise to a commutative, idempotent monoid with the partition consisting of singleton blocks as the identity element.

**Example 9.** Let $X = \{x_0, x_1, x_2, x_3\}$, $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$ such that $\tilde{T}_1 = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}$ and $\tilde{T}_2 = \{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4\}$, where $\tilde{A}_i, \tilde{B}_i, i = 1, 4$, are as given below:

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.4), (x_2, 0.3), (x_3, 0.2)\}$$
$$\tilde{A}_2 = \{(x_0, 0.4), (x_1, 1.0), (x_2, 0.3), (x_3, 0.2)\}$$
$$\tilde{A}_3 = \{(x_0, 0.3), (x_1, 0.3), (x_2, 1.0), (x_3, 0.2)\}$$
$$\tilde{A}_4 = \{(x_0, 0.2), (x_1, 0.2), (x_2, 0.2), (x_3, 1.0)\}$$

and

$$\tilde{B}_1 = \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\}$$
$$\tilde{B}_2 = \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\}$$
$$\tilde{B}_3 = \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\}$$
$$\tilde{B}_4 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}$$
It is easy to verify that $\tilde{T}_1$ and $\tilde{T}_2$ are S-H fuzzy partitions of $X$.

Note that, $\hat{A}_i \cap \hat{B}_j$, for $i \neq j$ is weakly empty, for example,

$$\hat{A}_1 \cap \hat{B}_2 = \{(x_0, 0.3), (x_1, 0.4), (x_2, 0.2), (x_3, 0.1)\}.$$  

We have, following the definition given above, for example,

$$\tilde{T}_3 = \tilde{T}_1 \ast \tilde{T}_2 = \{\hat{A}_1 \cap \hat{B}_1, \hat{A}_2 \cap \hat{B}_2, \hat{A}_3 \cap \hat{B}_3, \hat{A}_4 \cap \hat{B}_4\} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$$

where

$$\tilde{C}_1 = \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\},$$

$$\tilde{C}_2 = \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\},$$

$$\tilde{C}_3 = \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\},$$

and

$$\tilde{C}_4 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}.$$  

Also, $\tilde{T}_1 \ast \tilde{T}_1 = \{\hat{A}_1 \cap \hat{A}_1, \hat{A}_2 \cap \hat{A}_2, \hat{A}_3 \cap \hat{A}_3, \hat{A}_4 \cap \hat{A}_4\} = \tilde{T}_1$; $\tilde{T}_1 \ast \tilde{T}_2 = \tilde{T}_2 \ast \tilde{T}_1$, etc. Similarly, results for various other combinations could be computed.

Thus $(\prod(X), \ast)$ is a commutative, idempotent monoid with $\hat{X} = \{\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3\}$, as the identity element.

It is interesting to see that another operation, denoted $\oplus$, can be defined on $\prod(X)$ such that $(\prod(X), \oplus)$ is also a monoid. We define $\oplus$ as follows: Let $X$ be a set and $\tilde{T}_1 = \{\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{A}_4\}$ and $\tilde{T}_2 = \{\hat{B}_1, \hat{B}_2, \hat{B}_3, \hat{B}_4\}$ be two S-H fuzzy partitions of $X$. A fuzzy subset $\tilde{T}$ of $X$ belongs to $\tilde{T}_1 \oplus \tilde{T}_2$ if:

i) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_1$,

ii) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_2$,

iii) No fuzzy subset of $\tilde{T}$ satisfies i) and ii) except $\tilde{T}$ itself.

It follows that $\oplus$ is both associative and commutative. The fuzzy partition consisting of single elements of $X$ is the identity of the operation $\oplus$ on $\prod(X)$. For example, $\{\hat{x}_0, \hat{x}_1, \hat{x}_2, \hat{x}_3\}$ is the identity with respect to $\oplus$ on $\prod(X)$ in the example considered above. On the same lines, the algebraic structure defined by the other operation described above, could be illustrated.

### 3.1. Concluding Remark

This research note is a contribution towards developing a fragment of fuzzy algebras that could be applied in the areas in which its counterpart in set algebras has been applied [5]. We wish to indicate that developing a variant of extant algorithms (See [8], for related reference) to compute monoidal structures, particularly involving complex objects such as fuzzy partitions, could be a challenging problem.
References


