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MONOIDS OF S-H FUZZY PARTITIONS OF A SET

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Abstract: This research note constructs S-H fuzzy collections and S-H fuzzy partitions of a finite set. The main objective of the paper lies in defining two different operations on a class of S-H fuzzy partitions of a set and, in turn, proving that these give rise to a *monoid*.

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1. Introduction

Zadeh [1] is known to be the pioneering work on extending the concepts of *hard(i.e non-fuzzy)* partitions and *equivalence* relations defined for finite sets to *fuzzy* partitions and *similarity* relations. Concurrently as well as subsequently, this endeavour of Zadeh attracted serious attention from a number of

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researchers, particularly [2], which also contains most extant related references. The most intriguing problem has been to justify *transitivity* fragment of similarity relation in the context of fuzzy sets. In the recent years, some works such as [3, 4] have appeared which describe fuzzy equivalence relations and partitions at par with their hard counterparts, and name it *S-H fuzzy partition*.

It is known that some important applications of the concept of *equivalence, similarity, ordering*, etc., defined for fuzzy sets, have been found in both pure and applied mathematics [1]. In particular, in view of the applications of certain *monoids* of partitions of a set in the areas of computer arithmetic, formal languages, and sequential machine [5],the objective of this paper centres at describing certain monoids of S-H fuzzy patitions of a set.

2. Preliminaries

Definition 1. (Fuzzy set [6]) A fuzzy set $\tilde{A}$ in a nonempty universe set X is a function from X into $[0, 1]$. Let $\tilde{A}(x)$ denote the degree of x in $\tilde{A}$. The fuzzy set $\tilde{A}$ is said to be contained in a fuzy set $\tilde{B}$ if and only if $\tilde{A}(x) \leq \tilde{B}(x)$, $\forall x \in X$. The union of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \cup \tilde{B}$, is defined by $(\tilde{A} \cup \tilde{B})(x) = max[\tilde{A}(x), \tilde{B}(x)]$, $\forall x \in X$. The intersection of $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A} \cap \tilde{B}$, is defined by $(\tilde{A} \cap \tilde{B})(x) = min[\tilde{A}(x), \tilde{B}(x)$, $\forall x \in X$. The compliment of $\tilde{A}$, denoted $\tilde{A}'$, defined by $\tilde{A}'(x) = 1 - \tilde{A}(x)]$, $\forall x \in X$.

Definition 2. (weak-separated fuzzy subsets [7]) Let $\tilde{F}$ be a collection of fuzzy subset of a nonempty set X. $\tilde{A}, \tilde{B} \in \tilde{F}$ with $\tilde{A} \neq \tilde{B}$. If $\mu_{\tilde{A} \cap \tilde{B}}(x) < 0.5$, $\forall x \in X$, then $\tilde{A}$ and $\tilde{B}$ are called weak-separated fuzzy subsets.

Definition 3. (Fuzzy Partition [3]) Let X be a nonempty set. A fuzzy partition $\tilde{T}$ of X is a set of nonempty fuzzy subsets of X such that

(i) If $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$, and

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$.

Definition 4. (Weakly Empty Fuzzy Subset, [3]) Let X be a nonempty set and $\tilde{A}$ be a fuzzy subset of X. $\tilde{A}$ is called a weakly empty fuzzy subset of X if $\mu_{\tilde{A}}(x) < 0.5$, $\forall x \in X$.

Definition 5. (Non-weakly empty fuzzy subset) A fuzzy subset of X is called *non-weakly empty fuzzy subset* if it is not weakly empty.

Definition 6. (S-H Fuzzy Collection [4]) Let $\tilde{F}$ be a collection of fuzzy subsets of a nonempty set X. $\tilde{F}$ is called a S-H collection if and only if $\tilde{B} \cap \tilde{A} \leq \tilde{B}(a)$, whenever $\tilde{A}, \tilde{B} \in \tilde{F}$ such that $\tilde{A}(a) = 1$.

Example 7. Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ and let $\tilde{F} = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\}$ be a collection of fuzzy subsets of X where

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.9), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_2 = \{(x_0, 0.9), (x_1, 1.0), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_3 = \{(x_0, 0.8), (x_1, 0.8), (x_2, 1.0), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_4 = \{(x_0, 0.7), (x_1, 0.7), (x_2, 0.7), (x_3, 1.0), (x_4, 0.1)\}, \text{and}$$

$$\tilde{A}_5 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 0.1), (x_4, 0.1)\}.$$

It is immediate to see that $\tilde{A}_i \cap \tilde{A}_j(x) \leq \tilde{A}_j(x)$, where $\tilde{A}_i(x) = 1$, $i = \overline{1, 4}$ and hence, $\tilde{F}$ is a S-H collection. Moreover, $\tilde{A}_5$ is a weak empty fuzzy set, and

$$\bigcup \tilde{A}_i = \{(x_0, 1), (x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0.1) \neq X.$$

Also $\mu_{\tilde{A}_1 \cap \tilde{A}_2}(x_0) = 0.9 > 0.5$, and similarly for other such combinations.

It is to be noted that S-H collections may contain both weakly empty and non-weakly empty fuzzy subsets of a given set.

Definition 8. (S-H Fuzzy Partition [4]) Let X be a nonempty set. An S-H fuzzy partition $\tilde{T}$ of X is defined as a set of non- weakly empty fuzzy subsets of X such that

(i) if $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$ i.e., $\tilde{A}, \tilde{B}$ are weak separated fuzzy subsets,

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$, and

(iii) $\tilde{T}$ is S-H collection.

3. Monoids of S-H Fuzzy Partitions of a Set

Let $\tilde{T}$ be a S-H fuzzy partition of a set X and, let $\tilde{A}_i$ $(i = \overline{1,n})$ denote the blocks of the partition $\tilde{T}$. Let $\prod(X)$ denote the collection of all S-H fuzzy partitions of X.

We define a binary operation $*$ on $\prod(X)$ as follows:

Given $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, let $\tilde{T}_1 * \tilde{T}_2$ be the fuzzy set consisting of all nonempty intersections of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, viz.,

$$\tilde{T}_1 * \tilde{T}_2 = \{\{\tilde{A}_i \cap \tilde{B}_j\}, i, j = \overline{1,n}\}$$

where $\tilde{T}_1 = \{\tilde{A}_i\}, \tilde{T}_2 = \{\tilde{B}_j\}, \tilde{X} = \{\tilde{x}_i\}$ and $\tilde{A}_i, \tilde{B}_j, i, j = \overline{1,n}$, are given by $\{(x_i, \mu_{\tilde{A}_i}(x_i))\}$ and $\{(x_i, \mu_{\tilde{B}_j}(x_i))\}, \forall x_i \in X$, respectively.

It is immediate to see that the operation $*$ on $\prod(X)$ is both associative and commutative, since the operation $\cap$ on fuzzy sets is both associative and commutative and also every element $\tilde{T}_k \in \prod(X)$ is idempotent with respect to $*$ i.e., $\tilde{T}_k * \tilde{T}_k = \tilde{T}_k$. Moreover, the identity element with respect to $*$ is the S-H fuzzy partition consisting of a single block. Thus $(\prod(X), *)$ is a commutative, idempotent monoid.

Similarly, if we define a binary operation $\circ$ on $\prod(X)$ such that every resulting fuzzy set consists of all non-empty union of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, for all $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, , then $(\prod(X), \circ)$ also gives rise to a commutative, idempotent monoid with the partition consisting of singleton blocks as the identity element.

Example 9. Let $X = \{x_0, x_1, x_2, x_3\}$, $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$ such that $\tilde{T}_1 = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}$ and $\tilde{T}_2 = \{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4\}$, where $\tilde{A}_i, \tilde{B}_i$, $i = \overline{1,4}$, are as given below:

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.4), (x_2, 0.3), (x_3, 0.2)\}$$

$$\tilde{A}_2 = \{(x_0, 0.4), (x_1, 1.0), (x_2, 0.3), (x_3, 0.2)\}$$

$$\tilde{A}_3 = \{(x_0, 0.3), (x_1, 0.3), (x_2, 1.0), (x_3, 0.2)\}$$

$$\tilde{A}_4 = \{(x_0, 0.2), (x_1, 0.2), (x_2, 0.2), (x_3, 1.0)\}$$

and

$$\tilde{B}_1 = \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\}$$

$$\tilde{B}_2 = \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\}$$

$$\tilde{B}_3 = \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\}$$

$$\tilde{B}_4 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}$$

It is easy to verify that $\tilde{T}_1$ and $\tilde{T}_2$ are S-H fuzzy partitions of X.

Note that, $\tilde{A}_i \cap \tilde{B}_j$, for $i \neq j$ is weakly empty, for example,

$$\tilde{A}_1 \cap \tilde{B}_2 = \{(x_0, 0.3), (x_1, 0.4), (x_2, 0.2), (x_3, 0.1)\}.$$

We have, following the definition given above, for example,

$$\tilde{T}_3 = \tilde{T}_1 * \tilde{T}_2 = \{\tilde{A}_1 \cap \tilde{B}_1, \tilde{A}_2 \cap \tilde{B}_2, \tilde{A}_3 \cap \tilde{B}_3, \tilde{A}_4 \cap \tilde{B}_4\} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$$

where

$$\begin{aligned}
\tilde{C}_1 &= \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\}, \\
\tilde{C}_2 &= \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\}, \\
\tilde{C}_3 &= \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\}, \text{and} \\
\tilde{C}_4 &= \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}.
\end{aligned}$$

Also, $\tilde{T}_1 * \tilde{T}_1 = \{\tilde{A}_1 \cap \tilde{A}_1, \tilde{A}_2 \cap \tilde{A}_2, \tilde{A}_3 \cap \tilde{A}_3, \tilde{A}_4 \cap \tilde{A}_4\} = \tilde{T}_1$; $\tilde{T}_1 * \tilde{T}_2 = \tilde{T}_2 * \tilde{T}_1$, etc. Similarly, results for various other combinations could be computed.

Thus $(\prod(X), *)$ is a commutative, idempotent monoid with $\tilde{X} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$, as the identity element.

It is intresting to see that another operation, denoted $\oplus$, can be defined on $\prod(X)$ such that $(\prod(X), \oplus)$ is also a monoid. We define $\oplus$ as follows: Let X be a set and $\tilde{T}_1 = \left\{\tilde{A}_1\ \tilde{A}_2, \tilde{A}_3\ \tilde{A}_4\right\}$ and $\tilde{T}_2 = \left\{\tilde{B}_1\ \tilde{B}_2, \tilde{B}_3\ \tilde{B}_4\right\}$ be two S-H fuzzy partitions of X. A fuzzy subset $\tilde{T}$ of X belongs to $\tilde{T}_1 \oplus \tilde{T}_2$ if:

i) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_1$,

ii) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_2$,

iii) No fuzzy subset of $\tilde{T}$ satisfies i) and ii) except $\tilde{T}$ itself.

It follows that $\oplus$ is both associative and commutative. The fuzzy partition consisting of single elements of X is the identity of the operation $\oplus$ on $\prod(X)$. For example, $\{\tilde{x_0}, \tilde{x_1}, \tilde{x_2}, \tilde{x_3}\}$ is the identity with respect to $\oplus$ on $\prod(X)$ in the example considered above. On the same lines, the algebraic structure defined by the other operation described above, could be illustrated.

3.1. Concluding Remark

This research note is a contribution towards developing a fragment of fuzzy algebras that could be applied in the areas in which its counterpart in set algebras has been applied [5]. We wish to indicate that developing a variant of extant algorithms (See [8], for related reference) to compute monoidal structures, particularly involving complex objects such as fuzzy partitions, could be a challenging problem.

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