ON P-SEMISIMPLE IN KK-ALGEBRAS

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Abstract: In the present paper, we introduce the notion of \(p\)-semisimple in KK-algebras and investigate some related properties. We also give some necessary and sufficient conditions under which a branchwise commutative KK-algebra is also commutative.

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1. Introduction and Preliminaries

In 1992, E.H. Roh et al. [3] studied the \(p\)-semisimple in BCI-algebras and obtained some related properties. In 2012, S. Asawasamrit and A. Sudprasert [2] introduced a new algebraic structure called a KK-algebra and described the relationships between ideals and congruences in this algebra. Furthermore, they defined a quotient KK-algebra and studied its properties. In this paper, we extend the notion of \(p\)-semisimple in BCI-algebras to KK-algebras, and investigate its properties.
We will now summarize some of the results from [2], that we require to prove the results in this paper.

A **KK-algebra** was defined as an algebra \((X, \ast, 0)\) with a binary operation \(\ast\) and a nullary element 0 such that for all \(x, y, z \in X\), the following properties are satisfied:

(KK-1) \((x \ast y) \ast ((y \ast z) \ast (x \ast z)) = 0;\)

(KK-2) \(0 \ast x = x;\)

(KK-3) \(x \ast y = 0\) and \(y \ast x = 0\) if and only if \(x = y.\)

As an example, let \(\ast\) be defined on an abelian group \(G\) by letting \(x \ast y = x^{-1} \cdot y,\) where \(x, y \in G,\) \(\cdot\) is the multiplication operator in \(G,\) and \(e\) is the unit element of \(G.\) Then \((G, \cdot, e)\) is a KK-algebra.

For a KK-algebra \((X, \ast, 0),\) we introduced in [2] a binary relation \(\leq\) on \(X\) by defining \(x \leq y\) if and only if \(y \ast x = 0\) and proved that \((X, \leq)\) is a partially ordered set. In the remainder of this paper, we will usually abbreviate \((X, \ast, 0)\) to \(X.\) It was then proved in [2] that the following properties are true for a KK-algebra. For any \(x, y, z \in X:\)

(P-1) \(x \ast ((x \ast y) \ast y) = 0;\)

(P-2) \(x \ast x = 0;\)

(P-3) \(x \ast (y \ast z) = y \ast (x \ast z);\)

(P-4) \(((x \ast y) \ast y) \ast y = x \ast y;\)

(P-5) \((x \ast y) \ast 0 = (x \ast 0) \ast (y \ast 0);\)

(P-6) \((x \ast y) \ast ((z \ast x) \ast (z \ast y))) = 0;\)

(P-7) If \(x \leq y\) then \(y \ast z \leq x \ast z;\)

(P-8) If \(x \leq y\) then \(z \ast x \leq z \ast y.\)

In [2], a **closed subset** \(A\) of a KK-algebra \(X\) was defined as a set \(A\) such that if \(x, y \in A\) then \(x \ast y \in A.\) Further, a non-empty subset \(A\) of a KK-algebra \(X\) was defined to be an **ideal** of \(X\) if it satisfies the following conditions:

(I-1) \(0 \in A.\)

(I-2) For any \(x, y \in X,\) if \(x \ast y \in A\) and \(x \in A,\) then \(y \in A.\)
In [2], we also defined $x$ to be a positive element of $X$ if $x \ast 0 = 0$, that is if $0 \leq x$. From this definition, the element 0 of $X$ is positive. Note that if $x$ is any element in a KK-algebra $X$, then $((x \ast 0) \ast 0) \ast x$ is a positive element of $X$ for every $x \in X$.

2. $p$-Semisimple KK-Algebras

In this section, we define a $p$-semisimple in a KK-algebra and prove some of its basic properties.

**Definition 2.1** A KK-algebra $X$ is called $p$-semisimple if $(x \ast 0) \ast 0 = x$ for all $x \in X$.

In [2], we defined an element $a$ in a KK-algebra $X$ to be a minimal element if $a \ast x = 0$ (i.e. $x \leq a$) implied $x = a$ for all $x \in X$.

**Proposition 2.2** The following conditions are equivalent for a KK-algebra $X$:

1. $a$ is a minimal element of $X$;
2. $(a \ast 0) \ast 0 = a$;
3. there is $x \in X$ such that $a = x \ast 0$;
4. for all $x \in X$, $x \ast a = (a \ast 0) \ast (x \ast 0)$;
5. for all $x \in X$, $x \ast a = (a \ast x) \ast 0$.

Let’s first give equivalent conditions of $p$-semisimplicity. From proposition 2.2, we have the following theorem.

**Theorem 2.3** Given a KK-algebra $X$, then the following conditions are equivalent:

1. $X$ is a $p$-semisimple;
2. every element $x$ in $X$ is minimal;
3. $X = \{x \ast 0 : x \in X\}$. 
Theorem 2.4 Given a KK-algebra $X$, then the following conditions are equivalent:

1. $X$ is a $p$-semisimple;
2. for any $x, y \in X$, $(x * 0) * y = (y * 0) * x$;
3. for any $x \in X$, $x * 0 = 0$ implies $x = 0$;
4. for any $a, x \in X$, $(x * a) * a = x$;
5. $X = \{x * a : x \in X\}$ for any $a \in X$.

Proof. Let $X$ be a KK-algebra and $a, x, y \in X$.

(1) $\Rightarrow$ (2). Let $X$ be a $p$-semisimple, by theorem 2.2, it follows that for any element in $X$ is minimal. Since $(y * 0) * x = (y * 0) * ((x * 0) * 0) = (x * 0) * ((y * 0) * 0) = (x * 0) y$.

(2) $\Rightarrow$ (3). Assume that $x * 0 = 0$, by (2) it follows that $x = 0 * x = (0 * 0) * x = (x * 0) * 0 = 0$, proving that $x = 0$.

(3) $\Rightarrow$ (1). Suppose that (3) holds, by proposition 2.12,$[2]$, $((x * 0) * 0) * x = 0$. Since $(x * 0) * 0$ is a minimal element of $X$, it follows that $x = (x * 0) * 0$. Therefore $X$ is a $p$-semisimple.

(1) $\Rightarrow$ (4). Suppose that $X$ be a $p$-semisimple. It follows that $a$ and $x$ are minimal elements of $X$. Since $(x * a) * a \leq x$, then $(x * a) * a = x$.

(4) $\Rightarrow$ (5). Suppose that (4) holds in $X$, then $x = (x * a) * a = y * a$, whenever $y = x * a$.

(5) $\Rightarrow$ (1). Obvious. \qed

Now, we give some examples and show some properties for a $p$-semisimple in KK-algebras.

Example 2.5 Suppose that $(G; \cdot, e)$ is an abelian group with $e$ is a unity element in $G$. Define a binary operation $*$ on $X$ by putting $x * y = x^{-1} \cdot y$. Then we get that $(G,*)$ is a KK-algebra. And since $(x * e) * e = (x^{-1} \cdot e)^{-1} \cdot e = (x^{-1})^{-1} = x$, for any $x \in X$. Hence $(G; *, e)$ is a $p$-semisimple algebra.

Example 2.6 Let $(X; *, 0)$ be a $p$-semisimple algebra. Define another binary operation $\cdot$ on $X$ as follow: $x \cdot y = (x * 0) \cdot y$. Then $(X; \cdot, 0)$ is an abelian group with $0$ as the unity element. In fact, by theorem 2.4, $x \cdot y = (x * 0) \cdot y = (y * 0) \cdot x = y \cdot x$, then the operation $\cdot$ satisfies the commutative law. Also since $x \cdot (y \cdot z) = (x * 0) \cdot ((y * 0) \cdot z) = (y * 0) \cdot ((x * 0) \cdot z) = y \cdot (x \cdot z)$, then the commutative law gives $(x \cdot y) \cdot z = z \cdot (x \cdot y) = x \cdot (z \cdot y) = x \cdot (y \cdot z)$. So, the operation
meets the associative law. Moreover, since \( x \cdot 0 = (x \cdot 0) \cdot 0 = x \), as \( X \) is a \( p \)-semisimple, so 0 is the unit element of \( X \). Finally, the inverse element of any element \( x \) in \( X \) is \( x \cdot 0 \). That is because \((x \cdot 0) \cdot x = (x \cdot 0) \cdot x = x \cdot x = 0 \)

Theorem 2.7 Given a KK-algebra \( X \), then the following conditions are equivalent:

1. \((x \cdot 0) \cdot 0 = x \) for any \( x \in X \);
2. \((x \cdot 0) \cdot y = (y \cdot 0) \cdot x \) for any \( x, y \in X \);
3. \( x \cdot 0 = 0 \) implies \( x = 0 \).

Proof. Let \( X \) be a KK-algebra and \( x, y \in X \).

1. \( \Rightarrow \) 2. Suppose that \( X \) has the property (1). It follows that \((x \cdot 0) \cdot y = (x \cdot 0) \cdot ((y \cdot 0) \cdot 0) = (y \cdot 0) \cdot ((x \cdot 0) \cdot 0) = (y \cdot 0) \cdot x\).

2. \( \Rightarrow \) 3. Suppose that \((x \cdot 0) \cdot y = (y \cdot 0) \cdot x \) such that \( x \cdot 0 = 0 \). Then \( x = (0 \cdot 0) \cdot x = (x \cdot 0) \cdot 0 = 0 \).

3. \( \Rightarrow \) 1. Suppose that (3) holds in \( X \), then \((x \cdot 0) \cdot (x \cdot 0) = 0 \). From (P-3), \( x \cdot ((x \cdot 0) \cdot 0) = 0 \), which means \((x \cdot 0) \cdot 0 \leq x \). And since \( 0 = x \cdot x \leq ((x \cdot 0) \cdot 0) \cdot x \). Therefore \[((x \cdot 0) \cdot 0) \cdot x \] \( = 0 \), and by hypothesis, it follows that \((x \cdot 0) \cdot 0 \) \( = 0 \). From KK-3, we conclude that \((x \cdot 0) \cdot 0 \) \( = 0 \).

Theorem 2.8 Given a KK-algebra \( X \), then the following conditions are equivalent: for all \( x, y, z, u \in X \),

1. \( X \) is a \( p \)-semisimple;
2. \((x \cdot y) \cdot (z \cdot u) = (u \cdot z) \cdot (y \cdot x)\);
3. \((x \cdot y) \cdot 0 = y \cdot x\);
4. \((x \cdot y) \cdot (z \cdot y) = z \cdot x\);
5. \( x \cdot z = y \cdot z \) implies \( x = y \);
6. \( x \cdot y = 0 \) implies \( x = y \).

Proof. Let \( X \) be a KK-algebra and \( x, y, z, u \in X \).
(1) ⇒ (2). Suppose that $X$ is a $p$-semisimple. From proposition 2.2 and (P-5), we get $(x*y)*(z*u) = ((z*u)*(x*y)) * 0 = ((z*u)*0)*(x*y)*0 = (u*z)*(y*x)$.

(2) ⇒ (3). Suppose that (2) holds, it follows that $(x*y)*0 = (x*y)*(0*0) = (0*0)*(y*x) = y*x$.

(3) ⇒ (4). From (KK-1), (P-5) and (P-6), we can write $(z*x)*((x*y)*(z*y)) = 0$ and $((x*y)*(y*z))*(z*x) = ((((y*x)*0)*(y*z)*0)*(z*x) = (((y*x)*(y*z))*(x*z)) = ((y*x)*(y*z))*0 = 0 = (x*z)*(y*x)*(y*z) = 0$.

(4) ⇒ (5). Assume that $x*z = y*z$, Substituting $y$ for $x$ and $z$ for $y$ and and $x$ for $z$ in (4), we obtain $x*y = (y*z)*(x*z) = 0$. And replacing $z$ by $y$ and $y$ by $z$ in (4), we have $y*x = (x*z)*(y*z) = 0$. Thus $x = y$ because (KK-3).

(5) ⇒ (6). Assume that $x*y = 0$, it follows that $x*y = y*y$. By (5), we have $x = y$.

(6) ⇒ (1). For $x \in X$, by (P-5), we get $x*((x*0)*0) = (x*0)*(x*0) = 0$. Using (6), it yields $x = (x*0)*0$, proving that $X$ is a $p$-semisimple. □

**Theorem 2.9** Let $X$ be a KK-algebra. Then $X$ is the $p$-semisimple if and only if one of the following conditions holds: for all $x, y, z \in X$,

1. $z*x = z*y$ implies $x = y$.
2. $(x*y)*(x*z) = y*z$.
3. $(y*x)*(z*x) = (y*z)*0$.
4. $(x*y)*z = (z*y)*x$.

**Proof.** Let $X$ be a KK-algebra and $x, y, z \in X$.

1. From (P-6), $(z*x)*(z*y) \leq x*y$. Then we have $0 \leq x*y$, it follows that $(x*y)*0 = 0$. By theorem 2.8, we obtain $y*x = 0$, so $x = y$. Conversely, since $x*x = 0 = x*((x*0)*0)$. From (1), then $x = (x*0)*0$, proving that $X$ is the $p$-semisimple.

2. Now, we will show $(x*y)*(x*z) = y*z$. We see that $(y*z)*((x*y)*(x*z)) = 0$ and $((x*y)*(x*z))*(y*z) = ((z*x)*(y*x))*(y*z) = (y*z)*(y*z) = 0$, these imply that $(x*y)*(x*z) = y*z$. On the other hand, by (2), we get $(x*0)*0 = (x*0)*(x*x) = 0*x = x$.

3. By theorem 2.8, $(y*x)*(z*x) = z*y = (y*z)*0$. Conversely, by (3), we obtain $(x*0)*0 = (x*x)*(0*x) = 0*0 = x*x = x$.

4. By proposition 2.2 and (P-5), we obtain $(x*y)*z = (z*0)*((x*y)*0) = (z*(x*y))*0 = (x*(z*y))*0$. Then theorem 2.8 implies $(x*y)*z = (z*y)*x$. 


On the other hand, we see that $(x \ast 0) \ast 0 = ((0 \ast x) \ast 0) \ast 0 = (0 \ast 0) \ast (0 \ast x) = x$. This completes the proof. □

**Proposition 2.10** Assume that $X$ is a KK-algebra. Then $B$ and $P$ are closed of $X$, where $B$ is the set of all positive of $X$, and $P$ is the set of all minimal.

**Proof.** Since 0 is a positive, then $B$ is non-empty set. Let $x, y \in B$, it follows that $x \ast 0 = 0$ and $y \ast 0 = 0$. By (P-5), $(x \ast y) \ast 0 = (x \ast 0) \ast (y \ast 0) = 0 \ast 0 = 0$, it means that $x \ast y \in B$. And similarly, $y \ast x \in B$, proving that $B$ is a closed of $X$.

Next, we will show that $P$ is closed of $X$. Since 0 is a minimal, then $P$ is non-empty set. Let $a, b \in P$ and $x \leq a \ast b$. Then $b \ast x \leq b \ast (a \ast b) = a \ast (b \ast b) = a \ast 0$. Thus $b \ast x \leq a \ast 0$ implies $0 = (a \ast 0) \ast (b \ast x) = b \ast ((a \ast 0) \ast x)$. It follows that $(a \ast 0) \ast x \leq b$ and since $b$ is minimal, so $(a \ast 0) \ast x = b$. We get that $x \ast b = x \ast ((a \ast 0) \ast x) = (a \ast 0) \ast (x \ast x) = (a \ast 0) \ast 0 \leq a$. By Corollary 2.10 [2], $a \ast b \leq x$ and $x \leq a \ast b$, so $x = a \ast b$, i.e., $a \ast b$ is a minimal. Therefore $a \ast b \in P$, proving that $P$ is a closed. □

Define the set $P$ of all minimal element of $X$ is called the $p$-semisimple part of $X$. From theorem 2.3 and proposition 2.10, we get the following proposition.

**Proposition 2.11** Assume that $X$ is a KK-algebra. Then the $p$-semisimple part $P$ of $X$ is a $p$-semisimple closed of $X$, and $P = \{x \ast 0 : x \in X\}$.

**Proposition 2.12** If $X$ is a $p$-semisimple KK-algebras, then every closed $A$ of $X$ is an ideal of $X$.

**Proof.** Let $X$ be a $p$-semisimple KK-algebras and $A$ be a closed of $X$. Since $A$ is a closed of $X$, then $0 \in A$. Now, to show that $A$ satisfies (I-2), which assume that $x, y \in X$ such that $x \ast y \in A$ and $x \in A$. By closeness of $A$, it follows that $x \ast 0 \in A$ and $(x \ast 0) \ast (x \ast y) \in A$. Then $y \ast ((x \ast 0) \ast (x \ast y)) = (x \ast 0) \ast (y \ast (x \ast y)) = (x \ast 0) \ast (x \ast (y \ast y)) = 0$, so $(x \ast 0) \ast (x \ast y) \leq y$. And since $y$ is a minimal of $X$, $(x \ast 0) \ast (x \ast y) = y$. Hence $y \in A$, this shows that $A$ is an ideal of $X$. □

For a KK-algebra $X$ and a minimal element $a$ of $X$, defined the set $V(a) := \{x \in X : x \ast a = 0\}$, which is called the branch of $X$ generated by $a$. 

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Proposition 2.13 For any $a \in P$ and $x \in X$, if $x \in V(a)$, then $x \ast 0 = a \ast 0$.

Proof. Since $x \in V(a)$, it follows that $x \ast a = 0$. Then $(x \ast 0) \ast (a \ast 0) = (x \ast a) \ast 0 = 0$. Since $a$ is a minimal, so $(a \ast 0) \ast (x \ast 0) = x \ast a = 0$. From (KK-3), proving that $x \ast 0 = a \ast 0$. □

The next theorem is interesting and useful.

Theorem 2.14 Assume that $P$ is a $p$-semisimple part of KK-algebra $X$. Then

1. $X = \bigcup_{a \in P} V(a)$ and $V(a) \cap V(b) = \emptyset$ whenever $a \neq b$ and $a, b \in P$;
2. if $x \in V(a)$ and $y \in V(b)$, then $x \ast y \in V(a \ast b)$;
3. if $a \in P$, then $x \ast a \in P$, for any $x \in X$;
4. if $a \in P$ and $x \in V(b)$, then $x \ast a = b \ast a$.

Proof. Let $P$ be a $p$-semisimple part of KK-algebra $X$.

1. For any $x \in X$, put $a' = (x \ast 0) \ast 0$, then $a'$ is a minimal element of $X$, and so $a' \in P$. Since $x \ast a' = x \ast ((x \ast 0) \ast 0) = 0$, it follows that $x \in V(a') \subseteq \bigcup_{a \in P} V(a)$. Accordingly, $X = \bigcup_{a \in P} V(a)$.

Suppose that $a, b$ are minimals of $X$ such that $a \neq b$. Now, assume that $V(a) \cap V(b) \neq \emptyset$, then there exists $x \in V(a) \cap V(b)$, it means that, $x \ast a = 0$ and $x \ast b = 0$. By proposition 2.13, we obtain that $a \ast 0 = x \ast 0 = b \ast 0$. Then $b \ast a = (a \ast 0) \ast (b \ast 0) = 0$. Likewise, $a \ast b = 0$. Therefore $a = b$, contradiction with $a \neq b$.

2. Assume that $x \in V(a), y \in V(b)$ and $a, b$ are minimals of $X$. Then $x \ast a = 0 = y \ast b$, it follows that $x \ast 0 = a \ast 0$ and $y \ast 0 = b \ast 0$. Since $a \ast b = (b \ast 0) \ast (a \ast 0) = (y \ast 0) \ast (x \ast 0) \leq x \ast y$, so $a \ast b \leq x \ast y$. And since $a \ast b \in P$, thus $x \ast y \in V(a \ast b)$.

3. Assume that $a \in P$ and $x \in X$. By proposition 2.2, it follows that $x \ast a = (a \ast 0) \ast (x \ast 0)$. Since $a \ast 0, x \ast 0 \in P$ and $P$ has closeness property, $x \ast a \in P$.

4. Assume that $a \in P$ and $x \in V(b)$. From proposition 2.13, we get that $b \ast 0 = x \ast 0$ and $x \ast a = (a \ast 0) \ast (x \ast 0) = (a \ast 0) \ast (b \ast 0) = b \ast a$, we conclude that if $a \in P$ and $x \in V(b)$, then $x \ast a = b \ast a$. □

For a KK-algebra $X$, we define a binary operation $\wedge$ by $x \wedge y = (x \ast y) \ast y$, for each $x, y \in X$. In particular $a_x = (x \ast 0) \ast 0$, and $L_p(X) := \{a \in X : a \ast x = 0\}$
0 ⇒ a = x, ∀x ∈ X}. We call the elements of \( L_p(X) \) the \( p \)–atoms of \( X \). For any \( a \in X \). Note that \( a_x \in L_p(X) \), i.e., \( (a_x \ast 0) \ast 0 = a_x \).

**Definition 2.15** A KK-algebras \( X \) is said to be commutative if \( x = x \wedge y \) whenever \( x \leq y \) for all \( x, y \in X \).

Note that, it can easily be checked that every \( p \)-semisimple KK-algebra is commutative.

**Definition 2.16** A KK-algebras \( X \) is said to be branchwise commutative if \( x \wedge y = y \wedge x \) for all \( x, y \in V(a) \) and all \( a \in L_p(X) \).

**Proposition 2.17** If \( X \) is a branchwise commutative KK-algebra, then \( X \) is a commutative.

**Proof.** Assume that a KK-algebra \( X \) is a branchwise commutative. We have that \( x \wedge y = y \wedge x \), for any \( x, y \in V(a) \) and \( a \in L_p(X) \). Now, let \( x, y \in X \) such that \( x \leq y \). Since \( (x \ast 0) \ast 0 \in L_p(X) \) and \( (x \ast 0) \ast 0 \leq 0 \ast x \leq y \). Hence \( x, y \in V((x \ast 0) \ast 0) \), implies that \( x \wedge y = y \wedge x = (y \ast x) \ast x = 0 \ast x = x \). Consequently, \( X \) is a commutative KK-algebra. \( \square \)

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**References**


