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AN IMPROVEMENT OF POINTWISE NEGATIVE BINOMIAL APPROXIMATION BY w-FUNCTIONS

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Abstract: The result in pointwise negative binomial approximation for a non-negative integer-valued random variable proposed by [5] could not be applied to the case of $r \in (0,1)$. In this study, we determine a non-uniform bound on this approximation for $r \in (0,1)$.

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Key Words: negative binomial approximation, negative binomial distribution, non-uniform bound, point metric, Stein's method, w-functions

1. Introduction

The negative binomial random variable Y with parameters $r > 0$ and $p \in (0,1)$ has the probability function as

$$p_Y(y) = \frac{\Gamma(r+y)}{\Gamma(r)y!} q^y p^r, \; y = 0, 1, ..., \tag{1.1}$$

and has mean $\mathbb{E}(Y) = \frac{rq}{p}$ and variance $\mathbb{V}ar(Y) = \frac{rq}{p^2}$. Let X be a non-negative integer-valued random variable with probability function $p_X(x) > 0$ for every x in the support of X, $\mathcal{S}(x)$. Observing that if parameters of the distribution of X and the negative binomial distribution are given under appropriate conditions, then the negative binomial distribution can be used as an approximation of the distribution of X. Furthermore, if we expect the distribution of X to be

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closer to the negative binomial distribution than other distributions, then it is reasonable to approximate the distribution of X by the negative binomial distribution. In the past few years, there has been some research on topics related to the negative binomial approximation for non-negative integer-valued random variable, which can be found in [1], [6] and [7]. Recently, [5] gave a result in pointwise negative binomial approximation to the distribution of X, when $r \in [1, \infty)$, as follows:

$$|p_X(0) - p_Y(0)| \le \frac{(1-p^r)p}{rq} \left\{ \mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right| \right\}$$

and if $\frac{rq}{p} = \mu$, then

$$|p_X(0) - p_Y(0)| \le \frac{(1-p^r)p}{rq} \mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right|$$

for $x_0 = 0$. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|p_X(x_0) - p_Y(x_0)| \le \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} p \left\{ \mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right| \right\}$$

and if $\frac{rq}{p} = \mu$, then

$$|p_X(x_0) - p_Y(x_0)| \le \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} p\mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right|,$$

where μ and $\sigma^2 \in (0, \infty)$ are mean and variance of X and $w(X)$ is the w-function associated with X. However, this result could not be applied to the case of $r \in (0,1)$. In this study, we focus on determining a non-uniform bound on this approximation for $r \in (0,1)$.

2. Method

We use the same methodology as in [5], which consists of Stein's method and w-functions. For w-functions, [2] first defined a function w associated with non-negative integer-valued random variable X and [3] expressed this function in the simple relation as follows:

$$w(x) = \frac{1}{\sigma^2}\left\{\mu + \frac{\sigma^2 w(x-1)p_X(x-1)}{p_X(x)} - x\right\}, \; x \in \mathcal{S}(x) \setminus \{0\},$$

where $w(0) = \frac{\mu}{\sigma^2}$ and $p_X(x) > 0$ for every $x \in \mathcal{S}(x)$. For Stein's method, [4] introduced a method for bounding the error in the normal approximation. The method was applied to the negative binomial approximation by [1]. Stein's equation for negative binomial distribution with parameters $r > 0$ and $p = (1-q) \in (0,1)$, for given h, of the form

$$h(x) - \mathcal{NB}_{r,p}(h) = q(r+x)f(x+1) - xf(x), \tag{2.1}$$

where $\mathcal{NB}_{r,p}(h) = \sum_{k=0}^{\infty} h(k)\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k$ and f and h are bounded real-valued functions defined on $\mathbb{N} \cup \{0\}$.

For $A \subseteq \mathbb{N} \cup \{0\}$, let $h_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } \ x \in A, \\ 0 & \text{if } \ x \notin A. \end{cases} \tag{2.2}$$

For $A = \{x_0\}$ as $x_0 \in \mathbb{N} \cup \{0\}$, following [1] and [6], the solution $f_{x_0} = f_{\{x_0\}}$ of (2.1) can be written as

$$f_{x_0}(x) = \begin{cases} \frac{-\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(h_{C_{x-1}})}{x\mathcal{NB}_{r,p}(h_x)} & \text{if } \ x \leq x_0, \\ \frac{\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(1-h_{C_{x-1}})}{x\mathcal{NB}_{r,p}(h_x)} & \text{if } \ x > x_0, \\ 0 & \text{if } \ x = 0, \end{cases} \tag{2.3}$$

where $C_x = \{0, ..., x\}$ and $h_{x_0} = h_{\{x_0\}}$.

Lemma 2.1. *For $x_0, x \in \mathbb{N}$, let $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$, then*

$$|f_0(x)| \leq \frac{1-p^r}{rq}, \tag{2.4}$$

$$|\Delta f_0(x)| \leq \frac{rq - (1-p^r)p}{r(r+1)q^2}, \tag{2.5}$$

$$|f_{x_0}(x)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\}, \tag{2.6}$$

$$|\Delta f_{x_0}(x)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\}. \tag{2.7}$$

Proof. The inequalities (2.4), (2.6) and (2.7) are directly obtained from [5]. In the next step, we have to show that (2.5) holds. From [1], it follows that $|\Delta f_0|$ is a decreasing function in $x \in \mathbb{N}$. Thus, we have

$$|\Delta f_0(x)| \leq f_0(1) - f_0(2) = \frac{1-p^r}{rq} - \frac{1-p^r-rqp^r}{r(r+1)q^2} = \frac{rq-(1-p^r)p}{r(r+1)q^2},$$

which implies that (2.5) holds. □

3. Result

The following theorem presents the desired result for $r \in (0,1)$.

Theorem 3.1. *For $r \in (0,1)$, then we have the following:*
1. For $x_0 = 0$,

$$|p_X(0) - p_Y(0)| \leq \frac{(1-p^r)p}{rq}\left\{\mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right|\right\}$$

and if $\frac{rq}{p} = \mu$, then $|p_X(0) - p_Y(0)| \leq \frac{rq-(1-p^r)p}{r(r+1)q^2}\mathbb{E}\left|(r+X)q - p\sigma^2 w(X)\right|$.

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|p_X(x_0) - p_Y(x_0)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\} p\left\{\mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right|\right\}$$

and if $\frac{rq}{p} = \mu$, then

$$|p_X(x_0) - p_Y(x_0)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\} p\mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right|.$$

Proof. [5] showed that

$$|p_X(x_0) - p_Y(x_0)| \leq \sup_{x\geq 1}|\Delta f_{x_0}(x)| p\mathbb{E}\left|\frac{(r+X)q}{p} - \sigma^2 w(X)\right| + \sup_{x\geq 1}|f_{x_0}(x)| p(1-p_X(0))\left|\frac{rq}{p} - \mu\right|.$$

Hence, using Lemma 2.1, the theorem is easily obtained. □

Corollary 3.1. *If $(n+x)q/p - \sigma^2 w(x) \geq / < 0$ for every $x \in \mathcal{S}(x)$, then*
1. For $x_0 = 0$,

$$|p_X(0) - p_Y(0)| \leq \frac{(1-p^r)p}{rq}\left\{\left|\frac{(r+\mu)q}{p} - \sigma^2\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right|\right\}$$

and if $\frac{rq}{p} = \mu$, then

$$|p_X(0) - p_Y(0)| \leq \frac{rq-(1-p^r)p}{r(r+1)q^2}\left|\mu - p\sigma^2\right|.$$

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|p_X(x_0) - p_Y(x_0)| \\ \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\} p \left\{\left|\frac{(r+\mu)q}{p} - \sigma^2\right| + (1-p_X(0))\left|\frac{rq}{p} - \mu\right|\right\}$$

and if $\frac{rq}{p} = \mu$, *then*

$$|p_X(x_0) - p_Y(x_0)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\} \left|\mu - p\sigma^2\right|.$$

4. Applications

4.1. Approximation of the Pólya distribution The Pólya random variable X with positive integer parameters N, m,c and s has the probability function as

$$p_X(x) = \frac{\binom{\frac{s}{c}+x-1}{x}\binom{\frac{N-s}{c}+m-x-1}{m-x}}{\binom{\frac{N}{c}+m-1}{m}}, \quad x = 0, 1, ..., m,$$

and has mean $\mu = \frac{sm}{N}$ and variance $\sigma^2 = \frac{sm(N+cm)(N-s)}{N^2(N+c)}$. Using Corollary 3.1, we can obtain the following corollary.

Corollary 4.1. *If* $r = \frac{s}{c} < 1$ *and* $p = \frac{N}{N+cm}$, *then we have the following:*

$$|p_X(x_0) - p_Y(x_0)| \leq \begin{cases} \frac{sm-(1-p^{\frac{s}{c}})N}{m(N+c)p} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^{\frac{s}{c}}}{\frac{s}{c}q}\right\} \frac{s(s+c)m}{N(N+c)} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

4.2. Approximation of the beta negative binomial distribution. The beta negative binomial random variable X with positive real parameters α, β and r has the probability function as

$$p_X(x) = \frac{\Gamma(r+\alpha)\Gamma(x+\beta)\Gamma(r+x)\Gamma(\alpha+\beta)}{\Gamma(r+x+\alpha+\beta)\Gamma(r)\Gamma(x+1)\Gamma(\alpha)\Gamma(\beta)},$$

and has mean $\mu = \frac{r\beta}{\alpha-1}$ and variance $\sigma^2 = \frac{r\beta(r+\alpha-1)(\alpha+\beta-1)}{(\alpha-2)(\alpha-1)^2}$, where $\alpha > 2$. Using Corollary 3.1, we can get the following result.

Corollary 4.2. *If* $r < 1$ *and* $p = \frac{\alpha-1}{\alpha+\beta-1}$, *then*

$$|p_X(x_0) - p_Y(x_0)| \leq \begin{cases} \frac{r\beta-(1-p^r)(\alpha-1)}{\beta(\alpha-2)p} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{rq}\right\} \frac{r(r+1)\beta}{(\alpha-2)(\alpha-1)} & \text{if } x_0 \in \mathbb{N}. \end{cases}$$

5. Conclusion

In this study, a result in pointwise negative binomial approximation to the distribution of a non-negative integer valued random variable for $r \in (0,1)$ was obtained. It could be applied to approximate the Pólya and beta-negative binomial distributions when their parameters satisfy the parameter $r \in (0,1)$, which could not be applied for $r \in [1,\infty)$. Therefore, the result makes this approximation to be more complete.

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