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GENERALIZED HYERS–ULAM STABILITY OF REFINED QUADRATIC FUNCTIONAL EQUATIONS

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Abstract: In this paper, we give a general solution of a refined quadratic functional equation and then investigate its generalized Hyers–Ulam stability in quasi-normed spaces and in non-Archimedean normed spaces.

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1. Introduction

In [28], S.M. Ulam proposed the stability problem for functional equations concerning the stability of group homomorphisms. A functional equation is called stable if any approximate solution to the functional equation is near a true solution of that functional equation. In [12], D.H. Hyers considered the case of approximate additive mappings with the Cauchy difference controlled by a positive constant in Banach spaces. D.G. Bourgin [5] and T. Aoki [2] treated this problem for approximate additive mappings controlled by unbounded functions. In [23], Th. M. Rassias provided a generalization of Hyers' theorem for linear mappings which allows the Cauchy difference to be unbounded. In 1994, P. Găvruta [9] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions. During

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the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam stability and generalized Hyers–Ulam stability to a number of functional equations and mappings [1, 6, 8, 13, 22].

A stability problem of Ulam for the quadratic functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

was first proved by F. Skof for mapping $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space [26]. In the paper [7], S. Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (1). In particular, J.M. Rassias [19, 20, 21] solved the stability problem of Ulam for the Euler–Lagrange type quadratic functional equation

$$f(rx+sy)+f(sx-ry)=(r^2+s^2)\big[f(x)+f(y)\big] \tag{2}$$

for fixed real numbers r, s with $r \neq 0, s \neq 0$.

In particular, P.L. Kannappan [15] introduced the following functional equation

$$\begin{aligned} &f(x+y)+f(y+z)+f(z+x) \\ &= f(x+y+z)+f(x)+f(y)+f(z) \end{aligned} \tag{3}$$

and proved that a function on a real vector space is a solution of (3) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x)=B(x,x)+A(x)$. In [3], the authors proved the generalized Hyers–Ulam stability of the functional equation

$$\begin{aligned} &f(x-y)+f(y-z)+f(z-x)+f(x+y+z) \\ &= 3[f(x)+f(y)+f(z)], \end{aligned} \tag{4}$$

which is equivalent to the quadratic equation (1). Recently, Zivari–Kazempour and M. Eshaghi Gordji [29] proved the general solution of the following quadratic functional equation

$$\begin{aligned} &f(x+2y)+f(y+2z)+f(z+2x) \\ &= 2f(x+y+z)+3[f(x)+f(y)+f(z)] \end{aligned} \tag{5}$$

and investigated the Hyers–Ulam stability of the equation (5) in Banach space.

In this paper, we consider the following functional equation

$$f(x+ny)+f(y+nz)+f(z+nx) \tag{6}$$

$$= nf(x+y+z) + (n^2 - n + 1)[f(x) + f(y) + f(z)],$$

where n is any fixed nonzero integer, and then investigate the Hyers–Ulam stability of the equation (6).

2. General Solution

First, we remark that the equation (6) is equivalent to (4) for the case $n = -1$ [3], and the equation (6) is trivially equivalent to (3) for the case $n = 1$. Thus we give the general solution of the equation (6) for the case $n \neq -1, 0, 1$ in the following Theorem 2.3.

Lemma 1. *Let X and Y be vector spaces and $f : X \to Y$ be an even function satisfying the functional equation (6). Then f is quadratic.*

Proof. Assume that a function $f : X \to Y$ satisfies (6). Letting $y = z := x$ in (6), we get

$$3f((n+1)x) = nf(3x) + 3(n^2 - n + 1)f(x) \tag{7}$$

for all $x \in X$, which implies $f(0) = 0$. Letting $y = z := 0$ in (6), we have

$$f(nx) = n^2 f(x) \tag{8}$$

for all $x \in X$. Putting $z := 0, y := x$ in (6) and using (8), then we get

$$f((n+1)x) = nf(2x) + (n-1)^2 f(x) \tag{9}$$

for all $x \in X$. Combining (7) and (9), we obtain

$$3f(2x) = f(3x) + 3f(x) \tag{10}$$

for all $x \in X$. Letting $(x, y, z) := (-x, x, 0)$ in (6), we have

$$f((n-1)x) = (n^2 - n)f(x) + (-n+1)f(-x) \tag{11}$$

for all $x \in X$. Replacing x by $-x$ in (11) and adding it with (11), we get

$$f((n-1)x) + f(-(n-1)x) = (n-1)^2[f(-x) + f(x)] \tag{12}$$

for all $x \in X$. Letting $y := x, z := -x$ in (6) and using (9) and (12), we get

$$f(2x) = 3f(x) + f(-x) \tag{13}$$

for all $x \in X$. Combining (10) and (13), we get

$$f(3x) = 6f(x) + 3f(-x) \tag{14}$$

for all $x \in X$.

Now, suppose that f is an even function satisfying (6). Then we get $f((n-1)x) = (n-1)^2 f(x)$ by (12) and $f((n+1)x) = (n+1)^2 f(x)$ by (9). Putting $z := -y$ in (6), we get

$$f(x+ny) + f(nx-y) = (n^2+1)[f(x) + f(y)] \tag{15}$$

for all $x, y \in X$. Letting $z := 0$ in (6), we have

$$f(x+ny) = nf(x+y) + (-n+1)f(x) + (n^2-n)f(y) \tag{16}$$

for all $x, y \in X$. Exchanging x and y in (16), we obtain

$$f(nx+y) = nf(x+y) + (-n+1)f(y) + (n^2-n)f(x) \tag{17}$$

for all $x, y, \in X$. Replacing y by $-y$ in (17), we arrive at

$$f(nx-y) = nf(x-y) + (-n+1)f(y) + (n^2-n)f(x) \tag{18}$$

for all $x, y \in X$. Applying (16) and (18) to (15), we get the desired equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Therefore f is quadratic. □

Lemma 2. *If $f : X \to Y$ is an odd function satisfying (6), then $f \equiv 0$.*

Proof. We get $f(2x) = 2f(x)$ and $f(3x) = 3f(x)$ from (13) and (14), respectively. Using these equations, we have $f((n+1)x) = (n^2+1)f(x)$ and $f((n-1)x) = (n^2-1)f(x)$ from (9) and (11), respectively. Putting $z := -y$ in (6), we get

$$f(x+ny) + f(nx-y) = (n^2+1)f(x) + (n^2-1)f(y) \tag{19}$$

for all $x, y \in X$. Letting $z := 0$ in (6), we have

$$f(x+ny) = nf(x+y) + (-n+1)f(x) + (n^2-n)f(y) \tag{20}$$

for all $x, y \in X$. Taking $x := -y$ and $y := x$ in (20), we obtain

$$f(nx-y) = nf(x-y) + (-n+1)f(y) + (n^2-n)f(x) \tag{21}$$

for all $x, y \in X$. Applying (20) and (21) to (19), then we get

$$f(x+y) + f(x-y) = 2f(x) \tag{22}$$

for all $x, y \in X$. Thus, f is additive and so $f(nx) = nf(x)$ for all $x \in X$. Associating the last equation with (8), we get $f \equiv 0$. □

Theorem 3. *Let $f : X \to Y$ be a function satisfying the functional equation (6). Then f is quadratic and so (6) is equivalent to (1) for the case $n \neq -1, 0, 1$.*

Proof. We can express $f(x) = f_e(x) + f_o(x)$, where $f_e(x) = \dfrac{f(x) + f(-x)}{2}$ is even and $f_o(x) = \dfrac{f(x) - f(-x)}{2}$ is odd. Thus one can easily find that f_e and f_o satisfy the equation (6). Therefore, $f_o \equiv 0$ and so $f = f_e$ is quadratic and so the equation is equivalent to (1). □

3. The Hyers–Ulam Stability in Quasi-Banach Spaces

In this section, we investigate the generalized Hyers–Ulam stability problem for the functional equation (6) in quasi-Banach space. First, we introduce some basic information concerning quasi-Banach spaces which are referred in [4] and [25]. Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \|x\|$ for any scalar λ and all $x \in X$;

(iii) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible M is called the modulus of concavity of the quasi-norm $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a q-norm$(0 < q \leq 1)$ if $\|x + y\|^q \leq \|x\|^q + \|y\|^q$ for all $x, y \in X$. In this case, a quasi-Banach space is called a q-Banach space. Let X be a quasi-Banach space. Given a q-norm, the formula $d(x, y) := \|x - y\|^q$ gives us a translation invariant metric on X. By Aoki–Rolewicz Theorem [25] (see also [4]), each quasi-norm is equavalent to some q-norm. Since it is much easier to work with q-norms than quasi-norms, here and subsequently, we restrict our attention mainly to q-norms. Moreover, generalized stability theorems of functional equations in quasi-Banach spaces have been investigated by a lot of authors [14, 18, 27].

Now we introduce an abbreviation D_nf for a given mapping $f : X \to Y$ as follows:

$$\begin{aligned} D_nf(x,y,z) \;&:=\; f(x+ny)+f(y+nz)+f(z+nx) \\ &\quad -nf(x+y+z)-(n^2-n+1)[f(x)+f(y)+f(z)] \end{aligned}$$

for all $x, y, z \in X$, where $n \neq -1, 0, 1$ is a fixed integer.

From now on, let X be a normed linear space with norm $\|\cdot\|$ and Y be a q-Banach space with norm $\|\cdot\|$. In this part, by using an direct method, we prove the stability theorem of the equation (6).

Theorem 4. *Let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} \frac{1}{n^{2jq}} \phi(n^j x, 0, 0)^q < \infty, \quad \lim_{j\to\infty} \frac{\phi(n^j x, n^j y, n^j z)}{n^{2j}} = 0 \tag{23}$$

for all $x, y, z \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_nf(x,y,z)\| \le \phi(x,y,z) \tag{24}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{n^2}\Big[\sum_{j=0}^{\infty} \frac{\phi(n^j x, 0, 0)^q}{n^{2jq}}\Big]^{\frac{1}{q}} \tag{25}$$

for all $x \in X$.

Proof. Replacing (x, y, z) by $(x, 0, 0)$ in (24), we have

$$\|f(nx) - n^2 f(x)\| \le \phi(x, 0, 0) \tag{26}$$

for all $x \in X$. Replacing x by $n^k x$ in (26) and then dividing both sides by n^{2k+2}, we get

$$\|\frac{1}{n^{2k}} f(n^k x) - \frac{1}{n^{2k+2}} f(n^{k+1}x)\| \le \frac{1}{n^2}\frac{\phi(n^k x, 0, 0)}{n^{2k}}$$

for all $x \in X$ and all integers $k \geq 0$. Then for any integers m, k with $m \geq k \geq 0$, we obtain

$$\begin{aligned}
&\left\|\frac{1}{n^{2m+2}} f(n^{m+1}x) - \frac{1}{n^{2k}} f(n^k x)\right\|^q \qquad (27)\\
&= \left\|\sum_{j=k}^{m}\left(\frac{1}{n^{2j+2}} f(n^{j+1}x) - \frac{1}{n^{2j}} f(n^j x)\right)\right\|^q\\
&\leq \sum_{j=k}^{m}\left\|\frac{1}{n^{2j+2}} f(n^{j+1}x) - \frac{1}{n^{2j}} f(n^j x)\right\|^q\\
&\leq \frac{1}{n^{2q}} \sum_{j=k}^{m} \frac{\phi(n^j x, 0, 0)^q}{n^{2jq}}
\end{aligned}$$

for all $x \in X$. Thus the sequence $\left\{\frac{f(n^k x)}{n^{2k}}\right\}_{k=1}^{\infty}$ is Cauchy by (23). Since Y is complete, this sequence converges for all $x \in X$. So one can define a mapping $Q : X \to Y$ by

$$\lim_{k\to\infty} \frac{f(n^k x)}{n^{2k}} = Q(x) \quad (x \in X). \tag{28}$$

It follows from (23) and (28) that

$$\begin{aligned}
\|D_n Q(x, y, z)\| &= \lim_{k\to\infty} \frac{1}{n^{2k}} \|D_n f(n^k x, n^k y, n^k z)\|\\
&\leq \lim_{k\to\infty} \frac{\phi(n^k x, n^k y, n^k z)}{n^{2k}} = 0
\end{aligned}$$

for all $x, y, z \in X$. Hence, the mapping Q satisfies (6) and so it is quadratic. Putting $k := 0$ and letting m go to infinity in (27), we see that (25) holds. For the uniqueness of Q, assume that there exists a quadratic mapping $Q' : X \to Y$ satisfying the inequality (25). Then, we find that

$$\begin{aligned}
\left\|Q(x) - Q'(x)\right\|^q &= \lim_{k\to\infty} \frac{1}{n^{2kq}} \left\|f(n^k x) - Q'(n^k x)\right\|^q\\
&\leq \lim_{k\to\infty} \frac{1}{n^{2q} n^{2kq}} \sum_{j=0}^{\infty} \frac{1}{n^{2jq}} \phi(n^{j+k} x, 0, 0)^q\\
&= \frac{1}{n^{2q}} \lim_{k\to\infty} \sum_{j=k}^{\infty} \frac{1}{n^{2kq}} \phi(n^k x, 0, 0)^q = 0
\end{aligned}$$

for all $x \in X$, which proves the uniqueness. $\square$

Theorem 5. *Let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{j=0}^{\infty} n^{2jq} \phi(n^{-j}x, 0, 0)^q < \infty, \quad \lim_{j\to\infty} n^{2j} \phi(n^{-j}x, n^{-j}y, n^{-j}z) = 0$$

for all $x, y, z \in X$. Suppose that $f : X \to Y$ is a mapping with $f(0) = 0$ satisfying the inequality

$$\|D_n f(x, y, z)\| \le \phi(x, y, z)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{n^2}\Big[\sum_{j=1}^{\infty} n^{2jq} \phi(n^{-j}x, 0, 0)^q\Big]^{\frac{1}{q}}$$

for all $x \in X$.

Proof. We observe that one can obtain the following inequality

$$\|n^{2k} f(\frac{x}{n^k}) - n^{2(m+1)} f(\frac{x}{n^{m+1}})\|^q \le \frac{1}{n^{2q}} \sum_{j=k}^{m} n^{2(j+1)q} \phi(n^{-(j+1)}x, 0, 0)^q$$

for all $x \in X$ and all integers k, m with $m \ge k \ge 0$ by use of (26). Thus, we see that the proof may be verified by applying similar argument to that of Theorem 4. $\square$

Corollary 6. *Let $\varepsilon \ge 0$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_n f(x, y, z)\| \le \varepsilon$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{\varepsilon}{\sqrt[q]{n^{2q} - 1}}$$

for all $x \in X$.

Corollary 7. *Let α, a_1, a_2, a_3 be positive real numbers such that either $a_i > 2$ or $a_i < 2$ for all $i \in \{1, 2, 3\}$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_n f(x, y, z)\| \leq \alpha\big(\|x\|^{a_1} + \|y\|^{a_2} + \|z\|^{a_3}\big)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\alpha\|x\|^{a_1}}{\sqrt[q]{|n^{2q} - n^{q \cdot a_1}|}}$$

for all $x \in X$.

4. The Hyers–Ulam Stability in Non-Archimedean Spaces

Hensel [11] has introduced a normed space which does not have the non-Archimedean spaces property. During the last three decades, the theory of non-Archimedean spaces has gain the interest of physicists for their research in problems coming from quantum physics, p-adic strings and superstrings [16].

A *valuation* is a function $|\cdot|$ from a field $\mathbb{K}$ to $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a *valued field* if $\mathbb{K}$ equips with a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. Alternatively, if the triangle inequality is replaced by the strong triangle inequality

$$|r + s| \leq \max\{|r|, |s|\}, \forall r, s \in \mathbb{K},$$

then the valuation $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Cleary $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Definition 8. Let X be a vector space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm on X if it satisfies the following conditions

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\|$ $(r \in \mathbb{K})$;

(iii) $\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X)$.

In this case $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Because of the fact

$$\|x_k - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq k-1\} \quad (k > m),$$

a sequence $\{x_m\}$ is Cauchy in the non-Archimedean normed space if and only if $\{x_{m+1} - x_m\}$ converges to zero with respect to the non-Archimedean norm. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Example 9. Let p be a prime number. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x - y|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of $\mathbb{Q}_p$ are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k| = p^{-n_x}$ is a non-Archimedean norm on $\mathbb{Q}_p$ and it makes $\mathbb{Q}_p$ a locally compact field (see [10, 24]).

Let X be a vector space and Y be a non-Archimedean Banach space. In the following, we now prove the generalized Hyers–Ulam stability of quadratic functional equation (6) over the non-Archimedean space. As corollaries, we obtain especially stability result over the p-adic field $\mathbb{Q}_p$. To avoid trivial case, we assume $|n| < 1$.

Theorem 10. *Let $\phi : X^3 \to [0, \infty)$ $\Big(\psi : X^3 \to [0, \infty)\Big)$ be a function such that*

$$\lim_{j \to \infty} \frac{\phi(n^j x, n^j y, n^j z)}{|n|^{2j}} = 0 \tag{29}$$
$$\Big(\lim_{j \to \infty} |n|^{2j} \psi(n^{-j}x, n^{-j}y, n^{-j}z) = 0, resp\Big)$$

for all $x, y, z \in X$ and the limit

$$\Phi(x) \equiv \lim_{k \to \infty} \max\left\{\frac{\phi(n^j x, 0, 0)}{|n|^{2j}} : 0 \leq j < k\right\} \tag{30}$$

$$\Big(\Psi(x) \equiv \lim_{k\to\infty} \max\Big\{|n|^{2j}\psi(n^{-j}x,0,0) : 1 \le j \le k\Big\}, resp\Big)$$

exists for each $x \in X$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality

$$\|D_n f(x,y,z)\| \le \phi(x,y,z) \tag{31}$$
$$\Big(\|D_n f(x,y,z)\| \le \psi(x,y,z), resp\Big),$$

for all $x, y, z \in X$. Then there exists a quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{|n|^2}\Phi(x) \tag{32}$$
$$\Big(\|f(x) - Q(x)\| \le \frac{1}{|n|^2}\Psi(x), resp\Big),$$

for all $x \in X$. Moreover, if

$$\lim_{m\to\infty}\lim_{k\to\infty} \max\Big\{\frac{\phi(n^j x,0,0)}{|n|^{2j}} : m \le j < k+m\Big\} = 0 \tag{33}$$
$$\Big(\lim_{m\to\infty}\lim_{k\to\infty} \max\Big\{|n|^{2j}\psi(n^{-j}x,0,0) : m < j \le k+m\Big\} = 0, resp\Big),$$

for all $x \in X$, then the quadratic mapping Q is unique.

Proof. Replacing (x, y, z) by $(x, 0, 0)$ in (31), we have

$$\|f(nx) - n^2 f(x)\| \le \phi(x,0,0) \tag{34}$$

for all $x \in X$. Replacing x by $n^k x$ in (34) and then dividing both sides by $|n|^{2k+2}$, we get

$$\|\frac{1}{n^{2k+2}}f(n^{k+1}x) - \frac{1}{n^{2k}}f(n^k x)\| \le \frac{1}{|n|^2}\frac{\phi(n^k x,0,0)}{|n|^{2k}} \tag{35}$$

for all $x \in X$. It follows from (35) and (29) that the sequence $\{\frac{f(n^k x)}{n^{2k}}\}_{k=1}^{\infty}$ is Cauchy in the non-Archimedean normed space Y. Since Y is complete, we may define a mapping $Q : X \to Y$ as $Q(x) := \lim_{k\to\infty} \frac{f(n^k x)}{n^{2k}}$ for all $x \in X$. Using induction, one can show that

$$\|\frac{f(n^k x)}{n^{2k}} - f(x)\| \le \frac{1}{|n|^2} \max\Big\{\frac{\phi(n^j x,0,0)}{|n|^{2j}} : 0 \le j < k\Big\} \tag{36}$$

for all $k \in \mathbb{N}$ and all $x \in X$. By taking k to approach infinity in (36) and using (30), one obtains (32). Replacing x, y and z by $n^{2k}x$,$n^{2k}y$ and $n^{2k}z$, respectively, in (31), we get

$$\|\frac{D_n f(n^k x, n^k y, n^k z)}{n^{2k}}\| \leq \frac{\phi(n^k x, n^k y, n^k z)}{|n|^{2k}} \tag{37}$$

for all $x, y, z \in X$. Taking the limit as $k \to \infty$ and using Theorem 3, we conclude that Q is quadratic. Moreover, to prove the uniqueness, we assume that there exists a quadratic mapping $Q' : X \to Y$ satisfying (32) and (33). Then we figure out

$$\begin{aligned}
&\|Q(x) - Q'(x)\| \\
&= \lim_{m\to\infty} \frac{1}{|n|^{2m}} \|Q(n^m x) - Q'(n^m x)\| \\
&\leq \lim_{m\to\infty} max\{\frac{\|Q(n^m x) - f(n^m x)\|}{|n|^{2m}}, \frac{\|f(n^m x) - Q'(n^m x)\|}{|n|^{2m}}\} \\
&\leq \lim_{m\to\infty} \lim_{k\to\infty} \frac{1}{|n|^2} max\{\frac{\phi(n^j x, 0, 0)}{|n|^{2j}} : m \leq j < m+k\} = 0
\end{aligned}$$

for all $x \in X$. This completes the proof. □

Corollary 11. *Let X be a non-Archimedean normed space, $r \neq 2$ and ε, θ be positive numbers, where $\varepsilon = 0$ if $r > 2$. Suppose that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_n f(x, y, z)\| \leq \varepsilon + \theta\big(\|x\|^r + \|y\|^r + \|z\|^r\big) \quad (x, y, z \in X).$$

Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \varepsilon + \dfrac{\theta}{|n|^r}\|x\|^r & if \quad r < 2 \\ \dfrac{\theta}{|n|^2}\|x\|^r & if \quad r > 2 \end{cases}$$

for all $x \in X$.

Corollary 12. *Let $r \neq 2$ and ε, θ be positive numbers, where $\varepsilon = 0$ if $r > 2$. Suppose that a mapping $f : \mathbb{Q}_p \to \mathbb{Q}_p$ with $f(0) = 0$ satisfies the inequality*

$$|D_p f(x, y, z)|_p \leq \varepsilon + \theta\big(|x|_p^r + |y|_p^r + |z|_p^r\big) \quad (x, y, z \in \mathbb{Q}_p).$$

Then there exists a unique quadratic mapping $Q : \mathbb{Q}_p \to \mathbb{Q}_p$ *such that*

$$|f(x) - Q(x)|_p \leq \begin{cases} \varepsilon + p^r\theta|x|_p^r & if \quad r < 2 \\ p^2\theta|x|_p^r & if \quad r > 2 \end{cases}$$

for all $x \in \mathbb{Q}_p$.

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References

[1] J. Aczel and J. Dhombres, *Functional Equations in Several Variables,* Cambridge University Press, (1989), Doi:10.1017/CBO9781139086578.

[2] T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan,* **2**, No. 1-2 (1950), 64-66, Doi:10.2969/jmsj/00210064.

[3] J-H. Bae and I-S. Chang, On the Ulam stability problem of a quadratic functional equation, *Korean J. Comput, Appl. Math.(Series A),* **8**, No. 2 (2001), 561-567.

[4] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society, (2000).

[5] D.G. Bourgin, Classes of transformations and bordering transformations, *Bull. Amer. Math. Soc.* **57** (1951), 223-237, Doi:10.1090/S0002-9904-1951-09511-7.

[6] Y-J. Cho, Th. M. Rassias, R. Saadati, *Stability of Functional Equations in Random Normed Spaces*, Springer Optimization and Its Applications, **86**, Springer, (2013), Doi:10.1007/978-1-4614-8477-6.

[7] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Bull. Abh. Math. Sem. Univ. Hamburg,* **62** (1992), 59-64, Doi:10.1007/BF02941618.

[8] S. Czerwik, *Stability of Functional Equations of Ulam–Hyers–Rassias Type*, Hadronic Press, Florida, (2003).

[9] P. Găvruta, A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436, Doi:10.1006/jmaa.1994.1211.

[10] F. Q. Gouvêa, *p-adic Numbers,* Springer-Verlag, Berlin, (1997), Doi:10.1007/978-3-642-59058-04.

[11] K. Hensel, Über eine neue Begrndüng der Theorie der algebraischen Zahlen, *Jahresber. Deutsch. Math. Verein,* **6** (1987), 83-88.

[12] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.* **27** (1941), 222-224, Doi:10.1073/pnas.27.4.222.

[13] S-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer Optimization and Its Applications, USA, (2011), Doi:10.1007/978-1-4419-9637-4.

[14] K-W. Jun and H-M. Kim, On the stability of Euler–Lagrange type cubic mappings in quasi-Banach spaces, *J. Math. Anal. Appl.* **332** (2) (2006), 1335-1350, Doi:10.1016/j.jmaa.2006.11.024.

[15] Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.* **27** (1995), 368-372, Doi:10.1007/BF03322841.

[16] H. Khodaei and Th. M. Rassias, Approximately generalized additive functions in several variables, *Int. J. Nonlinear Anal. Appl.* **1** (2010), 22-41.

[17] M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, *Bull. Brazilian Math. Soc.* **37** (2006), 361-376, Doi:10.1007/s00574-006-0016-z.

[18] C. Park, Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras, *Bull. Sci. Math.* **132**(2) (2008), 87-96, Doi:10.1016/j.bulsci.2006.07.004.

[19] J.M. Rassias, On the stability of the Euler–Lagrange functional equation, *Chinese J. Math.* **20** (1992), 185-190.

[20] J.M. Rassias, On the stability of the non-linear Euler–Lagrange functional equation in real normed linear spaces, *J. Math. Phys. Sci.* **28** (1994), 231-235.

[21] J.M. Rassias, On the stability of the general Euler–Lagrange functional equation, *Demonst. Math.* **29** (1996), 755-766.

[22] Th. M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London (2003), Doi:10.1007/978-94-017-0225-6.

[23] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300, Doi:10.1090/S0002-9939-1978-0507327-1.

[24] A. M. Robert, *A Course in p-adic Analysis,* Springer-Verlag, New-York, (2000), Doi:10.1007/978-1-4757-3254-2.

[25] S. Rolewicz, *Metric linear spaces, Second edition.* PWN-Polish Scientific Publishers, Warsaw:D. Reidel Publishing Co. Dordrecht, (1984).

[26] F. Skof, Local properties and approximations of operators, *Rend. Sem. Math. Fis. Milano*, **53** (1983), 113-129.

[27] J. Tabor, Stability of the Cauchy functional equation in quasi-Banach spaces, *Ann. Polon. Math.* **83** (2004), 243-255, Doi:10.4064/ap83-3-6.

[28] S. M. Ulam, *Problems in Modern Mathematics*, Chapter 6, Wiley Interscience, New York, (1964).

[29] A. Zivari–Kazempour and M. Eshaghi Gordji, Generalized Hyers–Ulam stabilities of an Euler–Lagrange–Rassias quadratic functional equation, *Asian-European Journal of Mathematics,* **5** (2012), Doi:10.1142/S1793557112500143.