ON THE DIVISORS OF ORDER r

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Abstract: N. Minculete has introduced the divisor-of-order- r function $\tau^{(r)}$ and the sum-of-divisors-of-order- r function $\sigma^{(r)}$. We investigate the asymptotic behaviour of $\sum_{n \leq x} \tau^{(r)}(n)$ and $\sum_{n \leq x} \sigma^{(r)}(n)$ and improve known estimates.

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1. Introduction

Recently N. Minculete in his PhD Thesis [8] introduced a concept of *divisors of order r* : integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order r of number $n = p_1^{a_1} \cdots p_k^{a_k}$ if d divides n in the usual sense and $b_j \in \{r, a_j\}$ for $j = 1, \dots, k$. We also suppose that 1 is a divisor of any order of itself (but not of any other number). Let us denote respective divisor and sum-of-divisor functions as $\tau^{(r)}$ and $\sigma^{(r)}$. These functions are multiplicative and

$$\tau^{(r)}(p^a) = \begin{cases} 1, & a \leq r, \\ 2, & a > r. \end{cases} \quad (1)$$

$$\sigma^{(r)}(p^a) = \begin{cases} p^a, & a \leq r, \\ p^a + p^r, & a > r. \end{cases} \quad (2)$$

In a special case of $r = 0$ we get well-studied *unitary divisors*, see [2, 11]. Below we consider only the case $r > 0$.

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In another special case of $r = 1$ we get so-called by Minculete *exponential semiproper divisors* and denote $\tau^{(e)s} := \tau^{(1)}$, $\sigma^{(e)s} := \sigma^{(1)}$. An integer d is an exponential semiproper divisor of n if $\ker d = \ker n$ and $(d/\ker n, n/d) = 1$, where $\ker n = \prod_{p|n} p$.

Minculete proved in [8, (3.1.17–19)] that

$$\limsup_{n \rightarrow \infty} \frac{\log \tau^{(r)}(n) \log \log n}{\log n} = \frac{\log 2}{r+1}, \quad (3)$$

$$\sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)} x + Ax^{1/(r+1)} + O(x^{1/(r+2)+\varepsilon}), \quad (4)$$

$$\limsup_{n \rightarrow \infty} \frac{\sigma^{(r)}(n)}{n \log \log n} = \frac{6e^\gamma}{\pi^2}. \quad (5)$$

In the present paper we improve the error term in (4) and establish asymptotic formulas for $\sum_{n \leq x} \sigma^{(r)}(n)$ with O - and Ω -estimates of the error term.

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o , big omegas Ω and $\Omega_\pm$, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument tends to the infinity.

Letter p with or without indexes denote rational prime.

As usual $\zeta(s)$ is the Riemann zeta-function. For complex s we denote $\sigma := \Re s$ and $t := \Im s$.

We use abbreviations $\log x := \log \log x$, $\text{llog } x := \log \log \log x$.

Letter γ denotes Euler–Mascheroni constant, $\gamma \approx 0.577$.

Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

We write $f \star g$ for Dirichlet convolution:

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

Function $\ker: \mathbf{N} \rightarrow \mathbf{N}$ stands for $\ker n = \prod_{p|n} p$.

For a set A notation $\#A$ means the cardinality of A .

2. Preliminary Estimates

Consider $\tau(a, b; n) = \sum_{k|a, l|b=n} 1$ and $T(a, b; x) = \sum_{n \leq x} \tau(a, b; n)$ for $1 \leq a \leq b$. One can directly check that $\sum_{n=1}^{\infty} \tau(a, b; n)n^{-s} = \zeta(as)\zeta(bs)$ for $\sigma > 1$.

Lemma 1. $T(a, b; x) = H(a, b; x) + \Delta(a, b; x)$, where

$$H(a, b; x) = \begin{cases} \zeta(b/a)x^{1/a} + \zeta(a/b)x^{1/b}, & 1 \leq a < b, \\ x^{1/a} \log x + (2\gamma - 1)x^{1/a}, & a = b, \end{cases}$$

$$x^{1/2(a+b)} \ll \Delta(a, b; x) \ll \begin{cases} x^{1/(2a+b)} & 1 \leq a < b, \\ x^{1/3a} \log x & a = b. \end{cases}$$

Proof. See [6, Th. 5.1, Th. 5.3, Th. 5.8]. $\square$

In fact $\Delta(a, b; x)$ can be estimated more precisely. For our goals we are primarily interested in the behaviour of $\Delta(1, b; x)$. Let us suppose that

$$\Delta(1, b; x) \ll x^{\theta_b} \log^{\theta'_b} x, \quad (6)$$

then due to [6, Th. 5.11] for $b \geq 7$ we can choose $\theta_b = 1/(b + 7/2)$ and $\theta'_b = 1$. Estimates for $b \leq 16$ are given in Table 1. Estimate for $b = 1$ belongs to Huxley [4], and estimate for $b = 2$ belongs to Graham and Kolesnik [3]. We have found no references on the best known results for $b \geq 3$, so we calculated them with the use of [6, Th. 5.11, Th. 5.12] selecting appropriate exponent pairs carefully. It seems that some of this estimates may be new.

Lemma 2. Let α and β be positive real numbers with $\beta + 1 = \alpha$. Then

$$\sum_{mn^\alpha \leq x} mn^\beta = \frac{\zeta(2\alpha - \beta)}{2} x^2 + \mathcal{D}(\alpha, \beta; x), \quad \mathcal{D}(\alpha, \beta; x) \ll x \log^{2/3} x.$$

Proof. See [10, Th. 1]. $\square$

For $k > 0$ one can define a multiplicative function μ_k implicitly by

$$\sum_{n=1}^{\infty} \mu_k(n) n^{-s} = 1/\zeta(ks)$$

for $\sigma > 1$. So $\mu_k(n^k) = \mu(n)$ and $\mu_k(m) = 0$ for all other arguments. Trivially $\mu_1 \equiv \mu$. Then

$$M_k(x) := \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x^{1/k}} \mu(n) \ll x^{1/k} \exp(-CN(x)),$$

where $C > 0$, $N(x) = \log^{3/5} x \log^{-1/5} x$. See [5, Th. 12.7] for the proof of the last estimate. Assuming Riemann hypothesis (RH) we get much better result $M_k(x) \ll x^{1/2k+\varepsilon}$ [12, Th. 14.25 (C)].

b	θ_b	θ'_b	Exponent pair or reference
1	$131/416 + \varepsilon \approx 0.314904$	0	[4]
2	$1057/4785 + \varepsilon \approx 0.220899$	0	[3]
3	$1486/8647 + \varepsilon \approx 0.171852$	0	$AB(AB)^2H$
4	≈ 0.140161	0	AH
5	≈ 0.121398	2	$A^2BA(ABAB^2A^2B^6)^\infty I$
6	$669/6305 \approx 0.106106$	1	$(A^2B)^3(AB)^3A^4BI$
7	≈ 0.094491	2	$A^2B^2(BAB^2A^2B^6A)^\infty I$
8	≈ 0.085314	1	$A^2B^4A(ABA^2)^\infty I$
9	≈ 0.077892	2	$A^2B^2(AB^6ABAB^2A)^\infty I$
10	≈ 0.071774	0	$(A^2B^2)^3ABH$
11	≈ 0.066283	0	A^2H
12	$64/1037 + \varepsilon \approx 0.061716$	0	A^2H
13	≈ 0.058089	0	$A^3BA^3BA^2BA^4B(AB)^2H$
14	$75/1373 \approx 0.054625$	1	$A^2(AB)^2BA^3BI$
15	≈ 0.051570	0	$A(A^2B)^3A^4B^7A^3BBA^4BH$
16	$15/307 \approx 0.048860$	1	$A^3BA^2BA^4BI$

Table 1: Values of θ_b and θ'_b in (6) for $b \leq 16$. Exponent pairs are given in terms of A - and B -processes [6, Th. 2.12, 2.13]. We abbreviate $\mathcal{B} := BA$. Here $I = (0, 1)$ and $H = (32/205 + \varepsilon, 269/410 + \varepsilon)$ is Huxley exponent pair from [4].

Lemma 3. *Let $K \in \mathbf{N}$, $J \in \mathbf{N} \cup \{0\}$, $m_1 \leq \dots \leq m_K$, $n_1 \leq \dots \leq n_J$, where all $m_k, n_j \in \mathbf{N}$, and suppose that*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{\zeta(m_1 s) \cdots \zeta(m_K s)}{\zeta(n_1 s) \cdots \zeta(n_J s)}.$$

Let $\alpha = K - 1/(2 \sum_{k=1}^K m_k)$. If $1/\alpha < 2n_j$ for all $j = 1, \dots, J$ then for arbitrary $H(x)$ of the form $H(x) = \sum_{i=1}^I x^{\beta_i} P_i(\log x)$, $\beta_i \in \mathbf{C}$, $\alpha < \Re \beta_i \leq 1$, P_i are polynomials, we have $\sum_{n \leq x} a_n = H(x) + \Omega(x^\alpha)$.

Proof. This is a simplified version of [7, Th. 2]. □

3. Asymptotic Properties of $\sum \tau^{(r)}(n)$

Lemma 4.

$$\sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s} = \frac{\zeta(s)\zeta((r+1)s)}{\zeta((2r+2)s)}, \quad \sigma > 1. \quad (7)$$

Proof. Let us transform Bell series for $\tau^{(r)}$:

$$\begin{aligned}\tau_p^{(r)}(x) &= \sum_{k=0}^{\infty} \tau^{(r)}(p^k) x^k = \sum_{k=0}^r x^k + 2 \sum_{k>r} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k>r} x^k = \\ &= (1 + x^{r+1}) \sum_{k=0}^{\infty} x^k = \frac{1 + x^{r+1}}{1 - x} = \frac{1 - x^{2r+2}}{(1 - x)(1 - x^{r+1})}.\end{aligned}$$

Identity $\sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s} = \prod_p \tau_p^{(r)}(p^{-s})$ completes the proof. $\square$

It follows from (7) that

$$\tau^{(r)} = \tau(1, r + 1; \cdot) \star \mu_{2r+2} \quad (8)$$

Theorem 5. *If Δ is estimated as in (6) then for $r > 0$*

$$\sum_{n \leq x} \tau^{(r)}(n) = Ax + Bx^{1/(r+1)} + \mathcal{E}_{r+1}(x),$$

where $\mathcal{E}_r(x) = O(x^{\max(\theta_r, 1/2r)} \log^{\theta'_r} x)$, constants A and B are specified below in (9).

Proof. Taking into account (8) we have for $r > 0$

$$\begin{aligned}\sum_{n \leq x} \tau^{(r)}(n) &= \sum_{n \leq x} \mu_{2r+2}(n) T(1, r + 1; x/n) = \\ &= \zeta(r + 1) x \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n} + \zeta(1/(r + 1)) x^{1/(r+1)} \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n^{1/(r+1)}} + \\ &\quad + \sum_{n \leq x} \mu_{2r+2}(n) \Delta(1, r + 1, x/n).\end{aligned}$$

But for $s \geq 1/k$

$$\sum_{n \leq x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} - \sum_{n > x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} + O(x^{1/k-s})$$

and

$$\sum_{n \leq x} \mu_{2k}(n) \Delta(1, k, x/n) = \sum_{n \leq x^{1/2k}} \mu(n) \Delta(1, k, x/n^{2k}) \ll$$

$$\ll \sum_{n \leq x^{1/2k}} \left(\frac{x}{n^{2k}} \right)^{\theta_k} \log^{\theta'_k} x \ll x^{\max(\theta_k, 1/2k)} \log^{\theta'_k} x.$$

So

$$\sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)} x + \frac{\zeta\left(\frac{1}{r+1}\right)}{\zeta(2)} x^{\frac{1}{r+1}} + E_{r+1}(x) \quad (9)$$

□

Lemma 6. *Let $r > 0$, $x^\varepsilon \leq y \leq x^{1/2r}$. Then under RH we have*

$$\mathcal{E}_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon). \quad (10)$$

Proof. We follow the approach of Montgomery and Vaughan (see [9] or [1]).

First of all consider $g_y(s) = 1/\zeta(s) - \sum_{d \leq y} \mu(d) d^{-s}$. Then for $\sigma > 1$ we have $g_y(s) = \sum_{d > y} \mu(d) d^{-s}$. Assuming RH we have by [12, Th. 14.25]

$$\sum_{d \leq y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O(y^{1/2-\sigma+\varepsilon} (|t|^\varepsilon + 1)) \quad \text{for } \sigma > 1/2 + \varepsilon,$$

so

$$g_y(s) \ll y^{1/2-\sigma+\varepsilon} (|t|^\varepsilon + 1) \quad \text{for } \sigma > 1/2 + \varepsilon. \quad (11)$$

Now let us split $\sum_{n \leq x} \tau^{(r-1)}(n)$ into two parts:

$$\sum_{n \leq x} \tau^{(r-1)}(n) = \sum_{d^{2r} \leq x} \mu(d) T(1, r; x/d^{2r}) = S_1 + S_2,$$

where

$$\begin{aligned} S_1 := \sum_{d \leq y} \mu(d) T(1, r; x/d^{2r}) &= \zeta(r) x \sum_{d \leq y} \frac{\mu(d)}{d^{2r}} + \\ &+ \zeta(1/r) x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} + \sum_{d \leq y} \mu(d) \Delta(1, r; x/d^{2r}) \end{aligned}$$

and S_2 is the rest of $\sum_{n \leq x} \tau^{(r-1)}(n)$. We note that under RH by taking into account $y \leq x^{1/2r}$ we have

$$x^{1/r} \sum_{d > y} \frac{\mu(d)}{d^2} \ll x^{1/r} y^{-3/2+\varepsilon} \ll x^{1/2} y^{1/2-r+\varepsilon}$$

and so

$$x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} = \frac{x^{1/r}}{\zeta(2)} + O(x^{1/2} y^{1/2-r+\varepsilon}).$$

Next, let $h_y(s) := \zeta(s)\zeta(rs)g_y(2rs)x^s s^{-1}$. Then by Perron formula with $c = 1 + \varepsilon$, $T = x^2$ one can estimate

$$S_2 = \frac{1}{2\pi i} \int_{1+\varepsilon-ix^2}^{1+\varepsilon+ix^2} h_y(s) ds + O(x^\varepsilon).$$

By moving line of integration to $[1/2 + \varepsilon - ix^2, 1/2 + \varepsilon + ix^2]$ we obtain

$$S_2 = \operatorname{res}_{s=1} h(s) + O(I_1 + I_2 + I_3),$$

where

$$I_1 = \int_{1+\varepsilon-ix^2}^{1/2+\varepsilon-ix^2} h(s) ds, \quad I_2 = \int_{1/2+\varepsilon-ix^2}^{1/2+\varepsilon+ix^2} h(s) ds, \quad I_3 = \int_{1/2+\varepsilon-ix^2}^{1+\varepsilon-ix^2} h(s) ds.$$

Due to (11) and estimates of ζ under RH we have

$$\begin{aligned} g_y(2rs) &\ll y^{1/2-r}(|t|^\varepsilon + 1) && \text{for } \sigma > 1/2 + \varepsilon, \\ h(s) &\ll y^{1/2-r}(|t|^\varepsilon + 1)x^s s^{-1} && \text{for } \sigma > 1/2 + \varepsilon, \end{aligned}$$

and

$$\begin{aligned} I_{1,3} &\ll y^{1/2-r+\varepsilon} \max_{\sigma \in [1/2+\varepsilon, 1+\varepsilon]} x^{\sigma-2} \ll y^{1/2-r+\varepsilon}, \\ I_2 &\ll y^{1/2-r+\varepsilon} \int_1^{x^2} x^{1/2} t^{-1} dt \ll y^{1/2-r+\varepsilon} x^{1/2+\varepsilon}. \end{aligned}$$

Identity

$$\operatorname{res}_{s=1} h(s) = \zeta(r)x \sum_{d>y} \frac{\mu(d)}{d^{2r}}$$

completes the proof. □

Theorem 7. *If Δ is estimated as in (6) and $\theta_r < 1/2r$ then under RH we get $\mathcal{E}_r(x) = O(x^\alpha)$, where $\alpha = (1 - \theta_r)/(2r + 1 - 4r\theta_r)$.*

Proof. Let us start with (10):

$$\mathcal{E}_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon) \ll$$

$$\begin{aligned} &\ll \sum_{n \leq y} \left(\frac{x}{n^{2r}} \right)^{\theta_r + \varepsilon} + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon) \ll \\ &\ll x^\varepsilon \left(x^{\theta_r} \left(1 + y^{1-2r\theta_r} \right) + x^{1/2} y^{1/2-r} + 1 \right). \end{aligned}$$

If $\theta_r < 1/2r$ then $\mathcal{E}_r(x) \ll x^\varepsilon (x^{\theta_r} y^{1-2r\theta_r} + x^{1/2} y^{1/2-r})$. Choice $y = x^\beta$, where $\beta = (1 - 2\theta_r)/(2r + 1 - 4r\theta_r)$, accomplishes the proof. $\square$

For the values of θ_b from Table 1 we have

$$\max(\theta_r, 1/2r) = \begin{cases} 1/2r, & r \leq 2, \\ \theta_r, & r > 2. \end{cases}$$

So currently the only non-trivial case of the previous theorem is an estimation for $\tau^{(1)} \equiv \tau^{(e)s}$. We get under assumption of RH that

$$\sum_{n \leq x} \tau^{(1)}(n) = \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(1/2)}{\zeta(2)} x^{1/2} + O(x^{\alpha+\varepsilon}),$$

$$\alpha = \frac{1 - \theta_2}{5 - 8\theta_2} = \frac{3728}{15469} \approx 0.241 < 1/4.$$

Theorem 8. $\mathcal{E}_r(x) = \Omega(x^{1/(2r+2)})$.

Proof. The statement is implied by the substitution $m_1 = 1$, $m_2 = r$, $n_1 = 2r$ into Lemma 3. The choice of parameters plainly follows from (7). We obtain $\alpha = 1/(2r + 2)$, which is an exponent in the required Ω -term. $\square$

4. Asymptotic Properties of $\sum \sigma^{(r)}$

Lemma 9.

$$\sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s} = \frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)} H_r(s), \quad \sigma > 2, \quad (12)$$

where Dirichlet series $H_r(s)$ converges absolutely for $\sigma > (2r + 2)/(2r + 3)$.

Proof. Consider Bell series for $\sigma^{(r)}$:

$$\sigma_p^{(r)}(x) := \sum_{k=0}^{\infty} \sigma^{(r)}(p^k) x^k = \sum_{k=0}^r p^k x^k + \sum_{k>r} (p^r + p^k) x^k =$$

$$= \frac{1}{1 - px} + \frac{p^r x^{r+1}}{1 - x}.$$

Then

$$\frac{(1-px)(1-p^r x^{r+1})}{1 - p^{r+1} x^{r+2}} \sigma_p^{(e)s}(x) = 1 + \frac{p^r x^{r+2}(1-px)(1-p^r x^r)}{(1-x)(1 - p^{r+1} x^{r+2})} := h_p(x).$$

For $\sigma > 1$ we have $h_p(p^{-s}) \ll p^{-2}$. For $1 \geq \sigma \geq (2r+2)/(2r+3) + \varepsilon$ we have $h_p(p^{-s}) \ll p^{2r+1-(2r+3)s} \ll p^{-1-\varepsilon}$. Now (12) follows from the representation $\sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s} = \prod_p \sigma_p^{(r)}(p^{-s})$. $\square$

Theorem 10.

$$\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + O(x \log^{5/3} x), \quad D = \frac{\zeta(r+2)H_r(2)}{2\zeta(r+3)}.$$

Proof. For a fixed r let $z(n)$ be the coefficient at n^{-s} of the Dirichlet series

$$\frac{\zeta(s-1)\zeta((r+1)s-r)}{\zeta((r+2)s-r-1)}$$

and let $h(n)$ be the coefficient of the Dirichlet series $H_r(s)$. It follows from (12) that $\sigma^{(r)} = z \star h$. One can verify that

$$z(n) = \sum_{ab^{r+1}c^{r+2}=n} ab^r c^{r+1} \mu(c).$$

Taking into account Lemma 2 with $(\alpha, \beta) = (r+1, r)$ we obtain

$$\begin{aligned} \sum_{n \leq x} z(n) &= \sum_{c \leq x^{1/(r+2)}} c^{r+1} \mu(c) \left(\frac{\zeta(r+2)}{2} \frac{x^2}{c^{2r+4}} + O(xc^{-r-2} \log^{2/3} x) \right) = \\ &= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 + O(x \log^{5/3} x). \end{aligned}$$

Now

$$\begin{aligned} \sum_{n \leq x} \sigma^{(r)}(n) &= \sum_{n \leq x} h(n) \left(\frac{\zeta(r+2)}{2\zeta(r+3)} \frac{x^2}{n^2} + O\left(\frac{x}{n} \log^{5/3} x\right) \right) = \\ &= \frac{\zeta(r+2)}{2\zeta(r+3)} x^2 \sum_{n \leq x} \frac{h(n)}{n^2} + O\left(x \log^{5/3} x \sum_{n \leq x} \frac{h(n)}{n}\right). \end{aligned}$$

But $H_r(s)$ converges absolutely at $\sigma \geq (2r+2)/(2r+3)+\varepsilon$, so $\sum_{n \leq x} \frac{h(n)}{n} \ll O(1)$ and

$$\sum_{n \leq x} \frac{h(n)}{n^2} = H_r(2) - \sum_{n > x} \frac{h(n)}{n^2} = H_r(2) + O(x^{-(2r+4)/(2r+3)+\varepsilon}).$$

□

Theorem 11. For a fixed $r > 0$ $\sum_{n \leq x} \sigma^{(r)}(n) = Dx^2 + \Omega_{\pm}(x \log x)$.

Proof. The proof almost replicates the proof of [10, Th. 3] with following changes (in notations of [10]):

$$\kappa(n) := \frac{\sigma^{(r)}(n)}{n}, \quad \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \frac{\zeta(s)\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),$$

$$v := \mu \star \kappa, \quad \sum_{n=1}^{\infty} \frac{v(n)}{n^s} = \frac{\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),$$

$$v(p^a) = \frac{\sigma^{(r)}(p^a)}{p^a} - \frac{\sigma^{(r)}(p^{a-1})}{p^{a-1}} = \begin{cases} 0, & a \leq r+1, \\ 1/p, & a = r+1, \\ p^{r-a} - p^{r-a+1}, & a > r+1. \end{cases}$$

We take $m := \log^{1/(4r+4)} x$ and $A := \prod_{p \leq m} p^{r+1} \sim e^{(r+1)m} \sim \exp(\log^{1/4} x)$, then

$$G = \sum_{k \leq u(x)} \frac{v(k)}{k} \gcd(A, k) = \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \leq u(x)/n^{r+1}}^* \frac{v(n^{r+1}k)}{v(n^{r+1})k}.$$

Here $\sum_k^*$ means summation over k such that for every $p \mid k$ we have $p \mid n$ or $p \nmid A$. Taking into account $v(p^{r+1}) = 1/p$ we get

$$\begin{aligned} G &= \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \geq 1}^* \frac{v(n^{r+1}k)}{v(n^{r+1})k} + o(1) = \\ &= \sum_{n^{r+1} | A} v(n^{r+1}) \prod_{p|n} \left(1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) \prod_{p > m} \left(1 + \sum_{\nu \geq r+1} \frac{v(p^\nu)}{p^\nu} \right) + o(1). \end{aligned}$$

Since $|v(p^\nu)| \leq 1/p$ we obtain $\sum_{\nu \geq r+1} v(p^\nu) p^{-\nu} \ll p^{-r-2}$. Since $v(n^{r+1}) = 1/n$ for $n^{r+1} \mid A$ and $\log m \asymp \log x$ we have

$$\begin{aligned}
 G &= (1 + o(1)) \sum_{n^{r+1}|A} \frac{1}{n} \prod_{p|n} \left(1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) = \\
 &= (1 + o(1)) \prod_{p \leq m} \left(1 + \frac{1}{p} + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-1}} \right).
 \end{aligned}$$

But $v(p^\nu) \geq 1/2p^{\nu-r-1}$ for $a \geq r+2$. So

$$\sum_{\nu \geq r+2} v(p^\nu) p^{-\nu+r+1} \geq \sum_{\nu \geq r+2} p^{2(-\nu+r+1)}/2 \geq p^{-2}/2.$$

Hence

$$G \gg \prod_{p \leq m} (1 + p^{-1} + p^{-2}/2) \gg \prod_{p \leq m} (1 + p^{-1}) \gg \log m \gg \log x. \quad \square$$

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