ON THE DIVISORS OF ORDER $r$

Andrew V. Lelechenko$^1$ §, Yakov A. Vorobyov$^2$

$^1,2$Computational Algebra and Discrete Mathematics
Odessa National University
Dvoryanskaya st. 2, Odessa, 65026, UKRAINE

Abstract: N. Minculete has introduced the divisor-of-order-$r$ function $\tau(r)$ and the sum-of-divisors-of-order-$r$ function $\sigma(r)$. We investigate the asymptotic behaviour of $\sum_{n \leq x} \tau(r)(n)$ and $\sum_{n \leq x} \sigma(r)(n)$ and improve known estimates.

AMS Subject Classification: 11A25, 11N37
Key Words: divisor of order $r$, exponential semiproper divisor, average order

1. Introduction

Recently N. Minculete in his PhD Thesis [8] introduced a concept of divisors of order $r$: integer $d = p_1^{b_1} \cdots p_k^{b_k}$ is called a divisor of order $r$ of number $n = p_1^{a_1} \cdots p_k^{a_k}$ if $d$ divides $n$ in the usual sense and $b_j \in \{r, a_j\}$ for $j = 1, \ldots, k$. We also suppose that 1 is a divisor of any order of itself (but not of any other number). Let us denote respective divisor and sum-of-divisor functions as $\tau(r)$ and $\sigma(r)$. These functions are multiplicative and

$$\tau(r)(p^a) = \begin{cases} 1, & a \leq r, \\ 2, & a > r. \end{cases}$$

$$\sigma(r)(p^a) = \begin{cases} p^a, & a \leq r, \\ p^a + p^r, & a > r. \end{cases}$$

In a special case of $r = 0$ we get well-studied unitary divisors, see [2, 11]. Below we consider only the case $r > 0$. 

Received: June 2, 2014

© 2015 Academic Publications, Ltd.

url: www.acadpubl.eu

§Correspondence author
In another special case of \( r = 1 \) we get so-called by Minculete exponential semiproper divisors and denote \( \tau^{(e)s} := \tau^{(1)} \), \( \sigma^{(e)s} := \sigma^{(1)} \). An integer \( d \) is an exponential semiproper divisor of \( n \) if \( \ker d = \ker n \) and \( (d/\ker n, n/d) = 1 \), where \( \ker n = \prod_{p|n} p \).

Minculete proved in [8, (3.1.17–19)] that

\[
\limsup_{n \to \infty} \frac{\log \tau^{(r)}(n) \log \log n}{\log n} = \frac{\log 2}{r + 1}, \tag{3}
\]

\[
\sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r + 1)}{\zeta(2r + 2)} x + Ax^{1/(r + 1)} + O(x^{1/(r + 2) + \varepsilon}), \tag{4}
\]

\[
\limsup_{n \to \infty} \frac{\sigma^{(r)}(n)}{n \log \log n} = \frac{6e^\gamma}{\pi^2}. \tag{5}
\]

In the present paper we improve the error term in (4) and establish asymptotic formulas for \( \sum_{n \leq x} \sigma^{(r)}(n) \) with \( O \)- and \( \Omega \)-estimates of the error term.

In asymptotic relations we use \( \sim, \asymp \), Landau symbols \( O \) and \( o \), big omegas \( \Omega \) and \( \Omega_\pm \), Vinogradov symbols \( \ll \) and \( \gg \) in their usual meanings. All asymptotic relations are given as an argument tends to the infinity.

Letter \( p \) with or without indexes denote rational prime.

As usual \( \zeta(s) \) is the Riemann zeta-function. For complex \( s \) we denote \( \sigma := \Re s \) and \( t := \Im s \).

We use abbreviations \( \log \log x := \log \log x, \ll \log x := \log \log \log x \).

Letter \( \gamma \) denotes Euler—Mascheroni constant, \( \gamma \approx 0.577 \).

Everywhere \( \varepsilon > 0 \) is an arbitrarily small number (not always the same even in one equation).

We write \( f \ast g \) for Dirichlet convolution:

\[
(f \ast g)(n) = \sum_{d|n} f(d)g(n/d).
\]

Function \( \ker : \mathbb{N} \to \mathbb{N} \) stands for \( \ker n = \prod_{p|n} p \).

For a set \( A \) notation \( \#A \) means the cardinality of \( A \).

2. Preliminary Estimates

Consider \( \tau(a, b; n) = \sum_{k^n p^n=1} 1 \) and \( T(a, b; x) = \sum_{n \leq x} \tau(a, b; n) \) for \( 1 \leq a \leq b \).

One can directly check that \( \sum_{n=1}^{\infty} \tau(a, b; n)n^{-s} = \zeta(as)\zeta(bs) \) for \( \sigma > 1 \).
Lemma 1. \( T(a, b; x) = H(a, b; x) + \Delta(a, b; x) \), where

\[
H(a, b; x) = \begin{cases} 
\zeta(b/a)x^{1/a} + \zeta(a/b)x^{1/b}, & 1 \leq a < b, \\
x^{1/a}\log x + (2\gamma - 1)x^{1/a}, & a = b,
\end{cases}
\]

\[
x^{1/2(a+b)} \ll \Delta(a, b; x) \ll \begin{cases} 
x^{1/(2a+b)}, & 1 \leq a < b, \\
x^{1/3a}\log x, & a = b.
\end{cases}
\]

Proof. See [6, Th. 5.1, Th. 5.3, Th. 5.8].

In fact \( \Delta(a, b; x) \) can be estimated more precisely. For our goals we are primarily interested in the behaviour of \( \Delta(1, b; x) \). Let us suppose that

\[
\Delta(1, b; x) \ll x^{\theta_b} \log^{\theta'_b} x,
\]

then due to [6, Th. 5.11] for \( b \geq 7 \) we can choose \( \theta_b = 1/(b + 7/2) \) and \( \theta'_b = 1 \). Estimates for \( b \leq 16 \) are given in Table 1. Estimate for \( b = 1 \) belongs to Huxley [4], and estimate for \( b = 2 \) belongs to Graham and Kolesnik [3]. We have found no references on the best known results for \( b \geq 3 \), so we calculated them with the use of [6, Th. 5.11, Th. 5.12] selecting appropriate exponent pairs carefully. It seems that some of this estimates may be new.

Lemma 2. Let \( \alpha \) and \( \beta \) be positive real numbers with \( \beta + 1 = \alpha \). Then

\[
\sum_{mn^\alpha \leq x} mn^\beta = \frac{\zeta(2\alpha - \beta)}{2} x^2 + D(\alpha, \beta; x), \quad D(\alpha, \beta; x) \ll x \log^{2/3} x.
\]

Proof. See [10, Th. 1].

For \( k > 0 \) one can define a multiplicative function \( \mu_k \) implicitly by

\[
\sum_{n=1}^{\infty} \mu_k(n)n^{-s} = 1/\zeta(ks)
\]

for \( \sigma > 1 \). So \( \mu_k(n^k) = \mu(n) \) and \( \mu_k(m) = 0 \) for all other arguments. Trivially \( \mu_1 \equiv \mu \). Then

\[
M_k(x) := \sum_{n \leq x} \mu_k(n) = \sum_{n \leq x^{1/k}} \mu(n) \ll x^{1/k} \exp(-CN(x)),
\]

where \( C > 0 \), \( N(x) = \log^{3/5} x \log^{-1/5} x \). See [5, Th. 12.7] for the proof of the last estimate. Assuming Riemann hypothesis (RH) we get much better result \( M_k(x) \ll x^{1/2k+\varepsilon} \) [12, Th. 14.25 (C)].
Table 1: Values of $\theta_b$ and $\theta'_b$ in (6) for $b \leq 16$. Exponent pairs are given in terms of $A$- and $B$-processes [6, Th. 2.12, 2.13]. We abbreviate $B := BA$. Here $I = (0, 1)$ and $H = (32/205 + \varepsilon, 269/410 + \varepsilon)$ is Huxley exponent pair from [4].

**Lemma 3.** Let $K \in \mathbb{N}$, $J \in \mathbb{N} \cup \{0\}$, $m_1 \leq \cdots \leq m_K$, $n_1 \leq \cdots \leq n_J$, where all $m_k, n_j \in \mathbb{N}$, and suppose that

$$
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{\zeta(m_1 s) \cdots \zeta(m_K s)}{\zeta(n_1 s) \cdots \zeta(n_J s)}.
$$

Let $\alpha = K - 1/(2 \sum_{k=1}^{K} m_k)$. If $1/\alpha < 2n_j$ for all $j = 1, \ldots, J$ then for arbitrary $H(x)$ of the form $H(x) = \sum_{i=1}^{J} x^{\beta_i} P_i(\log x)$, $\beta_i \in \mathbb{C}$, $\alpha < \Re \beta_i \leq 1$, $P_i$ are polynomials, we have $\sum_{n \leq x} a_n = H(x) + \Omega(x^\alpha)$.

**Proof.** This is a simplified version of [7, Th. 2].

---

3. Asymptotic Properties of $\sum \tau^{(r)}(n)$

**Lemma 4.**

$$
\sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s} = \frac{\zeta(s) \zeta((r + 1)s)}{\zeta((2r + 2)s)}, \quad \sigma > 1.
$$ (7)
Proof. Let us transform Bell series for $\tau^{(r)}$:

$$
\tau_p^{(r)}(x) = \sum_{k=0}^{\infty} \tau^{(r)}(p^k) x^k = \sum_{k=0}^{r} x^k + 2 \sum_{k>r} x^k = \sum_{k=0}^{\infty} x^k + \sum_{k>r} x^k = (1 + x^{r+1}) \sum_{k=0}^{\infty} x^k = \frac{1 + x^{r+1}}{1 - x} = \frac{1 - x^{2r+2}}{(1 - x)(1 - x^{r+1})}.
$$

Identity $\sum_{n=1}^{\infty} \frac{\tau^{(r)}(n)}{n^s} = \prod_p \tau_p^{(r)}(p^{-s})$ completes the proof. \qed

It follows from (7) that

$$
\tau^{(r)} = \tau(1, r + 1; \cdot) \ast \mu_{2r+2}
$$

(8)

**Theorem 5.** If $\Delta$ is estimated as in (6) then for $r > 0$

$$
\sum_{n \leq x} \tau^{(r)}(n) = Ax + Bx^{1/(r+1)} + \mathcal{E}_{r+1}(x),
$$

where $\mathcal{E}_r(x) = O\left(x^{\max(\theta_r,1/2r) \log^{\theta_r} x}\right)$, constants $A$ and $B$ are specified below in (9).

Proof. Taking into account (8) we have for $r > 0$

$$
\sum_{n \leq x} \tau^{(r)}(n) = \sum_{n \leq x} \mu_{2r+2}(n)T(1, r + 1; x/n) = \\
= \zeta(r + 1)x \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n} + \zeta(1/(r + 1))x^{1/(r+1)} \sum_{n \leq x} \frac{\mu_{2r+2}(n)}{n^{1/(r+1)}} + \\
+ \sum_{n \leq x} \mu_{2r+2}(n)\Delta(1, r + 1, x/n).
$$

But for $s \geq 1/k$

$$
\sum_{n \leq x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} - \sum_{n > x} \frac{\mu_k(n)}{n^s} = \frac{1}{\zeta(ks)} + O(x^{1/k-s})
$$

and

$$
\sum_{n \leq x} \mu_{2k}(n)\Delta(1, k, x/n) = \sum_{n \leq x^{1/2k}} \mu(n)\Delta(1, k, x/n^{2k}) \ll
$$
\[
\ll \sum_{n \leq x^{1/2k}} \left( \frac{x}{n^{2k}} \right)^{\theta_k} \log^{\theta_k'} x \ll x^{\max(\theta_k, 1/2k)} \log^{\theta_k'} x.
\]

So
\[
\sum_{n \leq x} \tau^{(r)}(n) = \frac{\zeta(r+1)}{\zeta(2r+2)} x + \frac{\zeta(1/r+1)}{\zeta(2)} x^{1/r+1} + E_{r+1}(x) \tag{9}
\]

**Lemma 6.** Let \( r > 0, x^\varepsilon \leq y \leq x^{1/2r} \). Then under RH we have
\[
E_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon} y^{1/2-r} + x^\varepsilon). \tag{10}
\]

**Proof.** We follow the approach of Montgomery and Vaughan (see [9] or [1]).

First of all consider \( g_y(s) = 1/\zeta(s) - \sum_{d \leq y} \mu(d)d^{-s} \). Then for \( \sigma > 1 \) we have \( g_y(s) = \sum_{d > y} \mu(d)d^{-s} \). Assuming RH we have by [12, Th. 14.25]
\[
\sum_{d \leq y} \frac{\mu(d)}{d^s} = \zeta^{-1}(s) + O\left( y^{1/2-\sigma+\varepsilon} (|t|^\varepsilon + 1) \right) \quad \text{for } \sigma > 1/2 + \varepsilon,
\]
so
\[
g_y(s) \ll y^{1/2-\sigma+\varepsilon} (|t|^\varepsilon + 1) \quad \text{for } \sigma > 1/2 + \varepsilon. \tag{11}
\]

Now let us split \( \sum_{n \leq x} \tau^{(r-1)}(n) \) into two parts:
\[
\sum_{n \leq x} \tau^{(r-1)}(n) = \sum_{d^{2r} \leq x} \mu(d) T(1, r; x/d^{2r}) = S_1 + S_2,
\]
where
\[
S_1 := \sum_{d \leq y} \mu(d) T(1, r; x/d^{2r}) = \zeta(r)x \sum_{d \leq y} \frac{\mu(d)}{d^{2r}} + \zeta(1/r)x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} + \sum_{d \leq y} \mu(d) \Delta(1, r; x/d^{2r})
\]
and \( S_2 \) is the rest of \( \sum_{n \leq x} \tau^{(r-1)}(n) \). We note that under RH by taking into account \( y \leq x^{1/2r} \) we have
\[
x^{1/r} \sum_{d > y} \frac{\mu(d)}{d^2} \ll x^{1/r} y^{-3/2+\varepsilon} \ll x^{1/2} y^{1/2-r+\varepsilon}
\]
and so
\[ x^{1/r} \sum_{d \leq y} \frac{\mu(d)}{d^2} = \frac{x^{1/r}}{\zeta(2)} + O(x^{1/2}y^{1/2-r+\varepsilon}). \]

Next, let \( h_y(s) := \zeta(s)\zeta(rs)g_y(2rs)x^s s^{-1} \). Then by Perron formula with \( c = 1 + \varepsilon \), \( T = x^2 \) one can estimate
\[
S_2 = \frac{1}{2\pi i} \int_{1+\varepsilon - ix^2}^{1+\varepsilon + ix^2} h_y(s) ds + O(x^\varepsilon).
\]

By moving line of integration to \([1/2 + \varepsilon - ix^2, 1/2 + \varepsilon + ix^2]\) we obtain
\[
S_2 = \text{res}_{s=1} h(s) + O(I_1 + I_2 + I_3),
\]
where
\[
I_1 = \int_{1+\varepsilon - ix^2}^{1/2+\varepsilon - ix^2} h(s) ds, \quad I_2 = \int_{1/2+\varepsilon - ix^2}^{1/2+\varepsilon + ix^2} h(s) ds, \quad I_3 = \int_{1+\varepsilon - ix^2}^{1/2+\varepsilon - ix^2} h(s) ds.
\]

Due to (11) and estimates of \( \zeta \) under RH we have
\[
g_y(2rs) \ll y^{1/2-r}(|t|^\varepsilon + 1) \quad \text{for } \sigma > 1/2 + \varepsilon,
\]
\[
h(s) \ll y^{1/2-r}(|t|^\varepsilon + 1)x^s s^{-1} \quad \text{for } \sigma > 1/2 + \varepsilon,
\]
and
\[
I_{1,3} \ll y^{1/2-r+\varepsilon} \max_{\sigma \in [1/2+\varepsilon, 1+\varepsilon]} x^{\sigma-2} \ll y^{1/2-r+\varepsilon},
\]
\[
I_2 \ll y^{1/2-r+\varepsilon} \int_1^{x^2} x^{1/2-t-1} dt \ll y^{1/2-r+\varepsilon} x^{1/2+\varepsilon}.
\]
Identity
\[
\text{res}_{s=1} h(s) = \zeta(r) x \sum_{d>y} \frac{\mu(d)}{d^{2r}}
\]
completes the proof.

**Theorem 7.** If \( \Delta \) is estimated as in (6) and \( \theta_r < 1/2r \) then under RH we get \( \mathcal{E}_r(x) = O(x^\alpha) \), where \( \alpha = (1 - \theta_r)/(2r + 1 - 4r\theta_r) \).

**Proof.** Let us start with (10):
\[
\mathcal{E}_r(x) = \sum_{n \leq y} \mu(n) \Delta(1, r, x/n^{2r}) + O(x^{1/2+\varepsilon}y^{1/2-r} + x^\varepsilon) \ll
\]
\[ \ll \sum_{n \leq y} \left( \frac{x}{n^{2r}} \right)^{\theta_r + \varepsilon} + O(x^{1/2 + \varepsilon} y^{1/2 - r} + x^\varepsilon) \ll \]
\[ \ll x^\varepsilon \left( x^{\theta_r} \left( 1 + y^{1 - 2r \theta_r} \right) + x^{1/2} y^{1/2 - r} + 1 \right). \]

If \( \theta_r < 1/2r \) then \( \mathcal{E}_r(x) \ll x^\varepsilon \left( x^{\theta_r} y^{1 - 2r \theta_r} + x^{1/2} y^{1/2 - r} \right) \). Choice \( y = x^\beta \), where \( \beta = (1 - 2 \theta_r)/(2r + 1 - 4r \theta_r) \), accomplishes the proof. \( \square \)

For the values of \( \theta_b \) from Table 1 we have
\[ \max(\theta_r, 1/2r) = \begin{cases} 1/2r, & r \leq 2, \\ \theta_r, & r > 2. \end{cases} \]
So currently the only non-trivial case of the previous theorem is an estimation for \( \tau^{(1)} \equiv \tau^{(e)s} \). We get under assumption of RH that
\[ \sum_{n \leq x} \tau^{(1)}(n) = \frac{\zeta(2)}{\zeta(4)} x + \frac{\zeta(1/2)}{\zeta(2)} x^{1/2} + O(x^{\alpha + \varepsilon}), \]
\[ \alpha = \frac{1 - \theta_2}{5 - 8 \theta_2} = \frac{3728}{15469} \approx 0.241 < 1/4. \]

**Theorem 8.** \( \mathcal{E}_r(x) = \Omega \left( x^{1/(2r + 2)} \right) \).

**Proof.** The statement is implied by the substitution \( m_1 = 1, m_2 = r, n_1 = 2r \) into Lemma 3. The choice of parameters plainly follows from (7). We obtain \( \alpha = 1/(2r + 2) \), which is an exponent in the required \( \Omega \)-term. \( \square \)

4. Asymptotic Properties of \( \sum \sigma^{(r)} \)

**Lemma 9.**
\[ \sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s} = \frac{\zeta(s - 1) \zeta(r + 1 - s)}{\zeta(r + 2) s - r - 1} H_r(s), \quad \sigma > 2, \quad (12) \]
where Dirichlet series \( H_r(s) \) converges absolutely for \( \sigma > (2r + 2)/(2r + 3) \).

**Proof.** Consider Bell series for \( \sigma^{(r)} \):
\[ \sigma^{(r)}_p(x) := \sum_{k=0}^{\infty} \sigma^{(r)}(p^k) x^k = \sum_{k=0}^{r} p^k x^k + \sum_{k>r} (p^r + p^k) x^k = \]
ON THE DIVISORS OF ORDER $r$

$$= \frac{1}{1 - px} + \frac{p^r x^{r+1}}{1 - x}.$$ 

Then

$$\frac{(1 - px)(1 - p^r x^{r+1})}{1 - p^{r+1}x^{r+2}} \sigma_p^{(e)}(x) = 1 + \frac{p^r x^{r+2} (1 - px)(1 - p^r x^r)}{(1 - x)(1 - p^{r+1}x^{r+2})} := h_p(x).$$

For $\sigma > 1$ we have $h_p(p^{-s}) \ll p^{-2}$. For $1 \geq \sigma \geq (2r + 2)/(2r + 3) + \varepsilon$ we have $h_p(p^{-s}) \ll p^{2r+1-(2r+3)s} \ll p^{-1-\varepsilon}$. Now (12) follows from the representation

$$\sum_{n=1}^{\infty} \frac{\sigma^{(r)}(n)}{n^s} = \prod_p \sigma_p^{(r)}(p^{-s}).$$

**Theorem 10.**

$$\sigma^{(r)}(n) = D x^2 + O(x \log^{5/3} x), \quad D = \frac{\zeta(r + 2) H_r(2)}{2 \zeta(r + 3)}.$$

**Proof.** For a fixed $r$ let $z(n)$ be the coefficient at $n^{-s}$ of the Dirichlet series

$$\frac{\zeta(s - 1)\zeta((r + 1)s - r)}{\zeta((r + 2)s - r - 1)}$$

and let $h(n)$ be the coefficient of the Dirichlet series $H_r(s)$. It follows from (12) that $\sigma^{(r)} = z \ast h$. One can verify that

$$z(n) = \sum_{ab^{r+1}c^{r+2} = n} ab^r c^{r+1} \mu(c).$$

Taking into account Lemma 2 with $(\alpha, \beta) = (r + 1, r)$ we obtain

$$\sum_{n \leq x} z(n) = \sum_{c \leq x^{1/(r+2)}} c^{r+1} \mu(c) \left( \frac{\zeta(r + 2)}{2} \frac{x^2}{c^{2r+4}} + O\left(x c^{-r-2} \log^{2/3} x\right) \right) = \frac{\zeta(r + 2)}{2 \zeta(r + 3)} x^2 + O(x \log^{5/3} x).$$

Now

$$\sum_{n \leq x} \sigma^{(r)}(n) = \sum_{n \leq x} h(n) \left( \frac{\zeta(r + 2)}{2 \zeta(r + 3)} \frac{x^2}{n^2} + O\left(\frac{x}{n} \log^{5/3} x\right) \right) = \frac{\zeta(r + 2)}{2 \zeta(r + 3)} x^2 \sum_{n \leq x} \frac{h(n)}{n^2} + O\left(x \log^{5/3} x \sum_{n \leq x} \frac{h(n)}{n}\right).$$
But $H_r(s)$ converges absolutely at $\sigma \geq (2r+2)/(2r+3)+\varepsilon$, so $\sum_{n \leq x} \frac{h(n)}{n} \ll O(1)$ and
\[
\sum_{n \leq x} \frac{h(n)}{n^2} = H_r(2) - \sum_{n > x} \frac{h(n)}{n^2} = H_r(2) + O\left(x^{-\frac{(2r+4)}{(2r+3)+\varepsilon}}\right).
\]

\[ \square \]

**Theorem 11.** For a fixed $r > 0$ $\sum_{n \leq x} \sigma^{(r)}(n) = D x^2 + \Omega_{\pm}(x \log x)$.

**Proof.** The proof almost replicates the proof of [10, Th. 3] with following changes (in notations of [10]):

\[
\kappa(n) := \frac{\sigma^{(r)}(n)}{n}, \quad \sum_{n=1}^{\infty} \frac{\kappa(n)}{n^s} = \frac{\zeta(s)\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),
\]
\[
u := \mu * \kappa, \quad \sum_{n=1}^{\infty} \frac{\nu(n)}{n^s} = \frac{\zeta((r+1)s+1)}{\zeta((r+2)s+1)} H_r(s+1),
\]
\[
u(p^a) = \frac{\sigma^{(r)}(p^a)}{p^a} - \frac{\sigma^{(r)}(p^{a-1})}{p^{a-1}} = \begin{cases} 
0, & a \leq r + 1, \\
1/p, & a = r + 1, \\
\frac{p^{r-a} - p^{r-a+1}}{a} & a > r + 1.
\end{cases}
\]

We take $m := \log^{1/(4r+4)} x$ and $A := \prod_{p \leq m} p^{r+1} \sim e^{(r+1)m} \sim \exp(\log^{1/4} x)$, then

\[
G = \sum_{k \leq u(x)} \frac{\nu(k)}{k} \gcd(A, k) = \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \leq u(x)/n^{r+1}}^{*} \frac{v(n^{r+1}k)}{v(n^{r+1})k}.
\]

Here $\sum_{k}^{*}$ means summation over $k$ such that for every $p | k$ we have $p | n$ or $p \nmid A$. Taking into account $\nu(p^{r+1}) = 1/p$ we get

\[
G = \sum_{n^{r+1} | A} v(n^{r+1}) \sum_{k \geq 1}^{*} \frac{v(n^{r+1}k)}{v(n^{r+1})k} + o(1) = \sum_{n^{r+1} | A} v(n^{r+1}) \prod_{p | n} \left(1 + \sum_{\nu \geq r+2} \frac{\nu(p^{\nu})}{p^{\nu(r-2)}}\right) \prod_{p > m} \left(1 + \sum_{\nu \geq r+1} \frac{\nu(p^{\nu})}{p^{\nu}}\right) + o(1).
\]

Since $|\nu(p^{\nu})| \leq 1/p$ we obtain $\sum_{\nu \geq r+1} \nu(p^{\nu})p^{-\nu} \ll p^{-r-2}$. Since $v(n^{r+1}) = 1/n$ for $n^{r+1} | A$ and $\log m \succ \log x$ we have
ON THE DIVISORS OF ORDER \( r \)

\[
G = (1 + o(1)) \sum_{n \leq m} \frac{1}{n} \prod_{p | n} \left( 1 + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-2}} \right) = (1 + o(1)) \prod_{p \leq m} \left( 1 + \frac{1}{p} + \sum_{\nu \geq r+2} \frac{v(p^\nu)}{p^{\nu-r-1}} \right).
\]

But \( v(p^\nu) \geq 1/2 p^{\nu-r-1} \) for \( a \geq r+2 \). So

\[
\sum_{\nu \geq r+2} v(p^\nu)p^{-\nu+r+1} \geq \sum_{\nu \geq r+2} p^{2(-\nu+r+1)/2} \geq p^{-2}/2.
\]

Hence

\[
G \gg \prod_{p \leq m} (1 + p^{-1} + p^{-2}/2) \gg \prod_{p \leq m} (1 + p^{-1}) \gg \log m \gg \log x. \quad \Box
\]

References


