PRIME EXTENSION FILTRATION OF MODULES

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Abstract: In this article we introduce a filtration for modules namely, regular prime extension filtration, which exist for all finitely generated modules over Noetherian rings. We show that any two such filtrations of a module have same length.

AMS Subject Classification: 13C05, 13E05, 13E15

Key Words: associated prime ideals, prime filtration, Noetherian ring, prime submodule and torsion free module

1. Introduction

Throughout this paper unless otherwise stated A is a commutative Noetherian ring with identity and all modules are finitely generated and unitary. For standard reference and notations see [3] and [4].

Received: September 4, 2014

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It is well known that for a finitely generated module M over a Noetherian ring A , there exists a filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

of submodules of M . The filtration $\mathcal{F}$ with the cyclic quotients $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \text{Supp}(M)$ is called a prime filtration of M . The set of prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\text{Supp}(\mathcal{F})$. We also have that [4, Theorem 6.5]

$$\text{Ass}(M) \subseteq \text{Supp}(\mathcal{F}) \subset \text{Supp}(M).$$

In [2] Dress called the filtration $\mathcal{F}$ as weak prime decomposition of M , if for every $1 \leq i \leq n$, there exists a prime ideal $\mathfrak{p} = \mathfrak{p}_i \in \text{Spec} A$ with $\mathfrak{p} = (M_{i-1} : x)$ for all $x \in M_i \setminus M_{i-1}$. Clearly prime filtrations are weak prime decomposition and therefore every Noetherian A -module has a weak prime decomposition. For any Noetherian A -module M always admits a weak prime decomposition $\mathcal{F}$ with $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$.

In this article for a proper submodule N of M , we say a submodule K of M is prime extension of N in M , if N is a prime submodule of K . We say the filtration $\mathcal{F}$ is prime extension filtration of M if M_i is a prime extension of M_{i-1} in M for $1 \leq i \leq n$. It is clear that prime extension filtrations are nothing but weak prime decomposition of M . We call a prime extension filtration of M is maximal prime extension (MPE) filtration if M_i is maximal prime extension of M_{i-1} for $1 \leq i \leq n$. We prove that if $\mathcal{F}$ is MPE filtration of M , then $\text{Supp}(\mathcal{F}) = \text{Ass}(M)$.

Next we call an MPE filtration $N = M_0 \subset M_1 \subset \cdots \subset M_n = M$ with $(M_{i-1} : M_i) = \mathfrak{p}_i$, as regular prime extension (RPE) filtration of M over N , if $\mathfrak{p}_i$ is a maximal element in $\text{Ass}(M/M_{i-1})$. We prove that any two RPE filtration of M over N have same length. Hence this length is a numerical invariant for finitely generated modules over Noetherian ring.

2. Prime Extension of Submodules

Definition 1. Let N be a proper submodule of M . A submodule K of M is called prime extension of N in M , if N is a prime submodule of K and we denote it as $N \overset{\text{p}}{\subset} K$, when $(N : K) = \mathfrak{p}$.

Note 2. Let M be an A -module and N be a proper submodule of M . A submodule K of M is a $\mathfrak{p}$ -prime extension of N in M if and only if $N \subsetneq K$ and $(N : x) = \mathfrak{p}$, for every $x \in K \setminus N$. Moreover, $\text{Ass}(K/N) = \{\mathfrak{p}\}$.

Lemma 3. *Let N be a proper submodule of an A -module M . Then $\mathfrak{p} \in \text{Ass}(M/N)$ if and only if there exists a $\mathfrak{p}$ -prime extension of N in M .*

Proof. Let $\mathfrak{p} \in \text{Ass}(M/N)$, then $\mathfrak{p} = \text{ann}(x + N)$ for some $x \in M \setminus N$. Then $K = N + Ax$ is a $\mathfrak{p}$ -prime extension of N in M .

Conversely, assume K is a $\mathfrak{p}$ -prime extension of N in M . Then by note 2, $\{\mathfrak{p}\} = \text{Ass}(K/N) \subset \text{Ass}(M/N)$. $\square$

Lemma 4. *Let M be an A -module and N be a proper submodule of M . If M_1 and M_2 are $\mathfrak{p}_1$ and $\mathfrak{p}_2$ -prime extensions of N respectively and $\mathfrak{p}_1 \neq \mathfrak{p}_2$, then $M_1 \cap M_2 = N$.*

Proof. Suppose $N \neq M_1 \cap M_2$. Since N is a $\mathfrak{p}_1$ -prime submodule of M_1 , it is also $\mathfrak{p}_1$ -prime submodule of $M_1 \cap M_2$ and hence $(N : M_1 \cap M_2) = \mathfrak{p}_1$. Similarly, since N is a $\mathfrak{p}_2$ -prime submodule of M_2 , $(N : M_1 \cap M_2) = \mathfrak{p}_2$, a contradiction. Hence $M_1 \cap M_2 = N$. $\square$

Definition 5. Let M be an A -module and N be a proper submodule of M . A prime extension K of N is called maximal, if there is no prime extension L of N in M , which properly contains K .

Example 6. Let M be a torsion free module over an integral domain A and let $\mathfrak{p} = (0)$. Then any nontrivial submodule of M is a $\mathfrak{p}$ -prime extension of zero submodule in M . But the maximal $\mathfrak{p}$ -prime extension of zero submodule in M is M itself.

Remark 7. Let N be a proper submodule of M and $\mathfrak{p} \in \text{Ass}(M/N)$. Since M is Noetherian, there exists a maximal $\mathfrak{p}$ -prime extension of N in M .

Generally, for a proper submodule N of M , $\text{Ass}(M/N)$ need not be contained in $\text{Ass}(M)$. Next we show that $\text{Ass}(M/N) \subseteq \text{Ass}(M)$, if N is a maximal prime extension of 0 in M . More generally, we prove the following theorem.

Theorem 8. *Let M be a finitely generated module over a Noetherian ring A and N be a proper submodule of M . If a submodule K of M is a maximal prime extension of N in M , then $\text{Ass}(M/K) \subseteq \text{Ass}(M/N)$ and hence $\text{Ass}(M/N) = \text{Ass}(K/N) \cup \text{Ass}(M/K)$.*

Proof. Let $(N : K) = \mathfrak{p}$. Suppose $\mathfrak{q} \in \text{Ass}(M/K)$ and L be a $\mathfrak{q}$ -prime extension of K in M .

Case(i) $\mathfrak{p} \not\subseteq \mathfrak{q}$. For $x \in L \setminus K$ and $p \in \mathfrak{p} \setminus \mathfrak{q}$, we have that $px \notin K$ and $(K : px) = \mathfrak{q}$. Also $pqx \subseteq N$ and therefore $\mathfrak{q} \subseteq (N : px)$. Clearly $(N : px) \subset (K : px) = \mathfrak{q}$. Therefore $\mathfrak{q} = (N : px)$ and so $\mathfrak{q} \in \text{Ass}(M/N)$.

Case(ii) $\mathfrak{p} \subseteq \mathfrak{q}$. Then $\mathfrak{p}L \subseteq K$. Suppose $\mathfrak{p}L \not\subseteq N$, then for some $x \in L \setminus K$ and $p \in \mathfrak{p}$, $px \notin N$. Clearly, $pqx \subseteq N$ implies $\mathfrak{q} \subseteq (N : px)$. Since K is a $\mathfrak{p}$ -prime extension of N and $px \in K$, then by note 2, $(N : px) = \mathfrak{p} \subseteq \mathfrak{q}$. Therefore $\mathfrak{q} \in \text{Ass}(M/N)$.

Next assume that $\mathfrak{p}L \subseteq N$. Let $y \in L \setminus K$, then $\mathfrak{p} \subseteq (N : y)$. We have that $(N : y) \subseteq (K : y) = \mathfrak{q}$. Suppose $(N : y) \subsetneq \mathfrak{q}$, and let $q \in \mathfrak{q} \setminus (N : y)$. But $qy \in K$ implies $(N : qy) = \mathfrak{p}$, that is $(N : y) \subseteq \mathfrak{p}$. This implies $(N : y) = \mathfrak{p}$, for every $y \in L \setminus K$. Since $N \subset K$ is $\mathfrak{p}$ -prime extension for every $y \in K \setminus N$, $(N : y) = \mathfrak{p}$. Then for every $y \in L \setminus N$, $(N : y) = \mathfrak{p}$ and so by note 2, L is a $\mathfrak{p}$ -prime extension of N in M , a contradiction to the maximality of K . Hence $(N : y) = \mathfrak{q}$, and so $\mathfrak{q} \in \text{Ass}(M/N)$. $\square$

Taking $N = 0$ in the above theorem we have the following corollary.

Corollary 9. *If K is a maximal prime extension of 0 in M , then $\text{Ass}(M/K) \subset \text{Ass}(M)$ and hence $\text{Ass}(M) = \text{Ass}(K) \cup \text{Ass}(M/K)$.*

Example 10. Maximal prime extension of a proper submodule of M need not be unique. For example, let k be a field and x, y be indeterminates. Let $A = k[x, y]$ and $M = \frac{k[x, y]}{(x^2, xy)}$ and X, Y be images of x, y in M . Then $\mathfrak{p} = (x)$ belongs to $\text{Ass}M$, but AY , $A(X + Y)$ are two different maximal $\mathfrak{p}$ -prime extension of 0 in M .

Next we show that the maximal $\mathfrak{p}$ -prime extension of a submodule N is unique, if $\mathfrak{p}$ is a maximal element in $\text{Ass}(M/N)$.

Theorem 11. *Let M be an A -module, N be a proper submodule of M and $\mathfrak{p}$ be a maximal element in $\text{Ass}(M/N)$. Then*

$$L = \{x \in M \mid (N : x) \supseteq \mathfrak{p}\}$$

is a unique maximal $\mathfrak{p}$ -prime extension of N in M .

Proof. Clearly, L is a submodule of M . Since $\mathfrak{p} \in \text{Ass}(M/N)$, $\mathfrak{p} = (N : x)$ for some $x \in M \setminus N$. By assumption $x \in L$ and so N is properly contained in L . Now consider the nonempty set,

$$\Sigma = \{(N : z) \mid z \in L \setminus N\}$$

We have that, $\mathfrak{p} \subseteq (N : z)$ for every $(N : z) \in \Sigma$. Since A is Noetherian, Σ has a maximal element, say $(N : y)$. Then clearly $(N : y)$ is a prime ideal and so $(N : y) \in \text{Ass}(M/N)$. Since $(N : y) \supseteq \mathfrak{p}$ and $\mathfrak{p}$ is a maximal element in

$\text{Ass}(M/N)$, $(N : y) = \mathfrak{p}$. Therefore $\Sigma = \{\mathfrak{p}\}$ and hence for every $x \in L \setminus N$, $(N : x) = \mathfrak{p}$, then by note 2, L is a $\mathfrak{p}$ -prime extension of N in M .

Let K be any $\mathfrak{p}$ -prime extension of N in M , then by note 2, $(N : x) = \mathfrak{p}$, for every $x \in K \setminus N$ and by definition of L , $x \in L$. Hence $K \subseteq L$ and so L is unique maximal $\mathfrak{p}$ -prime extension of N in M . $\square$

3. Maximal Prime Extension Filtration

Next we define prime extension filtration of a module and we show that for every finitely generated module over a Noetherian ring prime extension filtration of M over N exists.

Definition 12. Let M be an A -module. A filtration of submodules of M , $N = M_0 \subset M_1 \subset \cdots \subset M_n = M$ is called prime extension filtration of M over N , if M_i is a prime extension of M_{i-1} in M , for $1 \leq i \leq n$. We say a prime ideal $\mathfrak{p}$ occurs in this prime extension filtration, if M_i is a maximal $\mathfrak{p}$ -prime extension of M_{i-1} for some i .

We may also write the filtration as $N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{n-1} \overset{\mathfrak{p}_n}{\subset} M_n = M$, where M_i is $\mathfrak{p}_i$ -prime extension of M_{i-1} .

If $N = 0$, then we say it as prime extension filtration of M .

For a proper submodule N of M , there exist a prime extension filtration of finite length of M over N .

Definition 13. A prime extension filtration of M over N is called maximal prime extension (MPE) filtration of M over N , if each prime extension in the filtration is maximal. If $N = 0$ then we call it as MPE filtration of M .

For example $0 \overset{(0)}{\subset} 2\mathbb{Z} \overset{(2)}{\subset} \mathbb{Z}$ is a prime extension filtration. But the only maximal prime extension filtration of $\mathbb{Z}$ is $0 \overset{(0)}{\subset} \mathbb{Z}$.

Note that by remark 7, each finitely generated module over a Noetherian ring has a MPE filtration of finite length.

Dress [2, PD7] proved that each finitely generated A -module M admits a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$, where each M_i is $\mathfrak{p}_i$ -prime extension of M_{i-1} such that $\text{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Now we show that the MPE filtrations of M are such filtration for M .

Proposition 14. Let N be the proper submodule of M . If

$$N = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a MPE filtration of M over N , with $(M_{i-1} : M_i) = \mathfrak{p}_i$ for every i , then $\text{Ass}(M/M_i) \subseteq \text{Ass}(M/N)$, for $1 \leq i \leq n$, and $\text{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

Proof. If $n = 1$, the result is trivial. So we assume $n > 1$ and prove by induction on i . If $i = 1$, by theorem 8, $\text{Ass}(M/M_1) \subseteq \text{Ass}(M/N)$. Now we assume $\text{Ass}(M/M_{i-1}) \subseteq \text{Ass}(M/N)$. Since M_i is a maximal $\mathfrak{p}_i$ -prime extension of M_{i-1} in M , by theorem 8, $\text{Ass}(M/M_i) \subseteq \text{Ass}(M/M_{i-1})$. Therefore $\text{Ass}(M/M_i) \subseteq \text{Ass}(M/N)$.

We prove the other part by induction on length of MPE filtration. If $n = 1$ then by note 2, $\text{Ass}(M/N) = \{\mathfrak{p}_1\}$. Now we assume this result is true for the MPE filtration of length less than n . Considering M_{n-1}/N as a submodule of M/N , we have that $\text{Ass}(M/N) \subseteq \text{Ass}(M_{n-1}/N) \cup \text{Ass}(M/M_{n-1})$. Now by previous paragraph $\text{Ass}(M/M_{n-1}) \subseteq \text{Ass}(M/N)$, that is $\text{Ass}(M/N) = \text{Ass}(M_{n-1}/N) \cup \text{Ass}(M/M_{n-1})$. Since M is a $\mathfrak{p}_n$ -prime extension of M_{n-1} , by note 2, $\text{Ass}(M/M_{n-1}) = \{\mathfrak{p}_n\}$. Also

$$N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \dots \overset{\mathfrak{p}_{n-1}}{\subset} M_{n-1}$$

is a MPE filtration of M_{n-1} over N of length $n - 1$. Then by assumption $\text{Ass}(M_{n-1}/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_{n-1}\}$, and so $\text{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. $\square$

If we take $N = 0$ in proposition 14, then we have the following corollary.

Corollary 15. *Let M be an A -module and*

$$0 = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \dots \overset{\mathfrak{p}_n}{\subset} M_n = M$$

be a MPE filtration of M . Then $\text{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. In other words a prime ideal $\mathfrak{p}$ occurs in a MPE filtration of M if and only if $\mathfrak{p} \in \text{Ass}(M)$.

Note, the above result is not true for prime extension filtration which are not maximal. For example we have the prime extension filtration $0 \overset{(0)}{\subset} 2\mathbb{Z} \overset{(2)}{\subset} \mathbb{Z}$, but the prime ideal $(2) \notin \text{Ass}\mathbb{Z}$.

Proposition 16. *Let N be a proper submodule of M and let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be maximal element in $\text{Ass}(M/N)$ and $N \overset{\mathfrak{p}_1}{\subset} K_1 \overset{\mathfrak{p}_2}{\subset} L$ be a filtration, such that $N \subset K_1$ is maximal $\mathfrak{p}_1$ -prime extension in M and $K_1 \subset L$ is maximal $\mathfrak{p}_2$ -prime extension in M . Then there exist a submodule K_2 of M such that $N \overset{\mathfrak{p}_2}{\subset} K_2 \overset{\mathfrak{p}_1}{\subset} L$ is a filtration, with $N \subset K_2$ is maximal $\mathfrak{p}_2$ -prime extension in M and $K_2 \subset L$ is maximal $\mathfrak{p}_1$ -prime extension in M .*

Proof. Since $\mathfrak{p}_2$ is a maximal element in $\text{Ass}(M/N)$, by theorem 11, there exist a unique maximal $\mathfrak{p}_2$ -prime extension K_2 of N in M . We have to show that L is maximal $\mathfrak{p}_1$ -prime extension of K_2 in M .

Let $a \in (K_2 : L)$, then $aL \subseteq K_2$. Now $aK_1 \subseteq aL \subseteq K_2$ and $aK_1 \subseteq K_2 \cap K_1$, then by lemma 4, $K_2 \cap K_1 = N$. Therefore $aK_1 \subseteq N$, implies $a \in (N : K_1) = \mathfrak{p}_1$. So $(K_2 : L) \subseteq \mathfrak{p}_1$. Let $a \in \mathfrak{p}_1$. Suppose $ax \notin K_2$, for some $x \in L$. Since K_2 is the unique maximal $\mathfrak{p}_2$ -prime extension of N in M , by theorem 11, $\mathfrak{p}_2 \not\subseteq (N : ax)$. That is there exist an element $p_2 \in \mathfrak{p}_2$, such that $p_2(ax) \notin N$ and therefore $p_2x \notin N$ and $a \notin (N : p_2x)$. Since $x \in L$ and L is $\mathfrak{p}_2$ -prime extension of K_1 , $p_2x \in K_1$ and hence $(N : p_2x) = \mathfrak{p}_1$. This implies $a \notin \mathfrak{p}_1$, a contradiction. So $ax \in K_2$, for every $x \in L$, that is $a \in (K_2 : L)$ and therefore $\mathfrak{p}_1 \subseteq (K_2 : L)$. So we proved $\mathfrak{p}_1 = (K_2 : L)$.

Let $x \in K_2$. Since $N \subset K_2$ is maximal $\mathfrak{p}_2$ -prime extension by theorem 11, $(N : x) \supseteq \mathfrak{p}_2$. That is $\mathfrak{p}_2x \subseteq N \subset K_1$ and this implies $\mathfrak{p}_2 \subseteq (K_1 : x)$. Again by theorem 11, $x \in L$, so $K_2 \subseteq L$. Since $(K_2 : L) = \mathfrak{p}_1$, K_2 is proper submodule of L .

Let $ax \in K_2$ for some $x \in L$ and $a \in A$. Suppose $x \notin K_2$, then by theorem 11, $\mathfrak{p}_2 \not\subseteq (N : x)$. That is there exist an element $p_2 \in \mathfrak{p}_2$ such that $p_2x \notin N$. But $ax \in K_2$, implies $p_2ax \in N$. Since $p_2x \in K_1$, $a \in (N : p_2x) = \mathfrak{p}_1$. This implies that L is a $\mathfrak{p}_1$ -prime extension of K_2 .

Next we prove L is maximal $\mathfrak{p}_1$ -prime extension of K_2 in M . Since K_2 is maximal prime extension of N , and by theorem 8, $\text{Ass}(M/K_2) \subset \text{Ass}(M/N)$, so $\mathfrak{p}_1$ is a maximal element in $\text{Ass}(M/K_2)$. Then by theorem 11, there exist a unique maximal $\mathfrak{p}_1$ -prime extension M of K_2 in M . So it is enough to show $M \subseteq L$.

Let $x \in M$, then $\mathfrak{p}_1 \subseteq (K_2 : x)$. That is for every $p_1 \in \mathfrak{p}_1$, $p_1x \in K_2$ and again by theorem 11, on maximal $\mathfrak{p}_2$ -prime extension $N \subset K_2$, $\mathfrak{p}_2 \subseteq (N : p_1x)$. Therefore $p_1p_2x \in N$ for every $p_2 \in \mathfrak{p}_2$. In other words $\mathfrak{p}_1 \subseteq (N : p_2x)$, then by $N \subset K_1$ is $\mathfrak{p}_1$ -maximal prime extension, $p_2x \in K_1$ for every $p_2 \in \mathfrak{p}_2$, which implies $\mathfrak{p}_2 \subseteq (K_1 : x)$. Since $K_1 \subset L$ is maximal $\mathfrak{p}_2$ -prime extension, we have that $x \in L$. So $M \subseteq L$. Hence L is a maximal $\mathfrak{p}_1$ -prime extension of K_2 in M . This completes the proof. $\square$

Next we show that we can modify an MPE filtration of M over N without altering its length so that the occurrences of any two prime ideals $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ with $\mathfrak{p}_i \not\subseteq \mathfrak{p}_{i+1}$ and $\mathfrak{p}_{i+1} \not\subseteq \mathfrak{p}_i$ can be interchanged.

Corollary 17. *Let N be any proper submodule of an A -module M . If $N = M_0 \subset \cdots \subset M_{i-1} \xrightarrow{\mathfrak{p}_i} M_i \xrightarrow{\mathfrak{p}_{i+1}} M_{i+1} \subset \cdots \subset M_n = M$ is a MPE filtration of M over N , with $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ are distinct maximal elements in $\text{Ass}(M/M_{i-1})$,*

then there exist a submodule K_i of M , such that $N = M_0 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_{i+1}}{\subset} K_i \overset{\mathfrak{p}_i}{\subset} M_{i+1} \subset \cdots \subset M_n = M$ is a MPE filtration of M over N .

Proof. Take $N = M_{i-1}$, $K_1 = M_i$ and $L = M_{i+1}$ in proposition 16. $\square$

Example 18. Generally two MPE filtration of an A -module need not have same length. Consider the module M given in example 10. Then we have two MPE filtration

$$0 \overset{(x,y)}{\subset} AX \overset{(x)}{\subset} M \quad \text{and} \quad 0 \overset{(x)}{\subset} AY \overset{(x,y)}{\subset} AX + AY \overset{(x,y)}{\subset} M$$

of M with length 2 and 3 respectively.

4. Regular Prime Extension Filtration

Now we give the following definition.

Definition 19. Let N be a proper submodule of an A -module M . An MPE filtration $N = M_0 \subset M_1 \subset \cdots \subset M_n = M$ of M over N is called Regular Prime Extension (RPE) filtration of M over N , if $\mathfrak{p}_i$ is maximal element in $\text{Ass}(M/M_{i-1})$, for $1 \leq i \leq n$. If $N = 0$, then we call it as an RPE filtration of M .

In this section we will show all RPE filtration of M over N have unique length and for any prime ideal $\mathfrak{p}$ of A , the number of times $\mathfrak{p}$ occurs on any two RPE filtration are same.

Lemma 20. Let M be an A -module and N be the proper submodule of M . If

$$N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \subset M_{i-1} \overset{\mathfrak{p}_i}{\subset} M_i \overset{\mathfrak{p}_{i+1}}{\subset} M_{i+1} \subset \cdots \subset M_n = M \quad (1)$$

is a RPE filtration of M over N , and if $\mathfrak{p}_{i+1} \not\subseteq \mathfrak{p}_i$ for some i , then there exist a submodule K_i of M , such that

$$N = M_0 \overset{\mathfrak{p}_1}{\subset} M_1 \subset \cdots \overset{\mathfrak{p}_{i-1}}{\subset} M_{i-1} \overset{\mathfrak{p}_{i+1}}{\subset} K_i \overset{\mathfrak{p}_i}{\subset} M_{i+1} \overset{\mathfrak{p}_{i+2}}{\subset} \cdots \subset M_n = M \quad (2)$$

is a RPE filtration of M over N .

Proof. Since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_{i+1}$ and by assumption $\mathfrak{p}_{i+1} \not\subseteq \mathfrak{p}_i$ also M_{i+1} is maximal $\mathfrak{p}_{i+1}$ -prime extension of M_i , $\mathfrak{p}_{i+1}$ is maximal element in $\text{Ass}(M/M_i)$. This

implies $\mathfrak{p}_{i+1}$ is maximal element in $\text{Ass}(M/M_i) \cup \{\mathfrak{p}_i\} = \text{Ass}(M/M_{i-1})$. So both $\mathfrak{p}_i$ and $\mathfrak{p}_{i+1}$ are distinct maximal elements in $\text{Ass}(M/M_{i-1})$. Then by corollary 17, there exist a prime extension K_i of M_{i-1} such that (2) is MPE filtration of M over N . Since $\mathfrak{p}_{i+1}$ is maximal element in $\text{Ass}(M/M_{i-1})$, (2) is RPE filtration of M over N . $\square$

Corollary 21. *Let N be a proper submodule of an A -module M and $\mathfrak{p}$ be a maximal element in $\text{Ass}(M/N)$. If there exist an RPE filtration of M over N with length n , then there exist an RPE filtration of M over N with length n , such that $\mathfrak{p}$ occurs in the first place and for each prime ideal $\mathfrak{q}$ the number of times $\mathfrak{q}$ occurs in these two filtration are equal.*

Proof. Suppose

$$N = M_0 \xrightarrow{\mathfrak{p}_1} M_1 \subset \cdots \xrightarrow{\mathfrak{p}_{i-1}} M_{i-1} \xrightarrow{\mathfrak{p}} M_i \xrightarrow{\mathfrak{p}_{i+1}} M_{i+1} \subset \cdots \xrightarrow{\mathfrak{p}_n} M_n = M \quad (3)$$

is a RPE filtration of M over N , with i be the least integer such that $\mathfrak{p} = \mathfrak{p}_i$. Since $\mathfrak{p}$ is a maximal element in $\text{Ass}(M/N)$, $\mathfrak{p} \not\subset \mathfrak{p}_j$ for $1 \leq j \leq n$ and in particular $\mathfrak{p} \not\subset \mathfrak{p}_{i-1}$. Then by lemma 20, there exist a submodule K_{i-1} of M , such that $N = M_0 \xrightarrow{\mathfrak{p}_1} M_1 \subset \cdots \subset M_{i-2} \xrightarrow{\mathfrak{p}} K_{i-1} \xrightarrow{\mathfrak{p}_{i-1}} M_i \subset \cdots \xrightarrow{\mathfrak{p}_n} M_n = M$ is a RPE filtration of M over N . Continuing the process, finally we will have a filtration

$$N = M_0 \xrightarrow{\mathfrak{p}} K_1 \xrightarrow{\mathfrak{p}_1} \cdots \xrightarrow{\mathfrak{p}_{i-2}} K_{i-1} \xrightarrow{\mathfrak{p}_{i-1}} M_i \xrightarrow{\mathfrak{p}_{i+1}} M_{i+1} \subset \cdots \xrightarrow{\mathfrak{p}_n} M_n = M \quad (4)$$

is also a RPE filtration of M over N . Obviously, for any prime ideal $\mathfrak{q}$, the number of times $\mathfrak{q}$ occurs in (3) and (4) are same. $\square$

Next we show that any two RPE filtration of M over N have same length.

Theorem 22. *Let N be a proper submodule of M . For any prime ideal $\mathfrak{p}$ of A the number of times $\mathfrak{p}$ occurs in any two RPE filtration of M over N are equal, and hence any two RPE filtration of M over N have same length.*

Proof. Let

$$N = M_0 \xrightarrow{\mathfrak{p}_1} M_1 \subset \cdots \subset M_{n-1} \xrightarrow{\mathfrak{p}_n} M_n = M \quad (5)$$

$$N = K_0 \xrightarrow{\mathfrak{q}_1} K_1 \subset \cdots \subset K_{m-1} \xrightarrow{\mathfrak{q}_m} K_m = M \quad (6)$$

be the two RPE filtration of M over N with length n and m respectively. We prove this result by induction on n .

If $n = 1$, then it is trivial. We assume that the result is true, if the length of the RPE filtration (5) is less than n . Now $\mathfrak{p}_1 \in \text{Ass}(M/N) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_m\}$, implies $\mathfrak{p}_1 = \mathfrak{q}_i$ for some i . Since $\mathfrak{p}_1$ is maximal element in $\text{Ass}(M/N)$, then by corollary 21, we can assume without loss of generality $\mathfrak{p}_1 = \mathfrak{q}_1$. Then by theorem 11, $K_1 = M_1$ and we have a RPE filtration

$$M_1 = M_1 \subset M_2 \subset \dots \subset M_{n-1} \subset M_n = M \quad (7)$$

and

$$M_1 = K_1 \subset K_2 \subset \dots \subset K_{m-1} \subset K_m = M \quad (8)$$

of M over M_1 . Then by assumption for any prime ideal $\mathfrak{p}$ the number of times $\mathfrak{p}$ occurs in the RPE filtration (7) and (8) are same. So the number of times $\mathfrak{p}$ occurs in the RPE filtration (5) and (6) are same. And so any two RPE filtration of M over N have same length. $\square$

Acknowledgments

The third author was financially supported by SRF of CSIR, India.

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