EDGE C_k SIGNED GRAPHS

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Abstract: In this paper, we define the edge C_k signed graph of a given signed graph and shown that for any signed graph Σ , its edge C_k signed graph $E_k(\Sigma)$ is balanced. We then give structural characterization of edge C_k signed graphs. Further, we obtain a structural characterization of signed graphs that are switching equivalent to their the edge C_k signed graphs.

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1. Introduction

For standard terminology and notation in graph theory we refer Harary [4] and Zaslavsky [20] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Within the rapid growth of the Internet and the Web, and in the ease with which global communication now takes place, connectedness took an important place in modern society. Global phenomena, involving social networks, incentives and the behavior of people based on the links that connect us appear in a regular manner. Motivated by these developments, there is a growing

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multidisciplinary interest to understand how highly connected systems operate [3].

In social sciences we often deal with relations of opposite content, e.g., “love”- “hatred”, “likes”-“dislikes”, “tells truth to”-“lies to” etc. In common use opposite relations are termed positive and negative relations. A signed graph is one in which relations between entities may be of various types in contrast to an unsigned graph where all relations are of the same type. In signed graphs edge-coloring provides an elegant and uniform representation of the various types of relations where every type of relation is represented by a distinct color.

In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs $+$ and $-$ are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs $+$ and $-$ are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. Formally, a signed graph $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function that assigns a sign $\sigma(e) \in \{+, -\}$, to each edge in Γ . σ is called the signature or sign function. In such a signed graph, a subset A of $E(\Gamma)$ is said to be positive if it contains an even number of negative edges, otherwise is said to be negative. Balance or imbalance is the fundamental property of a signed graph. A signed graph Σ is balanced if each cycle of Σ is positive. Otherwise it is unbalanced.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [7] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [2] provided a mathematical model for balance through graphs.

A *marking* of Σ is a function $\zeta : V(\Gamma) \rightarrow \{+, -\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ .

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$,

the disjoint subsets may be empty.

Theorem 1. *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

- (i) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2 , and every negative edge joins a vertex in V_1 and a vertex in V_2 (Harary [5]).*
- (ii) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [12]).*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\bar{\Sigma} = (\bar{\Gamma}, \sigma')$, where for any edge $e = uv \in \bar{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\bar{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.

A switching function for Σ is a function $\zeta : V \rightarrow \{+, -\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w . The switched signed graph is $\Sigma^\zeta := (\Sigma | \sigma^\zeta)$. We say that Σ switched by ζ . Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1, 11, 13, 15]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cutset $E(X, X^c)$. The switched signed graph is Σ^X . This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2 . The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [17]) or *cycle isomorphic* (see [19]) if there exists an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2 . The following result is well known (see [19]):

Theorem 2. (see T. Zaslavsky, [19]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [14], the authors introduced the switching and cycle isomorphism for signed digraphs.

2. Edge C_k Signed Graph of a Signed Graph

The *edge C_k graph* $E_k(\Gamma)$ of a graph Γ is defined in [9] as follows: The edge C_k graph of a graph $\Gamma = (V, E)$ is a graph $E_k(\Gamma) = (V', E')$, with vertex set $V' = E(\Gamma)$ such that two vertices e and f are adjacent if, and only if, the corresponding edges in Γ either incident or opposite edges of some cycle C_k . In this paper, we extend the notion of $E_k(\Gamma)$ to realm of signed graphs: Given a signed graph $\Sigma = (\Gamma, \sigma)$ its *edge C_k signed graph* $E_k(\Sigma) = (E_k(\Gamma), \sigma')$ is that signed graph whose underlying graph is $E_k(\Gamma)$, the edge C_k graph of Γ , where for any edge e_1e_2 in $E_k(\Sigma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. When $k = 3$, the definition coincides with triangular line signed graph of a graph [18], and when $k = 4$, the definition coincides with the edge E_4 signed graph of a graph [10].

Hence, we shall call a given signed graph an edge C_k signed graph if there exists a signed graph Σ' such that $\Sigma \cong E_k(\Sigma')$. In the following subsection, we shall present a characterization of edge C_k signed graphs.

The following result indicates the limitations of the notion of edge C_k signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be edge C_k signed graphs.

Theorem 3. *For any signed graph $\Sigma = (\Gamma, \sigma)$, its edge C_k signed graph $E_k(\Sigma)$ is balanced.*

Proof. Let σ' denote the signing of $E_k(\Sigma)$ and let the signing σ of Σ be treated as a marking of the vertices of $E_k(\Sigma)$. Then by definition of $E_k(\Sigma)$, we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge e_1e_2 of $E_k(\Sigma)$ and hence, by Theorem 1, the result follows. $\square$

When $k = 3$ and $k = 4$, we can deduce the following results:

Corollary 4. (see M.S. Subramanya and P. Siva Kota Reddy, [18]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its triangular line signed graph $\mathcal{T}(\Sigma)$ is balanced.*

Corollary 5. (see R. Rangarajan and P. Siva Kota Reddy, [10]) *For any signed graph $\Sigma = (\Gamma, \sigma)$, its edge C_4 signed graph $E_4(\Sigma)$ is balanced.*

For any positive integer i , the i^{th} iterated edge C_k signed graph, $E_k^i(\Sigma)$ of Σ is defined as follows:

$$E_k^0(\Sigma) = \Sigma, E_k^i(\Sigma) = E_k(E_k^{i-1}(\Sigma)).$$

Corollary 6. *For any signed graph $\Sigma = (\Gamma, \sigma)$ and any positive integer i , $E_k^i(\Sigma)$ is balanced.*

In [16], we obtained the characterizations for the edge C_k graph of a graph Γ is connected, complete, bipartite etc. We have also proved that the edge C_4 graph has no forbidden subgraph characterization. The dynamical behavior such as convergence, periodicity, mortality and touching number of $E_k(\Gamma)$ are also discussed.

Recall that, the edge C_k graph coincides with the line graph for any acyclic graph. As a case, for a connected graph Γ , $E_k(\Gamma) = \Gamma$ if, and only if, $\Gamma = C_n, n \neq k$ [9].

We now characterize signed graphs that are switching equivalent to their the edge C_k signed graphs.

Theorem 7. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\Sigma \sim E_k(\Sigma)$ if, and only if, $\Gamma \cong C_n$, where $n \geq 5$ and Σ is balanced.*

Proof. Suppose $\Sigma \sim E_k(\Sigma)$. This implies, $\Gamma \cong E_k(\Gamma)$ and hence by the above observation we see that the graph Γ must be isomorphic to C_n , where $n \geq 5$. Now, if Σ is signed graph on C_n , Theorem 3 implies that $E_k(\Sigma)$ is balanced and hence if Σ is unbalanced its $E_k(\Sigma)$ being balanced cannot be switching equivalent to Σ in accordance with Theorem 2. Therefore, Σ must be balanced.

Conversely, suppose that Σ is balanced signed graph on C_n , where $n \geq 5$. Then, since $E_k(\Sigma)$ is balanced as per Theorem 3 and since $\Gamma \cong C_n$, where $n \geq 5$, the result follows from Theorem 2 again. $\square$

In [16], we obtained the following result.

Theorem 8. (P. Siva Kota Reddy and P. S. Hemavathi, [10]) *For a graph Γ , $E_k(\Gamma) \cong L(\Gamma)$ if, and only if, Γ is C_k -free.*

In view of the above result, we have the following characterization:

Theorem 9. *For any signed graph Σ , $E_k(\Sigma) \cong L(\Sigma)$ if, and only if, Γ is C_k -free.*

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [6] as follows: $\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ . However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(\cdot)$ of taking the negation of Σ .

Theorem 7 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

Corollary 10. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $\eta(\Sigma) \sim E_k(\Sigma)$ if, and only if, $\Gamma \cong C_n$, where $n \geq 5$ and Σ is unbalanced.*

Corollary 11. *For any signed graph $\Sigma = (\Gamma, \sigma)$, $E_k(\eta(\Sigma)) \sim E_k(\Sigma)$.*

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $E_k(\Sigma)$ is balanced (Theorem 3). We now examine, the conditions under which negation $\eta(\Sigma)$ of $E_k(\Sigma)$ is balanced.

Proposition 12. *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. If $E_k(\Gamma)$ is bipartite then $\eta(E_k(\Sigma))$ is balanced.*

Proof. Since, by Theorem 3, $E_k(\Sigma)$ is balanced, if each cycle C in $E_k(\Sigma)$ contains even number of negative edges. Also, since $E_k(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $E_k(\Sigma)$ is also even. Hence $\eta(E_k(\Sigma))$ is balanced. $\square$

In [8], the authors proved that for a connected complete multipartite graph Γ , $E_k(\Gamma)$ is complete. The following result follows from the above observation and Theorem 3.

Theorem 13. *For a connected signed graph $\Sigma = (\Gamma, \sigma)$, $E_k(\Sigma)$ is complete balanced signed graph if, and only if, Γ is complete multipartite graph.*

In [16], the authors proved that: For a connected graph Γ , $E_k(\Gamma)$ is bipartite if, and only if, Γ is either a path or an even cycle of length $r \neq k$. The following result follows from the above result and Theorem 3.

Theorem 14. *For a connected signed graph $\Sigma = (\Gamma, \sigma)$, $E_k(\Sigma)$ is bipartite balanced signed graph if, and only if, Γ is isomorphic to either path or C_{2n} , where $n \geq 3$.*

2.1. Characterization of Edge C_k Signed Graphs

The following result characterize signed graphs which are edge C_k signed graphs.

Theorem 15. *A signed graph $\Sigma = (\Gamma, \sigma)$ is an edge C_k signed graph if, and only if, Σ is balanced signed graph and its underlying graph Γ is an edge C_k graph.*

Proof. Suppose that Σ is balanced and Γ is an edge C_k graph. Then there exists a graph Γ' such that $E_k(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ' , $\sigma'(e)$ is the marking of the corresponding vertex in Γ . Then clearly, $E_k(\Sigma') \cong \Sigma$. Hence Σ is an edge C_k signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is an edge C_k signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $E_k(\Sigma') \cong \Sigma$. Hence Γ is the edge C_k graph of Γ' and by Theorem 3, Σ is balanced. $\square$

If we take $k = 3$ and $k = 4$ in $E_k(\Sigma)$, then we can deduce the triangular line signed graph and edge C_4 signed graph respectively. In [10, 18], the authors obtained structural characterizations of edge C_4 signed graphs and triangular line signed graphs and clearly Theorem 15 is the generalization of above said notions.

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