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HYPER PATHS AND HYPER CYCLES

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Abstract: In graphs, paths are walks with no repeated vertex. A fortiori, paths cannot have any repeated edge. But in hypergraphs, hyperedges can repeat in vertex-to-vertex walks without causing repetition of any vertex. This is the crux of the idea of generalizing paths and cycles (from graphs to hypergraphs) presented in this short article.

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1. Introduction

A *simple hypergraph* [3] is an ordered couple $H = (V, E)$ where: (i) V is a nonempty finite set and (ii) E is a set of nonempty subsets of V such that $\bigcup_{X \in E} X = V$. Each member of V is a *vertex*; and each member of E is a *hyperedge* (or, an *edge*). The hypergraph H is *Sperner* if the members of E have the property that none is a subset of another.

The cardinality [5] of a set X is denoted by $\mid X \mid$. A hyperedge X with $\mid X \mid = 1$ is a *loop*. A hypergraph is *loop-free* if $\mid X \mid > 1$ for each hyperedge X.

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Two distinct vertices x and y are *adjacent* if there is a hyperedge that contains both x and y. A *simple loop-free graph* is a simple loop-free hypergraph with the additional stipulation that $\mid X \mid= 2$ for each hyperedge X. In the context of graphs, hyperedges are usually called edges.

The contributions of this article are generalizations of paths and cycles (from graph theory) to hyper paths and hyper cycles, respectively. Much of the motivation for this research work comes from: (i) varying definitions of acyclic hypergraphs (section 5 of [2]), and (ii) path formation in hypergraphs by sequences of hyperedges (section 3 of [2]). Perforce, more stipulations have been incorporated into the latter (see section 2 that follows) so that it can be successfully particularized to graphs (section 3). All the hypergraphs in the coming discussions are assumed Sperner and loop-free unless there are explicit and unambiguous indications to the contrary.

2. Definitions

A *hyper path* in $H = (V, E)$ between two distinct vertices x_1 and x_k is a sequence $x_1, A_1, ..., x_{k-1}, A_{k-1}, x_k$ with the following properties: (i) k is a positive integer ≥ 2; (ii) $x_1, ..., x_k$ are distinct vertices; (iii) $A_1, ...A_{k-1}$ are hyperedges (not necessarily distinct); and (iv) $x_j, x_{j+1} \in A_j$ for $j = 1$ through $k - 1$. This hyper path is also denoted by: (i) $P(x_1, ..., x_k)$; or (ii) $P = x_1, ..., x_k$; or (iii) $P(x_1, x_k)$; or, simply, (iv) P. The set $r(P) = \{x_1, ..., x_k\}$ is the *range* of P. The positive integer $\mid r(P) \mid -1$ is the *length* of P, and is denoted by $\parallel P \parallel$. The points x_1 and x_k are the *end points* (or, *terminal points*) of P. If the hyperedges $A_1, ..., A_{k-1}$ are also distinct, then the hyper path P is a *path* between x_1 and x_k. H is *connected* if there is a hyperpath between x and y whenever x and y are distinct vertices in H.

A *hyper cycle* in $H = (V, E)$ at a vertex x_1 is a sequence $x_1, A_1, ..., x_{k-1}, A_{k-1}, x_k, A_k, x_1$ with the following properties: (i) k is a positive integer ≥ 3; (ii) $x_1, A_1, ..., x_{k-1}, A_{k-1}, x_k$ is a hyper path between x_1 and x_k; (iii) at least one among A_1 through A_{k-1} is distinct from A_k; (iv) x_j, $x_{j+1} \in A_j$ for $j = 1$ through $k - 1$; and (v) x_k, $x_1 \in A_k$. This hyper cycle is also denoted by: (i) $C(x_1, ..., x_k, x_1)$; or (ii) $C = x_1, ..., x_k, x_1$; or (iii) $C(x_1)$; or, simply, (iv) C. The set $r(C) = \{x_1, ..., x_k\}$ is the *range* of C. The positive integer $\mid r(C) \mid$ is the *length* of C, and is denoted by $\parallel C \parallel$. If $A_1, ..., A_k$ are also distinct, then the hyper cycle C is a *cycle* at x_1. H is *hyper acyclic* if it has no hyper cycles.

Clearly: (i) a hyper cycle C begins and ends at y for each vertex $y \in r(C)$; and

(ii) an *acyclic graph* is a hyper acyclic hypergraph with $\mid X \mid= 2$ for each hyperedge X.

The *distance* between two vertices x and y in $H = (V, E)$ is denoted by $d(x, y)$, and is defined as follows: $d(x, y) = 0$ if $x = y$; $d(x, y) = \min \{\parallel P \parallel$: P is a hyper path between x and $y\}$ if $x \neq y$. Evidently: (i) $d(x, y)$ defines a metric [1] on V; (ii) $d(x, y) = 1$ if $x \neq y$ and x is adjacent to y; and (iii) in a simple loop-free graph, $d(x, y)$ is the number of edges in any shortest path between x and y.

3. Main Results

Proposition 3.1. *If P is a hyper path between x_1 and x_k then P contains a path between x_1 and x_k.*

Proof. Induction is used on k. For $k = 2$, any hyper path x_1, A_1, x_2 is clearly a path. Assume the result for $k = r$. Then let P be a given hyper path between x_1 and x_{r+1}, written $x_1, A_1, ..., x_r, A_r, x_{r+1}$. Then $x_1, A_1, ..., A_{r-1}, x_r$ is a hyper path (call it Q) between x_1 and x_r. By induction hypothesis, Q contains a path between x_1 and x_r, say: $x_1, B_1, ..., B_t, x_r$ where the B's, like the x's, are distinct. At this point:

(i) if $A_r =$ some B_s $(1 \leq s \leq t)$, then $x_1, B_1, ..., B_s, x_{r+1}$ is a path between x_1 and x_{r+1}; or,
(ii) if A_r equals none of B_1 through B_t, then $x_1, B_1, ..., B_t, x_r, A_r, x_{r+1}$ is a path between x_1 and x_{r+1}.

Proposition 3.2. *In a simple, loop-free graph $G = (V, E)$, each hyper path is a path.*

Proof. In G, if an edge repeats in a hyper path, then the vertices in that edge must repeat in the hyper path. But this is contradictory to the definition of a hyper path.

Example 3.3. A hyper cycle need not contain a cycle. Consider $H = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7\}$, $E = \{A_1, A_2, A_3\}$ with $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 3, 4, 5\}$ and $A_3 = \{3, 5, 6, 7\}$. Then $C(1, A_1, 2, A_2, 3, A_1, 1)$ is a hypercycle, but C contains no cycles - because omission of any hyperedge from C leaves not more than two vertices standing, thus ruling out any chance of a cycle.

Proposition 3.4. *In a simple, loop-free graph $G = (V, E)$, each hyper cycle is a cycle.*

The proof is similar to that of 3.2.

4. Summing Up

Generalizing results from graph theory is one of the main purposes of the theory of hypergraphs [4]. For a generalization of this kind to be successful, its particularization from hypergraphs to graphs has to follow unambiguously when the additional stipulation for graphs (that $| X |= 2$ for each hyperedge X) is factored in. The generalizations of paths and cycles (to hyper paths and hyper cycles, respectively) are successful in this regard, corroborated by 3.2 and 3.4. What seems promising, at this point, is that more ideas from graph theory (degree and neighborhood of a vertex, to mention two) could be generalized, besides, if at all, identifying graph-theoretical ideas that might not be amenable to generalization to hypergraphs.

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