Abstract: An image of a plane graph, $G = (V, E)$ of order $n$ and size $m$, is said to be a vertex-edge-magic plane graph if there is a bijection $f : V \cup E \to \{1, 2, \ldots, n + m\}$ such that for all $s$-side faces of $G$, except the infinite face, the sum of the labels of its vertices and edges is a constant $k(s)$. Such a bijection will be called a vertex-edge-magic plane labeling of $G$. In case that all the finite sides of a graph $G$ having the same size we will be interested in determining the minimum and the maximum number, $k$, such that there exists a vertex-edge-magic labeling of $G$, in which $k$ is the sum of the vertex and edge labeling of each face. In this paper we find such a minimum and maximum numbers for a wheel with even order.

Key Words: magic graph, plane graph, wheel, minimal magic graph, maximal magic graph, $(1,1,0)$ magic

1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph $G$; $V(G), E(G), v(G)$ and $e(G)$ stands for the set of vertices, the set of edges, the order (number of vertices) and the size (number of edges) of $G$. $K_n$, and $C_n$ stand for the complete graph and the cycle of order $n$. For two graphs $G$ and $H$ we denote by $G + H$ the graph obtained from the disjoint union $G \cup H$ by adding all edges between $G$ and $H$. 

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A wheel, $W_n$, is a graph of order $n + 1$ composed of a vertex, which will be
called the hub, adjacent to all vertices of a cycle of order $n$. The cycle will be
called the rim of the wheel, and the edges connecting the hub to the vertices of
the rim will be called the spokes. i.e., $W_n = C_n + K_1$.

### 1.1. Magic Plane Graphs

Koh wei lih defined in [7] the notions of magic labeling of a plane graph. In
this paper, we will use the term *edge-magic plane graph* for what was defined as
edge-magic graph in [7], to differ it from other definitions of edge-magic graph.

**Definition.** Let $G$ be a plane graph of size $m$. A bijection $f : E(G) \to
\{1, 2, \ldots, m\}$ is called *edge-magic labeling* of $G$ if the sum of the edge labels sur-
rounding each $s$-sided face of $G$ is a constant.

**Definition.** A plane graph $G$ is called *edge-magic plane graph* if there exist
an edge-magic labeling of $G$.

**Definition.** Let $G$ be a plane graph such that all its bounded faces having
the same size. $G$ will be called $k$-*edge-magic plane graph* if there exist an edge-
magic labeling of $G$, such that the sum of labels surrounding each face of $G$ is $k$.

**Notation.** For a plane graph $G$, such that all its bounded faces having the
same size, we denote by $EM(G)$ the set of natural numbers, $k$, such that $G$ as
$k$-*edge-magic labeling*.

Two results have been shown in [2] regarding these concepts:

**Theorem 1.1.1** For any odd natural number $n \geq 3$,

$$min(EM(W_n)) = \frac{n + 1}{2} + 2n + 1.$$  

**Theorem 1.1.2** For any odd natural number $n \geq 3$,

$$max(EM(W_n)) = \frac{3n + 1}{2} + 2n + 1.$$
**Definition.** Let $G$ be a plane graph of order $n$ and size $m$. A bijection $f : V(G) \cup E(G) \to \{1, 2, \ldots, n + m\}$ is called *vertex-edge-magic labeling* if the sum of the edge labels surrounding each $s$-sided face of $G$ is a constant.

**Definition.** A plane graph $G$ is called *vertex-edge-magic plane graph* if there exist a vertex-edge-magic labeling of $G$.

**Definition.** Let $G$ be a plane graph such that all its bounded faces having the same size. $G$ will be called *$k$-vertex-edge-magic plane graph* if there exist a vertex-edge-magic labeling of $G$, such that the sum of labels surrounding each face of $G$ is $k$.

**Notation.** For a plane graph $G$, such that all its bounded faces having the same size, we denote by $VEM(G)$ the set of natural numbers, $k$, such that $G$ as $k$-vertex-edge-magic labeling.

Ko Wei Lih shows in [7] that for all $n \geq 3$, $W_n$ has a consecutive vertex labeling if and only if $n \not\equiv 2 \mod 4$. In addition he shows that for all $n \geq 3$, $W_n$ has a consecutive edge labeling if and only if $n \not\equiv 2 \mod 4$. From these last two results of K.W. Lih it is easy to deduce that for all $n \geq 3$, $n \not\equiv 2 \mod 4$, $W_n$ has an edge-vertex magic labeling.

On this paper we will find $\min(VEM(W_n))$ and $\max(VEM((W_n))$ for all odd natural number $n$.

### 2. Labeling of Wheels

Let $(a_1, \ldots, a_n)$ be the labeling of the spokes, $(b_1, \ldots, b_n)$ the labeling of the rim edges, $(c_1, \ldots, c_n)$ the labeling of the rim vertices and $c_{n+1}$ the labeling of the hub of $W_n$, such that the sum of the labels on each face of the wheel is $k$. Since each spoke and each rim vertex belongs to two faces, each rim edge belongs to only one face and the hub belongs to $n$ faces, we conclude that:

$$nk = nc_{n+1} + 2 \sum_{i=1}^{n} a_i + 2 \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} b_i.$$ 

Therefore

$$nk = 2[c_{n+1} + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} b_i] + (n - 2)c_{n+1} - \sum_{i=1}^{n} b_i.$$
Thus
\[ nk = (3n + 1)(3n + 2) + (n - 2)c_{n+1} - \sum_{i=1}^{n} b_i. \] (1)

Hence, it is easy to derive the following inequalities
\[
\frac{(3n + 1)(3n + 2) + (n - 2)}{n} \leq k \leq \frac{(3n + 1)(3n + 2) + (n - 2)(3n + 1) - \frac{(n+1)n}{2}}{n}.
\]

Since,
\[
1 \leq c_{n+1} \leq 3n + 1 \text{ and } 1 + 2 + \ldots + n \leq \sum_{i=1}^{n} b_i \leq (2n+2) + (2n+3) + \ldots + (3n+1).
\]

Thus,
\[
\left\lceil \frac{13n + 17}{2} \right\rceil \leq k \leq \left\lfloor \frac{23n + 7}{2} \right\rfloor.
\] (2)

In the case of odd \( n \), we will show that \( k \) attain these bounds.

**Theorem 2.1.** For any odd natural number \( n \geq 3 \),
\[
\text{min}(VEM(W_n)) = \frac{13n + 17}{2}.
\]

**Proof.** Let \( m \) be the natural number, such that \( n = 2m+1 \). From inequality (2) it is sufficient to point out a \( 13m + 15 \) vertex-edge-magic labeling of \( W_{2m+1} \) for all natural \( m \).

Such a labeling of \( W_{2m+1} \) can be described as followed. We label the hub vertex by 1 and the spokes edges by 2, 3, 4, .., 2m + 1, 2m + 2 clockwise. We label the rim vertices 2m + 3, 2m + 4, .., 4m + 3 counter clockwise skipping one edge every time starting by labeling 2m + 3 the rim vertex, which is contained in the spoke labeled 2m + 1. The rim edges we label 4m + 4, 4m + 5, .., 6m + 4) counter clockwise, starting by labeling by 4m + 4 the rim edge containing the vertices, labeled by 4m + 3 and 3m + 3. Such a labeling is demonstrated by Figure 1. Notice that the sum of labels on the triangle which his vertices labeled by 1, 4m + 3, 3m + 3 is indeed 13m + 15 and that is also the sum of labels on the adjacent triangle counter clockwise. It is easy to see that from there on, Moving from a triangle to the adjacent triangle counter clockwise, the sum of the labels stays constant. \( \square \)
Theorem 2.2. For any odd natural number \( n \geq 3 \),
\[
\max(\text{VEM}(W_n)) = \frac{23n + 7}{2}.
\]

Proof. Let \( m \) be the natural number, such that \( n = 2m + 1 \). From inequality (2) it is sufficient to point out a \( 13m + 15 \) vertex-edge-magic labeling of \( W_{2m+1} \) for all natural \( m \).

Such a labeling of \( W_{2m+1} \) can be described as followed. We label the hub vertex by \( 6m + 4 \) and the spokes edges by \( 4m + 3, 4m + 4, 4m + 5, \ldots, 6m + 3 \) clockwise. We label the rim vertices \( 2m + 2, 2m + 3, \ldots, 4m + 2 \) counter clockwise skipping one vertex every time starting by labeling the rim vertex, which is contained in the spoke labeled \( 6m + 2 \). The rim edges we label \( 1, 2, \ldots, 2m + 1 \) counter clockwise, starting with the rim edge containing the vertices, labeled by \( 4m + 2 \) and \( 3m + 2 \). Such a labeling is demonstrated by Figure 3. Notice that the sum of the labels on the triangle which his vertices labeled by \( 6m + 4, 4m + 2, 3m + 2 \) is indeed \( 13m + 15 \) and that is also the sum of labels on the adjacent triangle counter clockwise. It is easy to see that from there on, Moving from a triangle to the adjacent triangle counter clockwise, the sum of the labels stays constant.

Figure 1. describes an edge-vertex magic labeling of \( W_{2m+1} \) with a minimum magic number. Figure 2. demonstrate such a labeling on \( W_7 \).

Figure 3. describes an edge-vertex magic labeling of \( W_{2m+1} \) with a maximum magic number. Figure 4. demonstrate such a labeling on \( W_7 \).
Figure 1. min. labeling of $W_{2n+1}$

Figure 2. min. labeling of $W_7$
3. Discussion

We saw that for any odd natural number \( n \geq 3 \),

\[
\min(VEM(W_n)) = \left\lceil \frac{13n + 17}{2} \right\rceil, \quad \max(VEM(W_n)) = \left\lfloor \frac{23n + 7}{2} \right\rfloor.
\]

The question is whether these formulas are valid also in the case of even numbers. Moreover, for \( n \equiv 2 \mod 4 \) it is needed first to prove that for any such \( n \) there exist an edge-vertex magic labeling of \( W_n \). The following figures shows that these minimum and maximum values are valid at least for \( n = 4 \).
Figure 5. min. labeling of $W_4$.

Figure 6. max. labeling of $W_4$.

References


