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WHEEL AS A VERTEX-EDGE-MAGIC PLANE GRAPH

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Abstract: An image of a plane graph, $G = (V, E)$ of order n and size m, is said to be a *vertex-edge-magic plane graph* if there is a bijection $f : V \cup E \to \{1, 2, .., n+m\}$ such that for all $s - side$ faces of G, except the infinite face, the sum of the labels of its vertices and edges is a constant $k(s)$. Such a bijection will be called a vertex-edge-magic plane labeling of G. In case that all the finite sides of a graph G having the same size we will be interested in determining the minimum and the maximum number, k, such that there exists a vertex-edge-magic labeling of G, in which k is the sum of the vertex and edge labeling of each face. In this paper we find such a minimum and maximum numbers for a wheel with even order.

Key Words: magic graph, plane graph, wheel, minimal magic graph, maximal magic graph, (1,1,0) magic

1. Introduction

We study undirected graphs without loops or multiple edges. Given a graph G; $V(G), E(G), v(G)$ and $e(G)$ stands for the set of vertices, the set of edges, the order (number of vertices) and the size (number of edges) of G. K_n, and C_n stand for the complete graph and the cycle of order n. For two graphs G and H we denote by $G + H$ the graph obtained from the disjoint union $G \cup H$ by adding all edges between G and H.

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A wheel, W_n, is a graph of order $n+1$ composed of a vertex, which will be called the *hub*, adjacent to all vertices of a cycle of order n. The cycle will be called the *rim* of the wheel, and the edges connecting the hub to the vertices of the rim will be called the *spokes.* i.e., $W_n = C_n + K_1$.

1.1. Magic Plane Graphs

Koh wei lih defined in [7] the notions of magic labeling of a plane graph. In this paper, we will use the term *edge-magic plane graph* for what was defined as edge-magic graph in [7], to differ it from other definitions of edge-magic graph.

Definition. Let G be a plane graph of size m. A bijection $f : E(G) \to \{1, 2, ., m\}$ is called *edge-magic labeling* of G if the sum of the edge labels surrounding each s-sided face of G is a constant.

Definition. A plane graph G is called *edge-magic plane graph* if there exist an edge-magic labeling of G.

Definition. Let G be a plane graph such that all its bounded faces having the same size. G will be called *k-edge-magic plane graph* if there exist an edge-magic labeling of G, such that the sum of labels surrounding each face of G is k.

Notation. For a plane graph G, such that all its bounded faces having the same size, we denote by $EM(G)$ the set of natural numbers, k, such that G as *k-edge-magic labeling.*

Two results have been shown in [2] regarding these concepts:

Theorem 1.1.1 *For any odd natural number $n \geq 3$,*

$$min(EM(W_n)) = \frac{n+1}{2} + 2n + 1.$$

Theorem 1.1.2 *For any odd natural number $n \geq 3$,*

$$max(EM(W_n)) = \frac{3n+1}{2} + 2n + 1.$$

Definition. Let G be a plane graph of order n and size m. A bijection $f : V(G) \cup E(G) \to \{1, 2, ., n+m\}$ is called *vertex-edge-magic labeling* if the sum of the edge labels surrounding each s-sided face of G is a constant.

Definition. A plane graph G is called *vertex-edge-magic plane graph* if there exist a vertex-edge-magic labeling of G.

Definition. Let G be a plane graph such that all its bounded faces having the same size. G will be called *k-vertex-edge-magic plane graph* if there exist a vertex-edge-magic labeling of G, such that the sum of labels surrounding each face of G is k.

Notation. For a plane graph G, such that all its bounded faces having the same size, we denote by $VEM(G)$ the set of natural numbers, k, such that G as *k-vertex-edge-magic labeling*.

Ko wei lih shows in [7] that for all $n \geq 3$, W_n has a consecutive vertex labeling if and only if $n \not\equiv 2 \ mod\, 4$. In addition he shows that for all $n \geq 3$, W_n has a consecutive edge labeling if and only if $n \not\equiv 2 \ mod\, 4$. From these last two results of K.W. Lih it is easy to deduce that for all $n \geq 3$, $n \not\equiv 2 \ mod\, 4$, W_n has an edge-vertex magic labeling.

On this paper we will find $min(VEM(W_n))$ and $max(VEM((W_n))$ for all odd natural number n.

2. Labeling of Wheels

Let $(a_1, \ldots, a_n)$ be the labeling of the spokes, $(b_1, \ldots, b_n)$ the labeling of the rim edges, $(c_1, \ldots, c_n)$ the labeling of the rim vertices and c_{n+1} the labeling of the hub of W_n, such that the sum of the labels on each face of the wheel is k. Since each spoke and each rim vertex belongs to two faces, each rim edge belongs to only one face and the hub belongs to n faces, we conclude that:

$$nk = nc_{n+1} + 2\sum_{i=1}^{n} a_i + 2\sum_{i=1}^{n} c_i + \sum_{i=1}^{n} b_i.$$

Therefore

$$nk = 2[c_{n+1} + \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} c_i + \sum_{i=1}^{n} b_i] + (n-2)c_{n+1} - \sum_{i=1}^{n} b_i.$$

Thus

$$nk = (3n+1)(3n+2) + (n-2)c_{n+1} - \sum_{i=1}^{n} b_i. \tag{1}$$

Hence, it is easy to derive the following inequalities

$$\frac{(3n+1)(3n+2) + (n-2) - \frac{(5n+3)n}{2}}{n} \leq k$$
$$\leq \frac{(3n+1)(3n+2) + (n-2)(3n+1) - \frac{(n+1)n}{2}}{n}.$$

Since,

$$1 \leq c_{n+1} \leq 3n+1 \ \ and \ \ 1+2+\ldots n \leq \sum_{i=1}^{n} b_i \leq (2n+2)+(2n+3)+\cdots+(3n+1).$$

Thus,

$$\lceil \frac{13n+17}{2} \rceil \leq k \leq \lfloor \frac{23n+7}{2} \rfloor. \tag{2}$$

In the case of odd n, we will show that k attain these bounds.

Theorem 2.1. *For any odd natural number $n \geq 3$,*

$$min(VEM(W_n)) = \frac{13n+17}{2}.$$

Proof. Let m be the natural number, such that $n = 2m+1$. From inequality (2) it is sufficient to point out a $13m+15$ vertex-edge-magic labeling of W_{2m+1} for all natural m.

Such a labeling of W_{2m+1} can be described as followed. We label the hub vertex by 1 and the spokes edges by $2, 3, 4, .., 2m+1, 2m+2$ clockwise. We label the rim vertices $2m+3, 2m+4, .., 4m+3$ counter clockwise skipping one edge every time starting by labeling $2m+3$ the rim vertex, which is contained in the spoke labeled $2m+1$. The rim edges we label $4m+4, 4m+5, \ldots, 6n+4)$ counter clockwise, starting by labeling by $4m+4$ the rim edge containing the vertices, labeled by $4m+3$ and $3m+3$. Such a labeling is demonstrated by Figure 1. Notice that the sum of labels on the triangle which his vertices labeled by $1, 4m+3, 3m+3$ is indeed $13m+15$ and that is also the sum of labels on the adjacent triangle counter clockwise. It is easy to see that from there on, Moving from a triangle to the adjacent triangle counter clockwise, the sum of the labels stays constant. □

Theorem 2.2. *For any odd natural number $n \geq 3$,*

$$max(VEM(W_n)) = \frac{23n+7}{2}.$$

Proof. Let m be the natural number, such that $n = 2m+1$. From inequality (2) it is sufficient to point out a $13m+15$ vertex-edge-magic labeling of W_{2m+1} for all natural m.

Such a labeling of W_{2m+1} can be described as followed. We label the hub vertex by $6m+4$ and the spokes edges by $4m+3, 4m+4, 4m+5, .., 6m+3$ clockwise. We label the rim vertices $2m+2, 2m+3, .., 4m+2$ counter clockwise skipping one vertex every time starting by labeling the rim vertex, which is contained in the spoke labeled $6m+2$. The rim edges we label $1, 2, \ldots, 2m+1$ counter clockwise, starting with the rim edge containing the vertices, labeled by $4m+2$ and $3m+2$. Such a labeling is demonstrated by Figure 3. Notice that the sum of the labels on the triangle which his vertices labeled by $6m+4, 4m+2, 3m+2$ is indeed $13m+15$ and that is also the sum of labels on the adjacent triangle counter clockwise. It is easy to see that from there on, Moving from a triangle to the adjacent triangle counter clockwise, the sum of the labels stays constant.

Figure 1. describes an edge-vertex magic labeling of W_{2m+1} with a minimum magic number. Figure 2. demonstrate such a labeling on W_7.

Figure 3. describes an edge-vertex magic labeling of W_{2m+1} with a maximum magic number. Figure 4. demonstrate such a labeling on W_7.

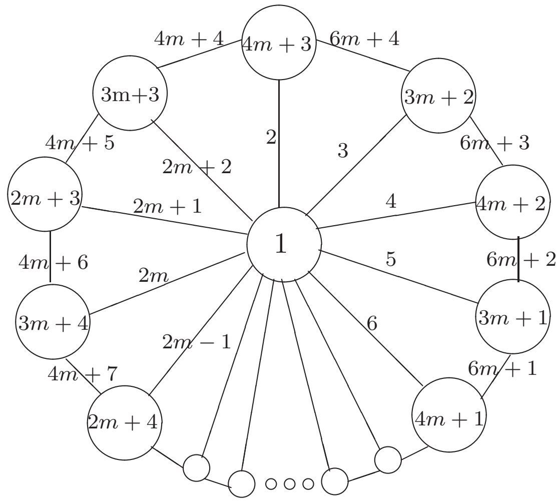

Figure 1. min. labeling of W_{2n+1}

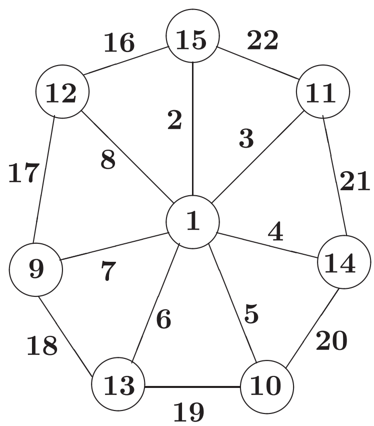

Figure 2. min. labeling of W_7

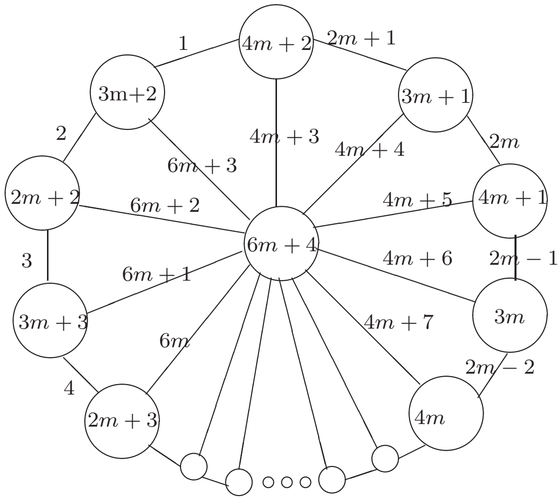

Figure 3. max. labeling of W_{2m+1}

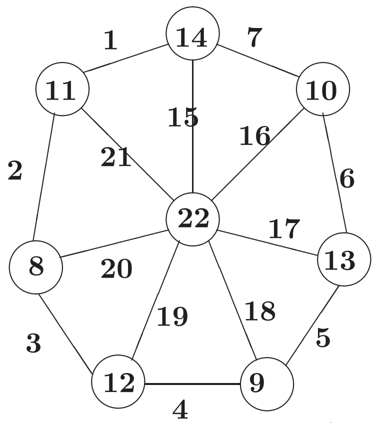

Figure 4. max. labeling of W_7.

3. Discussion

We saw that for any odd natural number $n \geq 3$,

$$\min(VEM(W_n)) = \lceil \frac{13n+17}{2} \rceil, \quad \max(VEM(W_n)) = \lfloor \frac{23n+7}{2} \rfloor.$$

The question is whether these formulas are valid also in the case of even numbers. Moreover, for $n \equiv 2 \ mod\, 4$ it is needed first to prove that for any such n there exist an edge-vertex magic labeling of W_n. the following figures shows that these minimum and maximum values are valid at least for $n = 4$.

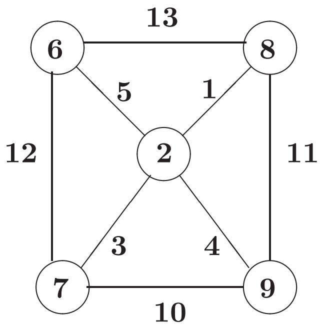

Figure 5. min. labeling of W_4.

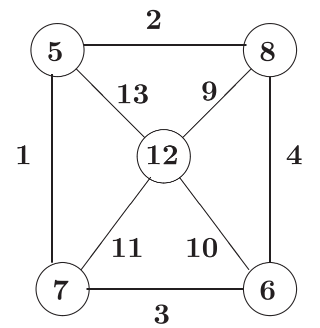

Figure 6. max. labeling of W_4.

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