ON WEIERTRASS $N$-SEMIGROUPS
ON SMOOTH CURVES

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Abstract: Let $X$ be a smooth curve of genus $g > 0$. For any $n \geq 2$ and any $n$ distinct points $P_1, \ldots, P_n \in X$ let $E(P_1, \ldots, P_n)$ be the set of all $(a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned. We study some abstract properties of the semigroup $E(P_1, \ldots, P_n)$ (e.g. its decompositions in irreducibles) which are upper bounded by a function of $g$ independent from $n$. We say that $E(P_1, \ldots, P_n)$ is symmetric if $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n) \cong \omega_X$ for some $(a_1, \ldots, a_n) \in \mathbb{N}^n$. We study symmetric $n$-semigroups of curves with general moduli.

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1. Introduction

Let $X$ be a smooth curve of genus $g$ defined over an algebraically closed base field $\mathbb{K}$. Fix $n$ distinct points $P_1, \ldots, P_n \in X$. Let $E(P_1, \ldots, P_n) \subset \mathbb{N}^n$ be the semigroup of non-gaps of the line bundles $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$, i.e. an $n$-ple $(a_1, \ldots, a_n) \in E(P_1, \ldots, P_n)$ if and only if the line bundle $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned (i.e. $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_i)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) - 1$ for all $i = 1, \ldots, n$, i.e. (Riemann-Roch) $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_i)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$ for all $i = 1, \ldots, n$). Set $G(P_1, \ldots, P_n) := \mathbb{N}^n \setminus \ldots \setminus \ldots \setminus \mathbb{N}$.

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$E(P_1, \ldots, P_n)$. $E(P_1, \ldots, P_n)$ is called the semigroup of all non-gaps, while $G(P_1, \ldots, P_n)$ is the set of all gaps of $(P_1, \ldots, P_n)$. There is an upper bound on $\sharp(G(P_1, \ldots, P_n))$ depending only on $g$ and $n$ ([1]). The main goal of this note is to point out that Weierstrass $n$-semigroups have invariants which are upper bounded by a function of $g$ not depending on $n$.

We recall the following definition ([4, Definition 2.3]). We will only need the case in which $M \subset \mathbb{N}^n$ (with componentwise addition as the monoid operation) with $\mathbb{N}^n \setminus M$ finite. Let $M$ be a commutative, cancellative monoid $M$ with $+$ as its operation. For any $a \in M$ the primality function $\pi_M(a) \in \mathbb{N} \setminus \{0\} \cup \{+\infty\}$ of $a$ in the following way. Let $\pi_M(a)$ be the minimal integer $t > 0$ with the following property. Assume that $a$ divides a sum $b_1 + \cdots + b_s$. Then there is a subset $S \subseteq \{1, \ldots, s\}$ with $\sharp(S) \leq \min\{t, s\}$ such that $a$ divides $\sum_{i \in S} b_i$. We write $\pi_M(a) = +\infty$ if there is no such an integer. Let $\pi(M, 1)$ be the supremum of all integer $\pi_M(a)$ with $a$ irreducible. Let $\pi(M, 2)$ be the minimal non-negative integer $t$ such that $\pi_M(a) \leq \pi_M(a_1) + \cdots + \pi_M(a_s) - t$ if $a = a_1 + \cdots + a_s$ and each $a_i$ is irreducible. For any $x \in M$ let $ir_{M, 1}(x)$ (resp. $ir_{M, 2}(x)$) be the minimal numbers of irreducible factors in a decomposition of $x$, with the convention $ir_{M, 2}(x) = +\infty$ if there is no such a maximum; let $\text{IRR}(M)$ be the supremum of all $ir_{M, 2}(x) - ir_{M, 1}(x)$ with $x \in M$. In our case when $M$ is an additive semigroup of $\mathbb{N}^n$ with a finite complement, we always have $ir_{M, 2}(x) < +\infty$. For any $b = (b_1, \ldots, b_n) \in \mathbb{N}^n$ set $\|b\| := b_1 + \cdots + b_n$.

We prove the following result.

**Corollary 1.** Let $E = E(P_1, \ldots, P_n)$ be an $n$-semigroup associated to a curve $X$ of genus $g$ with gonality $k$. Then $\pi_E(b) \leq [(\|b\| + 2g - 1)/k] + 1$ and $\pi(E, 1) \leq [(6g - 2)/k] + 1$.

Since $k \geq 2$, Corollary 1 gives $\pi(E, 1) \leq 3g$.

When $n = 1$ the genus $g$ semigroup $E(P_1)$ is called symmetric if $2g - 1$ is a gap (in this case it is the last gap), because $2g - 1$ is a gap if and only if $\mathcal{O}_X((2g - 2)P_1) \cong \omega_X$; Serre duality gives that symmetry is equivalent to a certain relation between gaps and non-gaps in degree $\leq 2g - 1$. We generalize this definition to the case $n \geq 2$ in the following way.

Call $\spadesuit(g)$ the condition that $a_1 + \cdots + a_n \leq 2g - 1$ for all $(a_1, \ldots, a_n) \in \mathbb{N}^n \setminus E$. The condition $\spadesuit(g)$ is satisfied if $E = E(P_1, \ldots, P_n)$ for $P_1, \ldots, P_n \in X$ and $X$ of genus $g$. Let $E \subset \mathbb{N}^n$ be any semigroup satisfying $\spadesuit(g)$. We say that $E$ is symmetric if there is $(a_1, \ldots, a_n) \in \mathbb{N}^n \setminus E$ with $a_1 + \cdots + a_n = 2g - 1$. Now assume $E = E(P_1, \ldots, P_n)$ and assume the existence of $(a_1, \ldots, a_n) \in G(P_1, \ldots, P_n)$ with $a_1 + \cdots + a_n = 2g - 1$. Since $(a_1, \ldots, a_n)$ is a gap, there is $h \in \{1, \ldots, n\}$ such that $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_h)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_h))$.
\( \cdots + a_n P_n \)). We have \( a_h > 0 \), because the points \( P_1, \ldots, P_n \) are distinct. Set \( b_i := a_i \) if \( i \neq h \) and \( b_h := a_h - 1 \). Since \( a_1 + \cdots + a_n = 2g - 1 \), we have \( h^0(\mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n)) = g \). Hence Riemann-Roch and Clifford’s inequality gives \( \omega_X \cong \mathcal{O}_X(b_1 P_1 + \cdots + b_n P_n) \). Riemann-Roch also gives the converse, i.e. it gives that \( E(P_1, \ldots, P_n) \) is symmetric if and only if \( \omega_X \cong \mathcal{O}_X(b_1 P_1 + \cdots + b_n P_n) \) for some \( (b_1, b_n) \in \mathbb{N}^n \). By Riemann-Roch we get that \( b_i \), \( i \neq h \), + \( e_i \in G(P_1, \ldots, P_n) \) for all \( i = 1, \ldots, n \). Therefore if \( n \geq 2 \) it is a very strong restriction for a symmetric semigroup to be of the form \( E(P_1, \ldots, P_n) \) for some curve \( X \) of genus \( g \) and an \( n \)-ple of distinct points on \( X \).

There are also many other obvious restrictions for the realization of an \( n \)-semigroup \( E \) as the semigroup of \( n \) points on a genus \( g \) curve (e.g. for each \( h = 1, \ldots, n \) taking \( a_j = 0 \) for all \( j \neq i \) we get a sub-semigroup of \( \mathbb{N} \) with exactly \( g \) gaps and this semigroup must be realized by some curve).

Set \( E_{can}(P_1, \ldots, P_n) := \{ (a_1, \ldots, a_n) \in \mathbb{N}^n : \mathcal{O}_X(a_1 P_1 + \cdots + a_n P_n) \cong \omega_X \} \). Since \( \omega_X \) is spanned, we have \( E_{can}(P_1, \ldots, P_n) \subset E(P_1, \ldots, P_n) \). We have \( E_{can}(P_1, \ldots, P_n) \neq \emptyset \) if and only if \( E(P_1, \ldots, P_n) \) is symmetric. The symmetric support \( \text{sym} \mathcal{O}_X(E(P_1, \ldots, P_n)) \) of a symmetric semigroup \( E(P_1, \ldots, P_n) \) is the set of all \( i \in \{1, \ldots, n\} \) such that there is \( (a_1, \ldots, a_n) \in E_{can}(P_1, \ldots, P_n) \) with \( a_i > 0 \). Write \( m := \#(\text{sym} \mathcal{O}_X(E(P_1, \ldots, P_n))) \) and \( \text{sym} \mathcal{O}_X(E(P_1, \ldots, P_n)) = \{ i_1, \ldots, i_m \} \). We get an \( m \)-semigroup \( \text{Sym}E(P_1, \ldots, P_n) \subset \mathbb{N}^m \). We say that \( E(P_1, \ldots, P_n) \) is full if \( n = \#(\text{sym} \mathcal{O}_X(E(P_1, \ldots, P_n))) \) and that it is totally full if \( \omega_X((2g - 2) P_i) \cong \omega_X \) for all \( i \), i.e. if \( E_{can}(P_1, \ldots, P_n) \) intersects the \( n \)-axis of \( \mathbb{N}^n \). Notice that if \( 1 \leq m < n \) and \( X \) has a totally full symmetric \( n \)-semigroup, then it has a totally full symmetric \( m \)-semigroup. For each \( X \) of genus \( g \) the maximal integer \( n \) (call it \( s(X) \)) of an \( n \)-totally full symmetric \( n \)-semigroup if just the number \( n \geq 0 \) of all \( P \in X \) with \( \mathcal{O}_X(2g - 2 P) \cong \omega_X \), i.e. the number \( s(X) \) of all symmetric Weierstrass points of \( X \). We have \( s(X) = 2g + 2 \) if \( X \) is hyperelliptic and \( s(X) = 0 \) for a general curve \( X \) of genus \( g \geq 3 \), because a curve with general moduli has only ordinary Weierstrass points.

We prove the following result.

**Proposition 1.** Assume \( \text{char} (\mathbb{K}) = 0 \) and let \( X \) be a general curve of genus \( g \geq 4 \). Let \( E(P_1, \ldots, P_n) \) be a symmetric semigroup.

(a) We have \( n \geq g - 1 \).

(b) Take any \( (a_1, \ldots, a_n) \in E_{can}(P_1, \ldots, P_n) \) and set \( S := \{ i \in \{1, \ldots, n\} : a_i > 0 \} \). Then \( \#(S) \geq g - 1 \) and the images of the points \( P_i, i \in S \), in the canonical embedding of \( X \) span a hyperplane of \( \mathbb{P}^{g-1} \).

(c) If there is no \( \{Q_1, \ldots, Q_m\} \subset \{P_1, \ldots, P_n\} \) with \( E(Q_1, \ldots, Q_m) \) symmetric, then \( \#(E_{can}(P_1, \ldots, P_n)) = 1 \).

(d) Fix \( (a_1, \ldots, a_n) \in \mathbb{N}^n \) such that \( a_1 + \cdots + a_n = 2g - 2 \), \( a_i = 1 \) for
at least \( g - 3 \) indices \( i \) and \( a_i > 0 \) for at least \( g - 1 \) indices \( i \). Then there are distinct points \( Q_1, \ldots, Q_n \in X \) such that \( (a_1, \ldots, a_n) \in E_{can}(Q_1, \ldots, Q_n) \).

### 2. The Proofs

**Proof of Proposition 1:** Part (b) follows from [3, Lemmas 2.2 and 2.3]. Part (b) implies the inequality \( n \geq g - 1 \). Part (c) follows from the last assertion of part (b) and the definition of non-gap. Now we prove part (d). Decreasing if necessary \( n \) we may assume \( a_i > 0 \) for all \( i \). The last assumption in part (d) implies \( n \geq g - 1 \). Without losing generality we may assume \( a_i \leq a_j \) for all \( i \geq j \). Let \( h \) be the maximal integer such that \( a_i \geq 2 \). Let \( u : X \rightarrow \mathbb{P}^{g-1} \) be the canonical embedding.

(i) Assume \( n = g - 1 \) and \( h = 1 \). We have \( a_1 = g \). Let \( Q_1 \) be a Weierstrass point of \( X \), i.e. a ramification point of \( u \). We get that the scheme \( u(gQ) \) is contained in a hyperplane \( H \subset \mathbb{P}^{g-1} \). By part (b) the scheme \( u(X) \cap H \) has support with cardinality \( \geq g - 1 \) and hence is the disjoint union of the scheme \( u(gQ_1) \) and of \( g - 2 \) degree one points \( u(Q_2), \ldots, u(Q_{g-1}) \) with \( Q_i \neq Q_1 \) for all \( i > 1 \).

(ii) Assume \( n = g - 1 \) and \( h > 1 \). We have \( h = 2 \) and \( a_2 = g + 1 - a_1 \). Fix a general \( Q_1 \in X \) and set \( R := \omega_X(-a_1Q_1) \). Since \( \text{char}(\mathbb{K}) = 0 \), the map \( u \) is ordinary in the sense of [2] ([2, Theorem 15]) and in particular for a general \( Q_1 \in X \) we have \( h^0(\omega_X(-a_1Q_1)) = g - a_1 \). Let \( B \) be the base scheme of \( \omega_X(-a_1Q_1) \) and \( v : X \rightarrow \mathbb{P}^{g-1-a_1} \) the morphism associated to \( \omega_X(-a_1Q_1 - B) \).

In characteristic zero we may apply the classical Brill-Segre formula and get the existence of \( Q_2 \in X \) such that \( h^0(\omega_X(-a_1Q_1 - B - a_2Q_2)) > 0 \) (we only need that \( v \) is not a linearly normal embedding of \( \mathbb{P}^1 \)). By [3, Lemmas 2.2 and 2.3] we get \( Q_2 \neq Q_1 \) and that any \( D \in |\omega_X| \) containing \( a_1Q_1 + a_2Q_2 \) is the sum of \( a_1Q_1 + a_2Q_2 \) and \( g - 3 \) distinct points \( Q_3, \ldots, Q_{g-1} \), all of them different from \( Q_1 \) and \( Q_2 \).

(iii) Assume \( n \geq g \) and \( h = n - g + 3 \). Take \( (b_1, \ldots, b_{g-1}) \in \mathbb{N}^{g-1} \) with \( b_1 = g - a_h + 1 \), \( b_2 = a_h \) and \( b_i = 1 \) for all \( i > 2 \). By step (ii) there are \( O_1, \ldots, O_{g-1} \in X \) such that \( b_1O_1 + \cdots + b_{g-1}O_{g-1} \in |\omega_X| \) and \( O_i \neq O_j \) for all \( i \neq j \). By [3, Theorem 1.1] we get as divisors in \( |\omega_X| \) all deformation of \( \sum b_iO_i \), and in particular we realize the \( n \)-ple \( (a_1, \ldots, a_n) \).

(iv) Assume \( n \geq g \) and \( h < n - g + 3 \). Take \( (b_1, \ldots, b_{g-1}) \in \mathbb{N}^{g-1} \) with \( b_1 = g \) and \( b_i = 1 \) for all \( i > 1 \). By step (i) there are \( O_1, \ldots, O_{g-1} \in X \) such that \( b_1O_1 + \cdots + b_{g-1}O_{g-1} \) and \( O_i \neq O_j \) for all \( i \neq j \). By [3, Theorem 1.1] we get as divisors in \( |\omega_X| \) all deformation of \( \sum b_iO_i \), and in particular we realize
the $n$-ple $(a_1, \ldots, a_n)$.

\square

**Lemma 1.** Fix integers $n > 0$ and $g \geq 2$. Let $E \subseteq \mathbb{N}^n$ such that $(a_1, \ldots, a_n) \in E$ if $a_1 + \cdots + a_n \geq 2g$, i.e. with property $\clubsuit(g)$. Fix $a, b \in E$ with $a$ irreducible. Then $\|a\| \leq 4g - 1$, $b - c \in E$ for each $c \in E$ with $\|c\| \geq \|b\| + 2g$, $\pi_E(b) \leq \|b\| + 2g$ and $\pi(E, 1) \leq 4g - 1$.

**Proof.** The first two assertions are obvious. Fix any $b \in E$ and take an integer $s \geq 2$ and $c_1, \ldots, c_s \in E$ such that $c_1 + \cdots + c_s - b \in E$, but $\sum_{i \in S} c_i - b \notin E$ for any $S \subsetneq \{1, \ldots, s\}$. With no loss of generality we may assume $c_1 \geq \cdots \geq c_s$. Since $c_1 + \cdots + c_{s-1} - b \notin E$, we have $c_1 + \cdots + c_{s-1} \leq \|b\| + 2g - 1$ and hence $s - 1 \leq \|b\| + 2g - 1$. Therefore $s \leq \|b\| + 2g$. Hence $\pi_E(b) \leq \|b\| + 2g$. This inequality for all irreducible $b \in E$ gives the bound $\pi(E, 1) \leq 4g - 1$. \square

The last equality in Lemma 1 is rather weak for an $n$-semigroup $E(P_1, \ldots, P_n)$ associated to a curve of genus $g$, but it does not depend from $n$ and it holds also for non-irreducible elements. So all semigroups $E$ associated to smooth curve of genus $g$ have a bound primality function. We may improve Lemma 1 in the following way.

**Lemma 2.** Fix integers $n > 0$ and $g \geq 2$. Let $E \subseteq \mathbb{N}^n$ such that $(a_1, \ldots, a_n) \in E$ if $a_1 + \cdots + a_n \geq 2g$. Let $k$ be the minimal integer $\|a\|$ with $a \in E \setminus \{0\}$. Then $\pi_E(b) \leq [\|(b\| + 2g - 1)/k\]| + 1 and $\pi(E, 1) \leq [(6g - 2)/k] + 1$.

**Proof.** Fix any $b \in E$ and take an integer $s \geq 2$ and $c_1, \ldots, c_s \in E$ such that $c_1 + \cdots + c_s - b \in E$, but $\sum_{i \in S} c_i - b \notin E$ for any $S \subsetneq \{1, \ldots, s\}$. With no loss of generality we may assume $c_1 \geq \cdots \geq c_s$. Since $c_1 + \cdots + c_{s-1} - b \notin E$, we have $c_1 + \cdots + c_{s-1} \leq \|b\| + 2g - 1$ and hence $s - 1 \leq \|(b\| + 2g - 1)/k\]|. Therefore $\pi_E(b) \leq [\|(b\| + 2g - 1)/k\]| + 1$. Since $\|b\| \leq 4g - 1$ for each irreducible $b$, we get $\pi(E, 1) \leq [(6g - 2)/k] + 1$. \square

**Proof of Corollary 1:** Apply Lemma 2 and the definition of gonality.

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References


