International Journal of Pure and Applied Mathematics
Volume 98 No. 3 2015, 333-338
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v98i3.5

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ON WEIERTRASS N-SEMIGROUPS ON SMOOTH CURVES

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Abstract: Let X be a smooth curve of genus $g > 0$. For any $n \geq 2$ and any n distinct points $P_1, \dots, P_n \in X$ let $E(P_1, \dots, P_n)$ be the set of all $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned. We study some abstract properties of the semigroup $E(P_1, \dots, P_n)$ (e.g. its decompositions in irreducibles) which are upper bounded by a function of g independent from n. We say that $E(P_1, \dots, P_n)$ is symmetric if $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n) \cong \omega_X$ for some $(a_1, \dots, a_n) \in \mathbb{N}^n$. We study symmetric n-semigroups of curves with general moduli.

AMS Subject Classification: 14N05
Key Words: Weiertrass n-semigroup, smooth curve, semigroup of non-gaps

1. Introduction

Let X be a smooth curve of genus g defined over an algebraically closed base field $\mathbb{K}$. Fix n distinct points $P_1, \dots, P_n \in X$. Let $E(P_1, \dots, P_n) \subset \mathbb{N}^n$ be the semigroup of non-gaps of the line bundles $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$, i.e. an n-ple $(a_1, \dots, a_n) \in E(P_1, \dots, P_n)$ if and only if the line bundle $\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)$ is spanned (i.e. $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_i)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) - 1$ for all $i = 1, \dots, n$, i.e. (Riemann-Roch) $h^1(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_i)) = h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n))$ for all $i = 1, \dots, n$). Set $G(P_1, \dots, P_n) := \mathbb{N}^n \setminus$

Received: August 24, 2014

$E(P_1, \dots, P_n)$. $E(P_1, \dots, P_n)$ is called the semigroup of all non-gaps, while $G(P_1, \dots, P_n)$ is the set of all gaps of $(P_1, \dots, P_n)$. There is an upper bound on $\sharp(G(P_1, \dots, P_n))$ depending only on g and n ([1]). The main goal of this note is to point out that Weierstrass n-semigroups have invariants which are upper bounded by a function of g not depending on n.

We recall the following definition ([4, Definition 2.3]). We will only need the case in which $M \subset \mathbb{N}^n$ (with componentwise addition as the monoid operation)with $\mathbb{N}^n \setminus M$ finite. Let M be a commutative, cancellative monoid M with $+$ as its operation. For any $a \in M$ the *primality function* $\pi_M(a) \in \mathbb{N} \setminus \{0\}) \cup \{+\infty\}$ of a in the following way. Let $\pi_M(a)$ be the minimal integer $t > 0$ with the following property. Assume that a divides a sum $b_1 + \cdots + b_s$. Then there is a subset $S \subseteq \{1, \dots, s\}$ with $\sharp(S) \le \min\{t, s\}$ such that a divides $\sum_{i \in S} b_i$. We write $\pi_M(a) = +\infty$ if there is no such an integer. Let $\pi(M, 1)$ be the supremum of all integer $\pi_M(a)$ with a irreducible. Let $\pi(M, 2)$ be the minimal non-negative integer t such that $\pi_M(a) \le \pi_M(a_1) + \cdots + \pi_M(a_s) - t$ if $a = a_1 + \cdots + a_s$ and each a_i is irreducible. For any $x \in M$ let $ir_{M,1}(x)$ (resp. $ir_{M,2}(x)$) be the minimal numbers of irreducible factors in a decomposition of x, with the convention $ir_{M,2}(x) = +\infty$ if there is no such a maximum; let $\mathrm{IRR}(M)$ be the supremum of all $ir_{M,2}(x) - ir_{M,1}(x)$ with $x \in M$. In our case when M is an additive semigroup of $\mathbb{N}^n$ with a finite complement, we always have $ir_{M,2}(x) < +\infty$. For any $b = (b_1, \dots, b_n) \in \mathbb{N}^n$ set $\|b\| := b_1 + \cdots + b_n$.

We prove the following result.

Corollary 1. *Let $E = E(P_1, \dots, P_n)$ be an n-semigroup associated to a curve X of genus g with gonality k.Then $\pi_E(b) \le \lfloor(\|b\| + 2g - 1)/k\rfloor + 1$ and $\pi(E, 1) \le \lfloor(6g - 2)/k\rfloor + 1$.*

Since $k \ge 2$, Corollary 1 gives $\pi(E, 1) \le 3g$.

When $n = 1$ the genus g semigroup $E(P_1)$ is called *symmetric* if $2g - 1$ is a gap (in this case it is the last gap), because $2g - 1$ is a gap if and only if $\mathcal{O}_X((2g - 2)P_1) \cong \omega_X$; Serre duality gives that symmetry is equivalent to a certain relation between gaps and non-gaps in degree $\le 2g - 1$. We generalize this definition to the case $n \ge 2$ in the following way.

Call $\spadesuit(g)$ the condition that $a_1 + \cdots + a_n \le 2g - 1$ for all $(a_1, \dots, a_n) \in \mathbb{N}^n \setminus E$. The condition $\spadesuit(g)$ is satisfied if $E = E(P_1, \dots, P_n)$ for $P_1, \dots, P_n \in X$ and X of genus g. Let $E \subseteq \mathbb{N}^n$ be any semigroup satisfying $\spadesuit(g)$. We say that E is symmetric if there is $(a_1, \dots, a_n) \in \mathbb{N}^n \setminus E$ with $a_1 + \cdots + a_n = 2g - 1$. Now assume $E = E(P_1, \dots, P_n)$ and assume the existence of $(a_1, \dots, a_n) \in G(P_1, \dots, P_n)$ with $a_1 + \cdots + a_n = 2g - 1$. Since $(a_1, \dots, a_n)$ is a gap, there is $h \in \{1, \dots, n\}$ such that $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n - P_h)) = h^0(\mathcal{O}_X(a_1P_1 +$

$\cdots + a_nP_n))$. We have $a_h > 0$, because the points $P_1, \dots, P_n$ are distinct. Set $b_i := a_i$ if $i \neq h$ and $b_h := a_h - 1$. Since $a_1 + \cdots + a_n = 2g - 1$, we have $h^0(\mathcal{O}_X(a_1P_1 + \cdots + a_nP_n)) = g$. Hence Riemann-Roch and Clifford's inequality gives $\omega_X \cong \mathcal{O}_X(b_1P_1 + \cdots + b_nP_n)$. Riemann-Roch also gives the converse, i.e. it gives that $E(P_1, \dots, P_n)$ is symmetric if and only if $\omega_X \cong \mathcal{O}_X(b_1P_1 + \cdots + b_nP_n)$ for some $(b_1, \dots, b_n) \in \mathbb{N}^n$. By Riemann-Roch we get that $(b_1, \dots, b_n) + e_i \in G(P_1, \dots, P_n)$ for all $i = 1, \dots, n$. Therefore if $n \geq 2$ it is a very strong restriction for a symmetric semigroup to be of the form $E(P_1, \dots, P_n)$ for some curve X of genus g and an n-ple of distinct points on X. There are also many other obvious restrictions for the realization of an n-semigroup E as the semigroup of n points on a genus g curve (e.g. for each $h = 1, \dots, n$ taking $a_j = 0$ for all $j \neq i$ we get a sub-semigroup of $\mathbb{N}$ with exactly g gaps and this semigroup must be realized by some curve). Set $E_{can}(P_1, \dots, P_n) := \{(a_1, \dots, a_n) \in \mathbb{N}^n : \mathcal{O}_X(a_1P_1 + \cdots + a_nP_n) \cong \omega_X\}$. Since ω_X is spanned, we have $E_{can}(P_1, \dots, P_n) \subset E(P_1, \dots, P_n)$. We have $E_{can}(P_1, \dots, P_n) \neq \emptyset$ if and only if $E(P_1, \dots, P_n)$ is symmetric. The symmetric support symsup$(E(P_1, \dots, P_n))$ of a symmetric semigroup $E(P_1, \dots, P_n)$ is the set of all $i \in \{1, \dots, n\}$ such that there is $(a_1, \dots, a_n) \in E_{can}(P_1, \dots, P_n)$ with $a_i > 0$. Write $m := \sharp(\text{symsup}(E(P_1, \dots, P_n)))$ and symsup$(E(P_1, \dots, P_n)) = \{i_1, \dots, i_m\}$. We get an m-semigroup $SymE(P_1, \dots, P_n) \subset \mathbb{N}^m$. We say that $E(P_1, \dots, P_n)$ is *full* if $n = \sharp(\text{symsup}(E(P_1, \dots, P_n)))$ and that it is *totally full* if $\omega_X((2g-2)P_i) \cong \omega_X$ for all i, i.e. if $E_{can}(P_1, \dots, P_n)$ intersects the n axis of $\mathbb{N}^n$. Notice that if $1 \leq m < n$ and X has a totally full symmetric n-semigroup, then it has a totally full symmetric m-semigroup. For each X of genus g the maximal integer n (call it $s(X)$) of an n-totally full symmetric n-semigroup if just the number $n \geq 0$ of all $P \in X$ with $\mathcal{O}_X((2g-2)P) \cong \omega_X$, i.e. the number $s(X)$ of all symmetric Weierstrass points of X. We have $s(X) = 2g + 2$ if X is hyperelliptic and $s(X) = 0$ for a general curve X of genus $g \geq 3$, because a curve with general moduli has only ordinary Weierstrass points.

We prove the following result.

Proposition 1. *Assume char$(\mathbb{K}) = 0$ and let X be a general curve of genus $g \geq 4$. Let $E(P_1, \dots, P_n)$ be a symmetric semigroup.*

(a) We have $n \geq g - 1$.

(b) Take any $(a_1, \dots, a_n) \in E_{can}(P_1, \dots, P_n)$ and set $S := \{i \in \{1, \dots, n\} : a_i > 0\}$. Then $\sharp(S) \geq g-1$ and the images of the points P_i, $i \in S$, in the canonical embedding of X span a hyperplane of $\mathbb{P}^{g-1}$.

(c) If there is no $\{Q_1, \dots, Q_m\} \subsetneq \{P_1, \dots, P_n\}$ with $E(Q_1, \dots, Q_m)$ symmetric, then $\sharp(E_{can}(P_1, \dots, P_n)) = 1$.

(d) Fix $(a_1, \dots, a_n) \in \mathbb{N}^n$ such that $a_1 + \cdots + a_n = 2g - 2$, $a_i = 1$ for

at least $g-3$ indices i and $a_i > 0$ for at least $g-1$ indices i. Then there are distinct points $Q_1, \dots, Q_n \in X$ such that $(a_1, \dots, a_n) \in E_{can}(Q_1, \dots, Q_n)$.

2. The Proofs

Proof of Proposition 1: Part (b) follows from [3, Lemmas 2.2 and 2.3]. Part (b) implies the inequality $n \geq g-1$. Part (c) follows from the last assertion of part (b) and the definition of non-gap. Now we prove part (d). Decreasing if necessary n we may assume $a_i > 0$ for all i. The last assumption in part (d) implies $n \geq g-1$. Without losing generality we may assume $a_i \leq a_j$ for all $i \geq j$. Let h be the maximal integer such that $a_i \geq 2$. Let $u : X \to \mathbb{P}^{g-1}$ be the canonical embedding.

(i) Assume $n = g-1$ and $h = 1$. We have $a_1 = g$. Let Q_1 be a Weierstrass point of X, i.e. a ramification point of u. We get that the scheme $u(gQ)$ is contained in a hyperplane $H \subset \mathbb{P}^{g-1}$. By part (b) the scheme $u(X) \cap H$ has support with cardinality $\geq g-1$ and hence is the disjoint union of the scheme $u(gQ_1)$ and of $g-2$ degree one points $u(Q_2), \dots, u(Q_{g-1})$ with $Q_i \neq Q_1$ for all $i > 1$.

(ii) Assume $n = g-1$ and $h > 1$. We have $h = 2$ and $a_2 = g+1-a_1$. Fix a general $Q_1 \in X$ and set $R := \omega_X(-a_1Q_1)$. Since $\text{char}(\mathbb{K}) = 0$, the map u is ordinary in the sense of [2] ([2, Theorem 15]) and in particular for a general $Q_1 \in X$ we have $h^0(\omega_X(-a_1Q_1)) = g - a_1$. Let B be the base scheme of $\omega_X(-a_1Q_1)$ and $v : X \to \mathbb{P}^{g-1-a_1}$ the morphism associated to $|\omega_X(-a_1Q_1 - B)|$. In characteristic zero we may apply the classical Brill-Segre formula and get the existence of $Q_2 \in X$ such that $h^0(\omega_X(-a_1Q_1 - B - a_2Q_2)) > 0$ (we only need that v is not a linearly normal embedding of $\mathbb{P}^1$). By [3, Lemmas 2.2 and 2.3] we get $Q_2 \neq Q_1$ and that any $D \in |\omega_X|$ containing $a_1Q_1 + a_2Q_2$ is the sum of $a_1Q_1 + a_2Q_2$ and $g-3$ distinct points $Q_3, \dots, Q_{g-1}$, all of them different from Q_1 and Q_2.

(iii) Assume $n \geq g$ and $h = n - g + 3$. Take $(b_1, \cdots, b_{g-1}) \in \mathbb{N}^{g-1}$ with $b_1 = g - a_h + 1$, $b_2 = a_h$ and $b_i = 1$ for all $i > 2$. By step (ii) there are $O_1, \dots, O_{g-1} \in X$ such that $b_1O_1 + \cdots + b_{g-1}O_{g-1} \in |\omega_X|$ and $O_i \neq O_j$ for all $i \neq j$. By [3, Theorem 1.1] we get as divisors in $|\omega_X|$ all deformation of $\sum b_iO_i$, and in particular we realize the n-ple $(a_1, \dots, a_n)$.

(iv) Assume $n \geq g$ and $h < n - g + 3$. Take $(b_1, \cdots, b_{g-1}) \in \mathbb{N}^{g-1}$ with $b_1 = g$ and $b_i = 1$ for all $i > 1$. By step (i) there are $O_1, \dots, O_{g-1} \in X$ such that $b_1O_1 + \cdots + b_{g-1}O_{g-1}$ and $O_i \neq O_j$ for all $i \neq j$. By [3, Theorem 1.1] we get as divisors in $|\omega_X|$ all deformation of $\sum b_iO_i$, and in particular we realize the n-ple $(a_1, \ldots, a_n)$. $\square$

Lemma 1. *Fix integers $n > 0$ and $g \geq 2$. Let $E \subseteq \mathbb{N}^n$ such that $(a_1, \ldots, a_n) \in E$ if $a_1 + \cdots + a_n \geq 2g$, i.e. with property $\spadesuit(g)$. Fix $a, b \in E$ with a irreducible. Then $\|a\| \leq 4g-1$, $b - c \in E$ for each $c \in E$ with $\|c\| \geq \|b\| + 2g$, $\pi_E(b) \leq \|b\| + 2g$ and $\pi(E, 1) \leq 4g - 1$.*

Proof. The first two assertions are obvious. Fix any $b \in E$ and take an integer $s \geq 2$ and $c_1, \ldots, c_s \in E$ such that $c_1 + \cdots + c_s - b \in E$, but $\sum_{i \in S} c_i - b \notin E$ for any $S \subsetneq \{1, \ldots, s\}$. With no loss of generality we may assume $c_1 \geq \cdots \geq c_s$. Since $c_1 + \cdots + c_{s-1} - b \notin E$, we have $c_1 + \cdots + c_{s-1} \leq \|b\| + 2g - 1$ and hence $s - 1 \leq \|b\| + 2g - 1$. Therefore $s \leq \|b\| + 2g$. Hence $\pi_E(b) \leq \|b\| + 2g$. This inequality for all irreducible $b \in E$ gives the bound $\pi(E, 1) \leq 4g - 1$. $\square$

The last equality in Lemma 1 is rather weak for an n-semigroup $E(P_1, \ldots, P_n)$ associated to a curve of genus g, but it does not depend from n and it holds also for non-irreducible elements. So all semigroups E associated to smooth curve of genus g have a bound primality function. We may improve Lemma 1 in the following way.

Lemma 2. *Fix integers $n > 0$ and $g \geq 2$. Let $E \subseteq \mathbb{N}^n$ such that $(a_1, \ldots, a_n) \in E$ if $a_1 + \cdot + a_n \geq 2g$. Let k be the minimal integer $\|a\|$ with $a \in E \backslash \{0\}$. Then $\pi_E(b) \leq \lfloor (\|b\| + 2g - 1)/k \rfloor + 1$ and $\pi(E, 1) \leq \lfloor (6g-2)/k \rfloor + 1$.*

Proof. Fix any $b \in E$ and take an integer $s \geq 2$ and $c_1, \ldots, c_s \in E$ such that $c_1 + \cdots + c_s - b \in E$, but $\sum_{i \in S} c_i - b \notin E$ for any $S \subsetneq \{1, \ldots, s\}$. With no loss of generality we may assume $c_1 \geq \cdots \geq c_s$. Since $c_1 + \cdots + c_{s-1} - b \notin E$, we have $c_1 + \cdots + c_{s-1} \leq \|b\| + 2g - 1$ and hence $s - 1 \leq \lfloor (\|b\| + 2g - 1)/k \rfloor$. Therefore $\pi_E(b) \leq \lfloor (\|b\| + 2g - 1)/k \rfloor + 1$. Since $\|b\| \leq 4g - 1$ for each irreducible b, we get $\pi(E, 1) \leq \lfloor (6g - 2)/k \rfloor + 1$. $\square$

Proof of Corollary 1: Apply Lemma 2 and the definition of gonality. $\square$

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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