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ZERO-DIMENSIONAL SCHEMES CONTAINED BETWEEN TWO CONSECUTIVE MULTIPLE POINTS

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Fix an integral projective variety, $P \in X_{reg}$, $m > 0$, $L \in \mathrm{Pic}(X)$. Let $W \subseteq H^0(X, L)$ be a linear subspace. Set $b := \dim(W(-mP))$ and $a := \dim(W(-(m+1)P))$. Fix an integer z such that $0 \le z \le b - a$. We prove the existence of a zero-dimensional scheme $Z \subset X$ such that $mP \subseteq Z \subseteq (m+1)P$, $\deg(Z) = \deg(mP) + z$ and $\dim(W(-Z)) = b - z$.

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Let X be an integral projective variety with $\dim(X) > 0$ over an algebraically closed base field K. Fix $P \in X$. For each integers $m > 0$ let mP be the closed subscheme of X with $(\mathcal{I}_X)^m$ as its ideal sheaf. For each line bundle R on X, any linear subspace $W \subseteq H^0(X, L)$ and any zero-dimensional scheme $Z \subset X$ set $W(-Z) := W \cap H^0(X, \mathcal{I}_Z \otimes L)$. In this note we prove the following result.

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Theorem 1. *Fix an integral projective variety, $P \in X_{reg}$, $m > 0$, $L \in Pic(X)$. Let $W \subseteq H^0(X, L)$ be a linear subspace. Set $b := \dim(W(-mP))$ and $a := \dim(W(-(m+1)P))$. Fix an integer z such that $0 \le z \le b - a$. Then there is a zero-dimensional scheme $Z \subset X$ such that $mP \subseteq Z \subseteq (m+1)P$, $\deg(Z) = \deg(mP) + z$ and $\dim(W(-Z)) = b - z$.*

Theorem 1 cannot be extended to scheme contained between mP and $(m+2)P$, even if X is a smooth curve, $m = 1$ and $W = H^0(X, L)$, because we fixed the point P and hence it may be an ordinary flex of the linear system $|L|$. In the case $m = 1$ when we take general P, see [1], [2] and [3].

Proof of Theorem 1. We may assume $n := \dim(X) > 0$ and $b - a > 0$. Fix a regular sequence $x_1, \dots, x_n$ of the local ring $\mathcal{O}_{X,P}$ and call $\mathbf{m}$ the maximal ideal of $\mathcal{O}_{X,P}$. Let $\mathcal{J}$ be the ideal sheaf of mP in $(m+1)P$. We have $\mathcal{O}_{(m+1)P} \cong K[x_1, \dots, x_n]/(x_1, \dots, x_n)^{m+1}$ and $\mathcal{O}_{mP} \cong K[x_1, \dots, x_n]/(x_1, \dots, x_n)^m$. Therefore $\mathcal{J}$ is a K-vector space with as a basis the set $\mathcal{B}$ of all monomials of degree $m+1$ in the variables $x_1, \dots, x_n$ (a K-vector space of dimension $\binom{n+m}{n-1}$). For any $E \subseteq \mathcal{B}$ let $\langle E \rangle$ be the K-linear subspace of $\mathcal{J}$ generated by E. Since $\mathcal{J}^2 = 0$ each $E \subseteq \mathcal{B}$ define a zero-dimensional scheme $Z_E \subseteq (m+1)P \subset X$ such that $Z_E \supseteq mP$ and $\deg(Z_E) = \deg(mP) + \sharp(E)$. Note that $b - a \le \binom{n+m}{n-1}$ and hence $z \le \binom{n+m}{n-1}$. We will find $Z = Z_E$ for some E. If $z = 0$, then take $Z = mP$, i.e. $E = \emptyset$. Hence we may assume $z > 0$ and use induction on z for the existence of Z of the form Z_E. Take $F \subset \mathcal{B}$ such that $\sharp(F) = z - 1$ and $\dim(W(-Z_F)) = a - z + 1$. Since $a - z + 1 > b$ and $\mathcal{B}$ is a basis of the K-vector space $\mathcal{B}$ there is $m \in \mathcal{B} \setminus F$ and $f \in W(-Z_F))$ such that $f|m \neq 0$. Take $E := F \cup \{m\}$. $\square$

Corollary 1. *Fix an integral projective variety of dimension $n > 0$, $L \in Pic(X)$ and a linear subspace $W \subseteq H^0(X, L)$ be a linear subspace. Fix integers $s > 0$ and $m_i > 0$, $1 \le i \le s$. Suppose that for general $P_1, \dots, P_s \in X_{reg}$ we have $\dim(W(-\cup_i m_i P_i)) = \dim(W) - \sum_i \binom{m_i+n-1}{n}$ and $W(-\cup_i (m_i+1)P_i) = 0$. Then for general $O_1, \dots, O_s \in X_{reg}$ there are schemes Z_i, $1 \le i \le s$, such that $m_i P_i \subseteq Z_i \subseteq (m_i+1)P_i$ for all i, $\sum_i \deg(Z_i) = \dim(W)$ and $W(-\cup_i Z_i) = 0$.*

Now we drop the assumption that K is algebraically closed. Easy examples show that there is no hope of a straightforward extension if $P \notin X(K)$, i.e. if the natural map $K \to \mathcal{O}_{X,P}/\mathbf{m}$ is not an isomorphism. If we may find a regular sequence defined $x_1, \dots, x_n$ defined over K, then the proof just given works verbatim. It exists, by Nakayama's lemma, if we assume that P is a smooth point of X. The definition of the schemes Z_E, $E \subseteq \mathcal{B}$ depends from the choice of the regular sequence. If we fix we get the following corollary of the proof of

Theorem 1

Corollary 2. *Fix an integral projective variety,* $P \in X_{reg}$, $m > 0$, $L \in$ *Pic*(X)*. Let* $W \subseteq H^0(X, L)$ *be a linear subspace. Fix a regular system of parameters* $x_1, \dots, x_n$ *and use it to define* $\mathcal{B}$ *and all* Z_E, $E \subseteq \mathcal{B}$*. Fix* $E \subseteq F \subseteq \mathcal{B}$ *and set* $a := \dim(W(-Z_F))$ *and* $b := \dim(W(-Z_E))$*. Fix an integer* z *such that* $0 \le z \le b - a$*. Then there is* G *such that* $E \subseteq G \subseteq F$, $\sharp(G) = \sharp(E) + z$ *and* $\dim(W(-Z_G)) = b - z$*.*

If we have a way to fix a regular system of parameters for all points in a open subset U of X (e.g. if U is an open subset of $\mathbb{A}^n$ with coordinates $z_1, \dots, z_n$ and we take $(z_1 - a_1, \dots, z_n - a_n)$ at $P = (a_1, \dots, a_n)$), then we are in the set-up of [2].

Remark 1. Take $P \in \mathrm{Sing}(X)$ and call $\mathbf{m}$ the maximal ideal of $\mathcal{O}_{X,P}$. Take $m > 0$ such that for $t = m, m+1$ the natural map $S^t(\mathbf{m}/\mathbf{m}^2) \to \mathbf{m}^t/\mathbf{m}^{t+1}$ is an isomorphism (this condition is always satisfied if $m = 1$). Then Theorem 1 is true for P.

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