PSEUDO t-NORMS AND FUZZY CONNECTIVES

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Abstract: In this paper, we investigated the properties of pseudo t-conorms and implications induced by pairs of negations and a pseudo t-norm. Moreover, we give their examples.

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1. Introduction

Many researchers [1-8] introduced non-commutative algebraic structures. This concept is motivated not only by the logically interest, but also by the relation with some remarkable mathematical structures. In [3], pseudo t-norms were investigated in bounded lattices in a sense as non-commutative property. Kim [10] introduced pairs of negations and implications. which are induced by non-commutative property.

In this paper, we investigated the properties of pseudo t-conorms and implications induced by pairs of negations and a pseudo t-norm. Moreover, we give their examples.

2. Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with a bottom element $\perp$ and a top element $\top$.

Definition 2.1.[3,6] A map $T : L \times L \rightarrow L$ is called a *pseudo t-norm* if it satisfies the following conditions:

- (T1) $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$,
- (T2) If $y \leq z$, $T(x, y) \leq T(x, z)$ and $T(y, x) \leq T(z, x)$,
- (T3) $T(x, \top) = T(\top, x) = x$.

A pseudo t-norm is called a *t-norm* if $T(x, y) = T(y, x)$ for $x, y \in L$.

A map $S : L \times L \rightarrow L$ is called a *pseudo t-conorm* if it satisfies the following conditions:

- (S1) $S(x, S(y, z)) = S(S(x, y), z)$ for all $x, y, z \in L$,
- (S2) If $y \leq z$, $S(x, y) \leq S(x, z)$ and $S(y, x) \leq S(z, x)$,
- (S3) $S(x, \perp) = S(\perp, x) = x$.

A pseudo t-conorm is called a *t-conorm* if $S(x, y) = S(y, x)$ for $x, y \in L$.

Definition 2.2.[10] A pair (n_1, n_2) with maps $n_i : L \rightarrow L$ is called a *pair of negations* if it satisfies the following conditions:

- (N1) $n_i(\top) = \perp, n_i(\perp) = \top$ for all $i \in \{1, 2\}$.
- (N2) $n_i(x) \geq n_i(y)$ for $x \leq y$ and $i \in \{1, 2\}$.
- (N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in L$.

Definition 2.3.[10] A pair (I_1, I_2) with maps $I_1, I_2 : L \times L \rightarrow L$ is called a *pair of implications* if it satisfies the following conditions:

- (I1) $I_i(\top, \top) = I_i(\perp, \top) = I_i(\perp, \perp) = \top, I_i(\top, \perp) = \perp$ for all $i \in \{1, 2\}$.
- (I2) If $x \leq y$, then $I_i(x, z) \geq I_i(y, z)$ for all $i \in \{1, 2\}$.
- (I3) $I_i(\top, x) = x$ for all $x \in L$ and $i \in \{1, 2\}$.

A pair (I_1, I_2) of implications is called a *pair of E-implications* if it satisfies the following exchange properties:

- (E) $I_1(x, I_2(y, z)) = I_2(y, I_1(x, z))$ for all $x, y, z \in L$.

A pair (I_1, I_2) of implications is called a *pair of S-implications* if it satisfies the following strong properties:

- (S) $I_1(I_2(x, \perp), \perp) = I_2(I_1(x, \perp), \perp) = x$.

A pair (I_1, I_2) of implications is called a *pair of SE-implications* if it satisfies conditions (E) and (S).

In this paper, sometimes composition function $f \circ g \circ h$ will be denoted by fgh .

3. Pseudo t-Norms and Fuzzy Connectives

Theorem 3.1. Let $T : L \times L \rightarrow L$ be a pseudo t-norm and (n_1, n_2) a pair of negations on L . We define $T^t, S_{12}, S_{21}, I_i : L \times L \rightarrow L$

$$T^t(x, y) = T(y, x),$$

$$S_{12}(x, y) = n_1(T(n_2(x), n_2(y))), S_{21}(x, y) = n_2(T(n_1(x), n_1(y))).$$

The the following properties hold.

- (1) T^t is a pseudo t-norm.
- (2) S_{12} and S_{21} are pseudo t-conorms.
- (3) $S_{12} = S_{21}$ iff $S_{12}^t = S_{21}^t$ iff

$$T(x, y) = n_2 \circ n_2(T(n_1(n_1(x)), n_1(n_1(y))))$$

where $S_{ij}^t(x, y) = S_{ij}(y, x)$.

- (4) If $n_1 = n_2$, then $S_{12} = S_{21}$ and $S_{12}^t = S_{21}^t$.

Proof (1) It is easily proved as a similar method as following (2).

- (2) (S1) $S_{12}(S_{12}(x, y), z) = S_{12}(x, S_{12}(y, z))$ from

$$\begin{aligned} & S_{12}(S_{12}(x, y), z) \\ &= n_1(T(n_2(S_{12}(x, y)), n_2(z))) \\ &= n_1(T(n_2 n_1(T(n_2(x), n_2(y))), n_2(z))) \\ &= n_1(T(T(n_2(x), n_2(y)), n_2(z))) \\ &= n_1(T(n_2(x), T(n_2(y), n_2(z)))), \\ & S_{12}(x, S_{12}(y, z)) \\ &= n_1(T(n_2(x), n_2(S_{12}(y, z)))) \\ &= n_1(T(n_2(x), n_2 n_1(T(n_2(y), n_2(z))))) \\ &= n_1(T(n_2(x), T(n_2(y), n_2(z)))). \end{aligned}$$

(S2) $S_{12}(x, \perp) = n_1(T(n_2(x), n_2(\perp))) = n_1(n_2(x)) = x$. Similarly, $S_{12}(\perp, x) = x$.

(S3) If $x \leq z$ and $y \leq w$, then $n_2(x) \geq n_2(z)$ and $n_2(y) \geq n_2(w)$. Thus $S_{12}(x, y) \leq S_{12}(z, w)$.

Hence S_{12} is a pseudo t-conorm. Similarly, S_{21} is a pseudo t-conorm.

- (3) It follows from:

$$\begin{aligned} & S_{12}(x, y) = S_{21}(x, y) \\ & \text{iff } n_1(T(n_2(x), n_2(y))) = n_2(T(n_1(x), n_1(y))) \\ & \text{iff } T(n_2(x), n_2(y)) = n_2(n_2(T(n_1(x), n_1(y)))) \\ & \text{iff } T(x, y) = n_2(n_2(T(n_1(n_1(x)), n_1(n_1(y))))) \\ & \text{iff } S_{12}^t(x, y) = S_{21}^t(x, y). \end{aligned}$$

(4) Since $n_1 = n_2$, then $n_1 n_1(x) = n_2 n_2(x) = x$. In (3), it is trivial.

Example 3.2. Put $L = \{(x, y) \in R^2 \mid (0, 1) \leq (x, y) \leq (2, 3)\}$ where $(0, 1)$ is the bottom element and $(2, 3)$ is the top element where

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow y_1 < y_2 \text{ or } y_1 = y_2, x_1 \leq x_2.$$

(1) A map $T : L \times L \rightarrow L$ is defined as

$$T((x_1, y_1), (x_2, y_2)) = (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2y_1, \frac{1}{3}y_1y_2) \vee (0, 1).$$

(T1) $T(T((x_1, y_1), (x_2, y_2)), (x_3, y_3)) = T((x_1, y_1), T((x_2, y_2), (x_3, y_3)))$ from:

$$\begin{aligned} & T(T((x_1, y_1), (x_2, y_2)), (x_3, y_3)) \\ &= T((x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2y_1, \frac{1}{3}y_1y_2) \vee (0, 1), (x_3, y_3)) \\ &= (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2y_1 - \frac{2}{9}y_1y_2 + \frac{1}{3}y_1y_2x_3, \frac{1}{9}y_1y_2y_3) \vee (0, 1). \\ & T((x_1, y_1), T((x_2, y_2), (x_3, y_3))) \\ &= T((x_1, y_1), (x_2 - \frac{2}{3}y_2 + \frac{1}{3}x_3y_2, \frac{1}{3}y_2y_3) \vee (0, 1)) \\ &= (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2y_1 - \frac{2}{9}y_1y_2 + \frac{1}{3}y_1y_2x_3, \frac{1}{9}y_1y_2y_3) \vee (0, 1). \end{aligned}$$

(T2) If $(x_1, y_1) \leq (x_2, y_2)$, then $y_1 < y_2$ or $y_1 = y_2, x_1 \leq x_2$. Thus

$$\begin{aligned} & T((x_1, y_1), (x_3, y_3)) = (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_3y_1, \frac{1}{3}y_1y_3) \vee (0, 1) \\ & \leq (x_2 - \frac{2}{3}y_2 + \frac{1}{3}x_3y_2, \frac{1}{3}y_2y_3) \vee (0, 1) = T((x_2, y_2), (x_3, y_3)). \end{aligned}$$

(T3)

$$T((x_1, y_1), (2, 3)) = (x_1, y_1) = T((2, 3), (x_1, y_1)).$$

Then T is a pseudo t-norm but not t-norm because

$$(1, \frac{4}{3}) = T((3, 2), (-1, 2)) \neq T((-1, 2), (3, 2)) = (-\frac{1}{3}, \frac{4}{3}).$$

(2) We obtain a pair (n_1, n_2) of negations where

$$n_1(x, y) = (2 - \frac{3x}{y}, \frac{3}{y}), \quad n_2(x, y) = (\frac{2-x}{y}, \frac{3}{y}).$$

$$n_1 n_2(x, y) = (x, y), \quad n_2 n_1(x, y) = (x, y).$$

(3) By Theorem 3.1(2), we obtain

$$\begin{aligned} & S_{12}((x_1, y_1), (x_2, y_2)) = n_1 T(n_2(x_1, y_1), n_2(x_2, y_2)) \\ &= n_1 T((\frac{2-x_1}{y_1}, \frac{3}{y_1}), (\frac{2-x_2}{y_2}, \frac{3}{y_2})) = n_1 (\frac{2-x_1}{y_1} - \frac{2}{y_1} + \frac{2-x_2}{y_1 y_2}, \frac{3}{y_1 y_2}) \wedge (2, 3) \\ &= (x_2 + x_1 y_2, y_1 y_2) \wedge (2, 3), \end{aligned}$$

$$\begin{aligned}
S_{21}((x_1, y_1), (x_2, y_2)) &= n_2 T(n_1(x_1, y_1), n_1(x_2, y_2)) \\
&= n_2 T((2 - \frac{3x_1}{y_1}, \frac{3}{y_1}), (2 - \frac{3x_2}{y_2}, \frac{3}{y_2})) = n_2(2 - \frac{3x_1}{y_1} - \frac{3x_2}{y_1 y_2}, \frac{3}{y_1 y_2}) \wedge (2, 3) \\
&= (x_2 + x_1 y_2, y_1 y_2) \wedge (2, 3).
\end{aligned}$$

(4) Since

$$n_1 n_1(x, y) = (3x - 2y + 2, y), \quad n_2 n_2(x, y) = (\frac{x + 2y - 2}{3}, y)$$

$$\begin{aligned}
&n_2 n_2(T(n_1(n_1(x_1, y_1)), n_1(n_1(x_2, y_2)))) \\
&= n_2 n_2(T((3x_1 - 2y_1 + 2, y_1), (3x_2 - 2y_2 + 2, y_2))), \\
&= n_2 n_2((\frac{1}{3}(3x_1 - 2y_1 + x_2 y_1), \frac{1}{3}y_1 y_2) \vee (0, 1)), \\
&= (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2 y_1, \frac{1}{3}y_1 y_2) \vee (0, 1) \\
&= T((x_1, y_1), (x_2, y_2)).
\end{aligned}$$

By Theorem 3.1 (3), $S_{12} = S_{21}$ and $S_{12}^t = S_{21}^t$.

Theorem 3.3. Let $T : L \times L \rightarrow L$ be a pseudo t-norm and (n_1, n_2) a pair of negations on L . We define $I_i : L \times L \rightarrow L$

$$\begin{aligned}
I_1(x, y) &= n_1(T(x, n_2(y))), \quad I_2(x, y) = n_1(T(n_2(y), x)), \\
I_3(x, y) &= n_2(T(x, n_1(y))), \quad I_4(x, y) = n_2(T(n_1(y), x)), \\
I_5(x, y) &= n_1(T(n_2(y), n_2(n_2(x)))), \quad I_6(x, y) = n_2(T(n_1(y), n_1(n_1(x)))), \\
I_7(x, y) &= n_1(T(n_2 n_2(x), n_2(y))), \quad I_8(x, y) = n_2(T(n_1 n_1(x), n_1(y))).
\end{aligned}$$

The the following properties hold.

(1) The pair (I_1, I_2) is a pair of E -implications such that

$$I_1(T(x, y), z) = I_1(x, I_1(y, z)),$$

$$I_2(T(x, y), z) = I_2(y, I_2(x, z)),$$

$$I_1(I_2(x, \perp), \perp) = I_2(I_1(x, \perp), \perp) = n_1(n_1(x)).$$

(2) If $n_1 = n_2$, then (I_1, I_2) is a pair of SE -implications.

(3) If $x \leq y$ iff $I_1(x, y) = \top$ iff $I_2(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_1(y, z)$ iff $y \leq I_2(x, z)$.

(4) The pair (I_3, I_4) is a pair of E -implications such that

$$I_3(T(x, y), z) = I_3(x, I_3(y, z)),$$

$$I_4(T(x, y), z) = I_4(y, I_4(x, z))$$

$$I_3(I_4(x, \perp), \perp) = I_4(I_3(x, \perp), \perp) = n_2(n_2(x)).$$

- (5) If $n_1 = n_2$, then (I_3, I_4) is a pair of implications.
 (6) If $x \leq y$ iff $I_3(x, y) = \top$ iff $I_4(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_3(y, z)$ iff $y \leq I_4(x, z)$.
 (7) The pair (I_2, I_3) is a pair of S -implications.
 (8) If it satisfies the following condition

$$\begin{aligned} &T(x, n_1 n_1(T(n_2(n_2 y, z)))) \\ &= n_1 n_1(T(n_2(n_2(T(x, y))))), z), \end{aligned}$$

then (I_2, I_3) is a pair of SE -implications and if $x \leq y$ iff $I_2(x, y) = \top$ iff $I_3(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_3(y, z)$ iff $y \leq I_2(x, z)$.

- (9) The pair (I_1, I_4) is a pair of S -implications.
 (10) If it satisfies the following condition

$$T(x, n_2 n_2(T(n_1(n_1(y))), z))) = n_2 n_2(T(n_1(n_1(T(x, y)))), z)),$$

then (I_1, I_4) is a pair of SE -implications and if $x \leq y$ iff $I_1(x, y) = \top$ iff $I_4(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_1(y, z)$ iff $y \leq I_4(x, z)$.

- (11) If $n_1 = n_2$, then (I_2, I_3) and (I_1, I_4) are pairs of SE -implications.
 (12) The pair (I_1, I_5) is a pair of SE -implications. If $x \leq y$ iff $I_1(x, y) = \top$ iff $I_5(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_1(y, z)$ iff $y \leq I_5(x, z)$.
 (13) The pair (I_3, I_6) is a pair of SE -implications. If $x \leq y$ iff $I_3(x, y) = \top$ iff $I_6(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_3(y, z)$ iff $y \leq I_6(x, z)$.
 (14) The pair (I_2, I_7) is a pair of SE -implications. If $x \leq y$ iff $I_2(x, y) = \top$ iff $I_7(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_7(y, z)$ iff $y \leq I_2(x, z)$.
 (15) The pair (I_4, I_8) is a pair of SE -implications. If $x \leq y$ iff $I_4(x, y) = \top$ iff $I_8(x, y) = \top$, then $T(x, y) \leq z$ iff $x \leq I_8(y, z)$ iff $y \leq I_4(x, z)$.

Proof (1) (I1) Since $T(\top, \top) = \top$ and $T(\top, \perp) = T(\perp, \top) = T(\perp, \perp) = \perp$, we have

$$I_i(\top, \perp) = \perp, I_i(\top, \top) = I_i(\perp, \top) = I_i(\perp, \perp) = \top.$$

(I2) If $x \geq z$ and $y \leq w$, then $n_i(x) \leq n_i(z)$ and $n_i(y) \geq n_i(w)$, for $i = 1, 2$. Thus, for $i = 1, 2$,

$$I_i(x, y) \leq I_i(z, w).$$

(I3) Since $I_1(\top, x) = n_1(T(\top, n_2(x))) = n_1 n_2(x) = x$ and $I_2(\top, x) = n_1(T(n_2(x), \top)) = n_1 n_2(x) = x$.

(I4) $I_2(x, I_1(y, z)) = I_1(y, I_2(x, z))$ from:

$$\begin{aligned} I_2(x, I_1(y, z)) &= n_1(T(n_2(I_1(y, z)), x)) \\ &= n_1(T(n_2 n_1(T(y, n_2(z))), x)) = n_1(T(T(y, n_2(z)), x)) \\ I_1(y, I_2(x, z)) &= n_1(T(y, n_2(I_2(x, z)))) \\ &= n_1(T(y, n_2 n_1 T(n_2(z), x))) = n_1(T(y, T(n_2(z), x))). \end{aligned}$$

Thus, (I_1, I_2) is a pair of E -implications. Since $T(x, n_2(y)) = n_2(I_1(x, y))$, we have

$$\begin{aligned} I_1(T(x, y), z) &= n_1(T(T(x, y), n_2(z))) \\ &= n_1(T(T(x, y), n_2(z))) = n_1(T(x, T(y, n_2(z)))) \\ &= n_1(T(x, n_2(I_1(y, z)))) = I_1(x, I_1(y, z)), \\ I_2(T(x, y), z) &= n_1(T(n_2(z), T(x, y))) \\ &= n_1(T(n_2(z), T(x, y))) = n_1(T(T(n_2(z), x), y)) \\ &= n_1(T(n_2(I_2(x, z)), y)) = I_2(y, I_2(x, z)). \end{aligned}$$

$$\begin{aligned} I_1(x, \perp) &= n_1(T(x, n_2(\perp))) = n_1(x) = I_2(x, \perp). \\ I_1(I_2(x, \perp), \perp) &= I_1(I_2(x, \perp), \perp) = n_1(n_1(x)). \end{aligned}$$

(2) If $n_1 = n_2$, then $n_1 n_1(x) = n_2 n_2(x) = x$. Thus

$$I_1(I_2(x, \perp), \perp) = n_1 n_1(x) = x.$$

Hence (I_1, I_2) is a pair of implications.

(3) From (1), since

$$I_1(T(x, y), z) = I_1(x, I_1(y, z)),$$

$$I_2(T(x, y), z) = I_2(y, I_2(x, z)),$$

then $T(x, y) \leq z$ iff $x \leq I_1(y, z)$ iff $y \leq I_2(x, z)$.

(4)-(6) are similarly proved as (1)-(3), respectively.

(7) It is easily proved from

$$\begin{aligned} I_2(x, \perp) &= n_1(T(n_2(\perp), x)) = n_1(x) \\ I_3(x, \perp) &= n_2(T(x, n_1(\perp))) = n_2(x) \\ I_2(I_3(x, \perp), \perp) &= n_1(n_2(x)) = x, \\ I_2(I_3(x, \perp), \perp) &= n_2(n_1(x)) = x. \end{aligned}$$

(8)

$$\begin{aligned}
& I_3(x, I_2(y, z)) \\
&= n_2(T(x, n_1(I_2(y, z)))) \\
&= n_2(T(x, n_1(n_1(T(n_2(z), y))))) \\
&= n_2(T(x, n_1(n_1(T(n_2(z), y))))) \\
& I_2(y, I_3(x, z)) \\
&= n_1(T(n_2(I_3(x, z)), y)) \\
&= n_1(T(n_2(n_2(T(x, n_1(z))))), y) \\
&= n_1(T(n_2(n_2(T(x, n_1(z))))), y)
\end{aligned}$$

Hence $I_3(x, I_2(y, z)) = I_2(y, I_3(x, z))$ iff

$$\begin{aligned}
& T(x, n_1 n_1(T(n_2(n_2 y, z))) \\
&= n_1 n_1(T(n_2(n_2(T(x, y))))), z)).
\end{aligned}$$

It follows that (I_2, I_3) is a pair of implications.

(9) and (10) are similarly proved as (7) and (8), respectively.

(11) It is trivial from $n_1(n_1(x)) = x = n_2(n_2(x))$, (12) and (14).

(12) $I_5(x, I_1(y, z)) = I_1(y, I_5(x, z))$ from:

$$\begin{aligned}
& I_5(x, I_1(y, z)) \\
&= n_1(T(n_2(I_1(y, z), n_2 n_2(x)))) \\
&= n_1(T(n_2 n_1(T(y, n_2(z)))), n_2 n_2(x)) \\
&= n_1(T(T(y, n_2(z))), n_2 n_2(x)) \\
& I_1(y, I_5(x, z)) \\
&= n_1(T(y, n_2(I_5(x, z)))) \\
&= n_1(T(y, n_2 n_1 T(n_2(z), n_2 n_2 x))) \\
&= n_1(T(y, T(n_2(z), n_2 n_2 x))).
\end{aligned}$$

Since

$$I_1(x, \perp) = n_1(T(x, n_2(\perp))) = n_1(x)$$

and

$$I_5(x, \perp) = n_1(T(n_2(\perp), n_2 n_2(x))) = n_1 n_2 n_2(x) = n_2(x),$$

$$I_1(I_5(x, \perp), \perp) = n_1 n_2(x) = x.$$

$$I_5(I_1(x, \perp), \perp) = n_2 n_1(x) = x.$$

(13) It is similarly proved as (12).

(14) $I_7(x, I_2(y, z)) = I_2(y, I_7(x, z))$ from:

$$\begin{aligned}
 & I_7(x, I_2(y, z)) \\
 &= n_1(T(n_2 n_2(x), n_2(I_2(y, z)))) \\
 &= n_1(T(n_2 n_2(x), n_2 n_1(T(n_2(z), y)))) \\
 &= n_1(T(n_2 n_2(x), T(n_2(z), y))) \\
 & I_2(y, I_7(x, z)) \\
 &= n_1(T(n_2(I_7(x, z)), y)) \\
 &= n_1(T(n_2 n_1 T(n_2 n_2 x, n_2(z)), y)) \\
 &= n_1(T(T(n_2 n_2 x, n_2(z)), y)).
 \end{aligned}$$

Since

$$I_2(x, \perp) = n_1(T(n_2(\perp), x)) = n_1(x)$$

and

$$I_7(x, \perp) = n_1(T(n_2(\perp), n_2 n_2(x))) = n_1 n_2 n_2(x) = n_2(x),$$

$$I_2(I_7(x, \perp), \perp) = n_1 n_2(x) = x.$$

$$I_7(I_2(x, \perp), \perp) = n_2 n_1(x) = x.$$

(15) It is similarly proved as (14).

Example 3.4. Put $L = \{(x, y) \in R^2 \mid (0, 1) \leq (x, y) \leq (2, 3)\}$, T a pseudo t-norm and (n_1, n_2) be a pair of negations in Example 3.2.

(1)

$$\begin{aligned}
 & I_1((x_1, y_1), (x_2, y_2)) = n_1 T((x_1, y_1), n_2(x_2, y_2)) \\
 &= n_1 T((x_1, y_1), (\frac{2-x_2}{y_2}, \frac{3}{y_2})) \\
 &= n_1(x_1 - \frac{2}{3}y_1 + \frac{2y_1-x_2y_1}{3y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
 &= (x_2 + 2y_2 - \frac{3x_1y_2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
 \end{aligned}$$

$$\begin{aligned}
 & I_2((x_1, y_1), (x_2, y_2)) = n_1 T(n_2(x_2, y_2), (x_1, y_1)) \\
 &= n_1 T((\frac{2-x_2}{y_2}, \frac{3}{y_2}), (x_1, y_1)) = n_1(\frac{x_1-x_2}{y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
 &= (2 - \frac{3(x_1-x_2)}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
 \end{aligned}$$

$$\begin{aligned}
 & I_3((x_1, y_1), (x_2, y_2)) = n_2 T((x_1, y_1), n_1(x_2, y_2)) \\
 &= n_2 T((x_1, y_1), (2 - \frac{3x_2}{y_2}, \frac{3}{y_2})) = n_2(x_1 - \frac{x_2y_1}{y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
 &= (x_2 + \frac{(2-x_1)y_2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
 \end{aligned}$$

$$\begin{aligned}
 & I_4((x_1, y_1), (x_2, y_2)) = n_2 T(n_1(x_2, y_2), (x_1, y_1)) \\
 &= n_2 T((2 - \frac{3x_2}{y_2}, \frac{3}{y_2}), (x_1, y_1)) \\
 &= n_2(2 - \frac{3x_2}{y_2} - \frac{2}{y_2} + \frac{x_1}{y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
 &= (\frac{3x_2-x_1+2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
 \end{aligned}$$

$$\begin{aligned}
I_5((x_1, y_1), (x_2, y_2)) &= n_1 T(n_2(x_2, y_2), n_2 n_2(x_1, y_1)) \\
&= n_1 T((\frac{2-x_2}{y_2}, \frac{3}{y_2}), (\frac{1}{3}(x_1 + 2y_1 - 2), y_1)) \\
&= n_1(\frac{-x_2}{y_2} + \frac{x_1+2y_1-2}{3y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
&= (\frac{3x_2-x_1+2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
\end{aligned}$$

$$\begin{aligned}
I_6((x_1, y_1), (x_2, y_2)) &= n_2 T(n_1(x_2, y_2), n_1 n_1(x_1, y_1)) \\
&= n_2 T((2 - \frac{3x_2}{y_2}, \frac{3}{y_2}), (3x_1 - 2y_1 + 2, y_1)) \\
&= n_2(2 - \frac{3x_2-3x_1+2y_1}{y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
&= (\frac{3x_2-3x_1+2y_1}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
\end{aligned}$$

$$\begin{aligned}
I_7((x_1, y_1), (x_2, y_2)) &= n_1 T(n_2 n_2(x_1, y_1), n_2(x_2, y_2)) \\
&= n_1 T((\frac{x_1+2y_1-2}{3}, y_1), (\frac{2-x_2}{y_2}, \frac{3}{y_2})) \\
&= n_1(\frac{x_1-2}{3} + \frac{(2-x_2)y_1}{3y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
&= (x_2 - \frac{(x_1-2)y_2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
\end{aligned}$$

$$\begin{aligned}
I_8((x_1, y_1), (x_2, y_2)) &= n_2 T(n_1 n_1(x_1, y_1), n_1(x_2, y_2)) \\
&= n_2 T((3x_1 - 2y_1 + 2, y_1), (2 - \frac{3x_2}{y_2}, \frac{3}{y_2})) \\
&= n_2(3x_1 - 2y_1 + 2 - \frac{x_2 y_1}{y_2}, \frac{y_1}{y_2}) \wedge (2, 3) \\
&= (x_2 + 2y_2 - \frac{3x_1 y_2}{y_1}, \frac{3y_2}{y_1}) \wedge (2, 3).
\end{aligned}$$

(2) A pair (I_1, I_2) is a pair of E -implications, but not a pair of SE -implications

$$I_1(I_2((2, 2), (0, 1)), (0, 1)) = n_1 n_1(2, 2) = (4, 2) \neq (2, 2).$$

By Theorem 3.1 (3), since $I_1((2, 2), (1, 2)) = (2, 3)$ but $(2, 2) \not\leq (1, 2)$, it does not satisfy the conditions in Theorem 3.1(3). Moreover,

$$(2, 2) = T((2, 3), (2, 2)) \not\leq (1, 2),$$

$$I_1(T((2, 3), (2, 2)), (1, 2)) = I_1((2, 3), I_1((2, 2), (2, 1))) = (2, 3).$$

(3)

$$\begin{aligned}
&T((x_1, y_1), n_1 n_1(T(n_2 n_2(x_2, y_2), (x_3, y_3)))) \\
&= T((x_1, y_1), n_1 n_1(T(\frac{x_2+2y_2-2}{3}, y_2), (x_3, y_3))) \\
&= T((x_1, y_1), n_1 n_1(\frac{x_2+2y_2-2}{3} - \frac{2}{3}y_2 + \frac{1}{3}x_3 y_2, \frac{1}{3}y_2 y_3) \wedge (1, 0)) \\
&= T((x_1, y_1), (x_2 + x_3 y_2 - \frac{2}{3}y_3 y_2, \frac{1}{3}y_2 y_3) \wedge (1, 0)) \\
&= (x_1 + \frac{1}{3}x_2 y_1 - \frac{2}{3}y_1 + \frac{1}{3}x_3 y_2 y_1 - \frac{2}{9}y_3 y_2 y_1, \frac{1}{9}y_1 y_2 y_3) \wedge (1, 0).
\end{aligned}$$

$$\begin{aligned}
& n_1 n_1 (T(n_2 n_2 T((x_1, y_1), (x_2, y_2)), (x_3, y_3))) \\
&= n_1 n_1 (T(n_2 n_2 (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2 y_1), \frac{1}{3}y_2 y_1), (x_3, y_3))) \\
&= n_1 n_1 (T(\frac{(x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2 y_1 + \frac{2}{3}y_2 y_1 - 2)}{3}, \frac{1}{3}y_2 y_1), (x_3, y_3))) \\
&= (x_1 + \frac{1}{3}x_2 y_1 - \frac{2}{3}y_1 + \frac{1}{3}x_3 y_2 y_1 - \frac{2}{9}y_3 y_2 y_1, \frac{1}{9}y_1 y_2 y_3) \wedge (1, 0).
\end{aligned}$$

Since

$$\begin{aligned}
& T((x_1, y_1), n_1 n_1 (T(n_2 n_2 (x_2, y_2), (x_3, y_3)))) \\
&= n_1 n_1 (T(n_2 n_2 T((x_1, y_1), (x_2, y_2)), (x_3, y_3))),
\end{aligned}$$

by Theorem 3.3(8), (I_2, I_3) is a pair of SE -implications ;i.e.

$$\begin{aligned}
& I_2((x_1, y_1), I_3((x_2, y_2), (x_3, y_3))) \\
&= I_3((x_2, y_2), I_2((x_1, y_1), (x_3, y_3))).
\end{aligned}$$

Moreover, since $(x_1, y_1) \leq (x_2, y_2)$ iff $y_1 < y_2$ or $y_1 = y_2, x_1 \leq x_2$ iff $I_2((x_1, y_1), (x_2, y_2)) = (2, 3)$ iff $I_3((x_1, y_1), (x_2, y_2)) = (2, 3)$, then $T((x_1, y_1), (x_2, y_2)) \leq (x_3, y_3)$ iff $(x_1, y_1) \leq I_3((x_2, y_2), (x_3, y_3))$ iff $(x_2, y_2) \leq I_2((x_1, y_1), (x_3, y_3))$.

(4) A pair (I_3, I_4) is a pair of E -implications, but not a pair of SE -implications

$$I_3(I_4((2, 2), (0, 1)), (0, 1)) = n_2 n_2(2, 2) = (\frac{4}{3}, 2) \neq (2, 2).$$

By Theorem 3.3 (8), since $I_4((\frac{3}{2}, \frac{3}{2}), (1, \frac{3}{2})) = (2, 3)$ but $(\frac{3}{2}, \frac{3}{2}) \not\leq (1, \frac{3}{2})$, it does not satisfy the conditions in Theorem 3.3(6). Moreover,

$$T((2, 3), (\frac{3}{2}, \frac{3}{2})) \not\leq (1, \frac{3}{2}),$$

$$I_4(T((2, 3), (\frac{3}{2}, \frac{3}{2})), (1, \frac{3}{2})) = I_4((2, 3), I_4((\frac{3}{2}, \frac{3}{2}), (1, \frac{3}{2}))) = (2, 3).$$

(5) Since

$$\begin{aligned}
& T((x_1, y_1), n_2 n_2 (T(n_1 n_1 (x_2, y_2), (x_3, y_3)))) \\
&= T((x_1, y_1), n_2 n_2 (3x_2 - 2y_2 + 2 - \frac{2}{3}y_2 + \frac{1}{3}x_3 y_2, \frac{1}{3}x_3 y_2)) \\
&= (x_1 - \frac{2}{3}y_1 + \frac{1}{3}x_2 y_1 - \frac{8}{27}y_1 y_2 + \frac{1}{27}x_3 y_1 y_2 + \frac{2}{27}y_1 y_2 y_3, \frac{1}{9}y_1 y_2 y_3) \vee (0, 1) \\
&= n_2 n_2 (T(n_1 n_1 T((x_1, y_1), (x_2, y_2)), (x_3, y_3))),
\end{aligned}$$

by Theorem 3.3(10), (I_1, I_4) is a pair of SE -implications ;i.e.

$$\begin{aligned}
& I_1((x_1, y_1), I_4((x_2, y_2), (x_3, y_3))) \\
&= I_4((x_2, y_2), I_1((x_1, y_1), (x_3, y_3))).
\end{aligned}$$

(6) From Theorem 3.3(13), the pair (I_3, I_6) is a pair of SE -implications. Moreover, since $(x_1, y_1) \leq (x_2, y_2)$ iff $y_1 < y_2$ or $y_1 = y_2, x_1 \leq x_2$ iff $I_3((x_1, y_1), (x_2, y_2)) = (2, 3)$ iff $I_6((x_1, y_1), (x_2, y_2)) = (2, 3)$, then $T((x_1, y_1), (x_2, y_2)) \leq (x_3, y_3)$ iff $(x_1, y_1) \leq I_3((x_2, y_2), (x_3, y_3))$ iff $(x_2, y_2) \leq I_6((x_1, y_1), (x_3, y_3))$.

(7) From Theorem 3.3(14), the pair (I_2, I_7) is a pair of SE -implications. Moreover, since $(x_1, y_1) \leq (x_2, y_2)$ iff $y_1 < y_2$ or $y_1 = y_2, x_1 \leq x_2$ iff $I_2((x_1, y_1), (x_2, y_2)) = (2, 3)$ iff $I_7((x_1, y_1), (x_2, y_2)) = (2, 3)$, then $T((x_1, y_1), (x_2, y_2)) \leq (x_3, y_3)$ iff $(x_1, y_1) \leq I_7((x_2, y_2), (x_3, y_3))$ iff $(x_2, y_2) \leq I_2((x_1, y_1), (x_3, y_3))$.

References

- [1] A. Dvurecenskij, Pseudo MV-algebras are intervals in l-groups, *J. Australian Math. Soc.*, **72** (2002), 427-445, **doi:** 10.1017/s1446788700036806.
- [2] A. Dvurecenskij, On pseudo MV-algebras, *Soft Computing*, **5** (2001), 347-354, **doi:** 10.1017/s005000100136.
- [3] P. Flonder, G. Georgescu, A. Iorgulescu, Pseudo t-norms and pseudo-BL algebras, *Soft Computing*, **5**(2001), 355-371. **doi:** 10.1017/s005000100136.
- [4] N. Galatos, C. Tsınakis, Generalized MV-algebras, *J. of Algebra*, **283** (2005), 254-291, **doi:** 10.1016/j. j.algebra,2004,07,002.
- [5] G. Georgescu, A. Iorgulescu, Pseudo MV-algebras, *Multiple-Valued Logics*, **6** (2001), 193-215.
- [6] G. Georgescu, A. Popescu, Non-commutative fuzzy Galois connections, *Soft Computing*, **7** (2003), 458-467, **doi:** 10.1017/s00500-003-0280-4.
- [7] G. Georgescu, A. Popescu, Non-commutative fuzzy structures and pairs of weak negations, *Fuzzy Sets and Systems*, **143** (2004), 129-155, **doi:** 10.1016/j. fss,2003,06,004.
- [8] G. Georgescu, A. Popescue, Non-dual fuzzy connections, *Arch. Math. Log.*, **43**(2004), 1009-1039, **doi:** 10.1007/s00153-004-0240-4.
- [9] U. Höhle, E. P. Klement, *Non-classical logic and their applications to fuzzy subsets*, Kluwer Academic Publisher, Boston, 1995, **doi:** 10.1007/978-94-011-0215-5.

- [10] Y.C. Kim, Pairs of interval negations and interval implications, *Int. J. Pure and Applied Math.* **88(2)** (2013), 305-319, **doi:** 10.12732/ijpam.v88i2.12.