NUMERICAL SOLUTION OF WAVE EQUATION USING HAAR WAVELET

Inderdeep Singh¹, Sangeeta Arora², Sheo Kumar³ §

^{1,3}Department of Mathematics

Dr. B.R. Ambedkar National Institute of Technology

Jalandhar, 144011, Punjab, INDIA

²PG Department Computer Science and IT

HMV College Jalandhar

144008, Punjab, INDIA

Abstract: Haar wavelet is a powerful mathematical tool used to solve various type of partial differential equations. The solutions obtained by Haar wavelet are more accurate and efficient. We present here a Haar wavelet method for numerical solution of wave equation. The numerical solution obtained here by the present method, are more accurate and better than that are presented in Shi [5].

AMS Subject Classification: 65M99

Key Words: wave equation, Haar wavelet, function approximation

1. Introduction

Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. Wave equation is a special partial differential equation arising in numerous engineering problems. Wave equation is a second order hyperbolic partial differential equation and it describes the phenomenon of wave vibration and wave propagation. Consider the partial differential equation of the form

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§Correspondence author

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x, t), 0 < x < 1, t > 0, \quad (1)$$

with initial conditions

$$u|_{t=0} = g(x), \frac{\partial u}{\partial t}|_{t=0} = 0, 0 < x < 1, \quad (2)$$

and boundary conditions

$$u|_{x=0} = 0, u|_{x=1} = 0, t > 0, \quad (3)$$

where $u(x, t)$ represents the displacement of every point at time t , $c^2 = F/\sigma$, F represents the tension of every point in the media, σ represents medium density, F and σ are constant, so c^2 is also constant, $f(x, t)$ represents the external force on per unit mass.

In Section 2, we briefly describe Haar wavelet method. In Section 3, we have described function approximation. Methods for solving wave equation have been presented in Section 4 and in Section 5, numerical examples have been solved using the Haar wavelet method to illustrate the efficiency and accuracy of present method over the method described in Shi [5].

2. Haar Wavelet Method

In recent decades the field of Haar wavelets for solving partial differential equations have attracted interest of researchers in several areas of science and engineering. A survey on differential equation is presented in Hariharan [6]. Wavelet analysis is a new branch of mathematics and widely applied in differential and integral equations. Several methods have been proposed to find the numerical solution of different linear and nonlinear partial differential equations. Wavelets have been applied extensively in mathematical problems related to scientific and engineering fields. There are many wavelet families such as Daubechies wavelet [8], Hermite-type trigonometric wavelet and many more. In 1910, Haar [7] introduced a function which presents a rectangular pulse pair. After that various generalizations were proposed. Among all these wavelet families, it is the simplest orthonormal wavelet with compact support. Haar wavelet is Daubechies wavelet of order one. Hariharan et al. [3] presented the numerical solution of Fisher's equation using Haar wavelet. Further using Haar wavelet, Hariharan and Kannan [4] solved numerically Fitzhugh-Nagumo equation. Berwal et al. [2] presented the numerical solution of Telegraph equation using Haar wavelet.

Lepik [9] and [10] presented methods based on Haar wavelet for numerical solution of differential and integral equations. The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0, 1]$.

$$h_i(x) = \begin{cases} 1, & \alpha \leq x < \beta, \\ -1, & \beta \leq x < \gamma, \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

where $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$ and $\gamma = \frac{k+1}{m}$. Integer $m = 2^j$, ($j = 0, 1, 2, 3, 4, \dots, J$) indicates the level of the wavelet, and $k = 0, 1, 2, 3, \dots, m-1$ is the translation parameter. Maximal level of resolution is J . The index i is calculated according to the formula $i = m + k + 1$. In the case of minimal values, $m = 1$, $k = 0$ we get $i = 2$. The maximal value of i is $i = 2M$, where $M = 2^J$. It is assumed that the value $i = 1$, corresponding to the scaling function in $[0, 1]$.

$$h_1(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Let us define the collocation points $x_l = \frac{(l-0.5)}{2M}$, where $l = 1, 2, 3, \dots, 2M$ and discretise the Haar function $h_i(x)$. Using the following four notations of Haar functions:

$h_1(x) = [1, 1, 1, 1]$, $h_2(x) = [1, 1, -1, -1]$, $h_3 = [1, -1, 0, 0]$ and $h_4 = [0, 0, 1, -1]$, we introduce the following notation:

$$H_4(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \quad (6)$$

Here $H_4(x)$ is called Haar coefficient matrix. It is a square matrix of order 4, and is defined as $H(i, l) = (h_i(x_l))$, which has dimension $2M \times 2M$. Let us integrate equation (4), we get

$$q_i(x) = \int_0^x h_i(t) dt. \quad (7)$$

In the collocation points, equation (7) gets the form $Q(i, l) = q_i(x_l)$, where Q is a $2M \times 2M$ matrix. Chen and Hsiao [1] presented this matrix in the form $Q_n = P_n H_n$, where $P_n H_n$ is interpreted as the product of the matrices

P_n and H_n , called Haar integration and coefficient matrix, respectively. The operational matrix of integration P , which is a $2M$ square matrix, is defined by the relations:

$$P_{i,1}(x) = \int_0^x h_i(t)dt, \quad (8)$$

$$P_{i,n+1}(x) = \int_0^x P_{i,n}(t)dt, \quad (9)$$

where $n = 1, 2, 3, 4, \dots$. These integrals can be evaluated using equation (4) and first four of them are given below:-

$$P_{i,1}(x) = \begin{cases} x - \alpha, & x \in [\alpha, \beta), \\ \gamma - x, & x \in [\beta, \gamma), \\ 0, & \text{elsewhere.} \end{cases} \quad (10)$$

$$P_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \alpha)^2, & x \in [\alpha, \beta), \\ \frac{1}{4m^2} - \frac{1}{2}(\gamma - x)^2, & x \in [\beta, \gamma), \\ \frac{1}{4m^2}, & x \in [\gamma, 1), \\ 0, & \text{elsewhere.} \end{cases} \quad (11)$$

$$P_{i,3}(x) = \begin{cases} \frac{1}{6}(x - \alpha)^3, & x \in [\alpha, \beta), \\ \frac{1}{4m^2}(x - \beta) - \frac{1}{6}(\gamma - x)^3, & x \in [\beta, \gamma), \\ \frac{1}{4m^2}(x - \beta), & x \in [\gamma, 1), \\ 0, & \text{elsewhere.} \end{cases} \quad (12)$$

$$P_{i,4}(x) = \begin{cases} \frac{1}{24}(x - \alpha)^4, & x \in [\alpha, \beta), \\ \frac{1}{8m^2}(x - \beta)^2 - \frac{1}{24}(\gamma - x)^4 + \frac{1}{192m^4}, & x \in [\beta, \gamma), \\ \frac{1}{8m^2}(x - \beta)^2 + \frac{1}{192m^4}, & x \in [\gamma, 1), \\ 0, & \text{elsewhere.} \end{cases} \quad (13)$$

3. Function Approximation

We know that all the Haar wavelets are orthogonal to each other:

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j}, & i = l = 2^j + k, \\ 0, & i \neq l. \end{cases} \quad (14)$$

Therefore, they construct a very good transform basis. Any square integrable function $y(x)$ in the interval $[0, 1]$ can be expanded by a Haar series of infinite terms as:

$$y(x) = \sum_{i=1} c_i h_i(x), \quad (15)$$

where the Haar coefficients c_i are determined as:

$$c_0 = \int_0^1 y(x) h_0(x) dx, \quad (16)$$

$$c_i = 2^j \int_0^1 y(x) h_i(x) dx, \quad (17)$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$, $x \in [0, 1]$ such that the following integral square error ε is minimized:

$$\varepsilon = \int_0^1 [y(x) - \sum_{i=1}^m c_i h_i(x)]^2 dx, \quad (18)$$

where $m = 2^j$ and $j = 0, 1, 2, 3, \dots$. Usually the series expansion of (15) contains infinite terms. If $y(x)$ is piecewise constant by itself or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite m terms. This means

$$y(x) \cong \sum_{i=1}^m c_i h_i(x) = c_m^T h_m(x), \quad (19)$$

where the coefficients c_m^T and the Haar function vectors $h_m(x)$ are defined as: $c_m^T = [c_1, c_2, c_3, \dots, c_m]$ and $h_m(x) = [h_1(x), h_2(x), h_3(x), \dots, h_m(x)]^T$, where T is the transpose.

4. Method for Solving Wave Equation

Consider the Wave equation (1) with initial conditions $u(x, 0) = f_0(x)$, $\frac{\partial u}{\partial t}(x, 0) = f_1(x)$ and boundary conditions $u(0, t) = g_0(t)$, $u(1, t) = g_1(t)$. Assume that $\ddot{u}(x, t)$ can be expanded in terms of Haar wavelets as follows:

$$\ddot{u}(x, t) = \sum_{i=1}^{2M} a_i h_i(x), \quad t \in [t_s, t_{s+1}]. \quad (20)$$

Integrating the above equation twice with respect to t from t_s to t , and twice with respect to x , from 0 to x , we get,

$$\dot{u}(x, t) = (t - t_s) \sum_{i=1}^{2M} a_i h_i(x) + \dot{u}(x, t_s), \quad (21)$$

$$u(x, t) = \frac{1}{2}(t - t_s)^2 \sum_{i=1}^{2M} a_i h_i(x) + (t - t_s) \dot{u}(x, t_s) + u(x, t_s), \quad (22)$$

$$u(x, t) = \frac{1}{2}(t - t_s)^2 \sum_{i=1}^{2M} a_i P_{1,i}(x) + (t - t_s) [\dot{u}(x, t_s) + \dot{u}(0, t_s)] \\ + u(x, t_s) - u(0, t_s) + u(0, t), \quad (23)$$

$$u(x, t) = \frac{1}{2}(t - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(x) + (t - t_s) [\dot{u}(x, t_s) - \dot{u}(0, t_s) - x \dot{u}(0, t_s)] \\ + u(x, t_s) - u(0, t_s) + x[u(0, t) - u(0, t_s)] + u(0, t). \quad (24)$$

Differentiating the above equation twice with respect to t , we get

$$\dot{u}(x, t) = (t - t_s) \sum_{i=1}^{2M} a_i P_{2,i}(x) + \dot{u}(x, t_s) + \dot{u}(0, t) - \dot{u}(0, t_s) + x[\dot{u}(0, t) - \dot{u}(0, t_s)], \quad (25)$$

$$\ddot{u}(x, t) = \sum_{i=1}^{2M} a_i P_{2,i}(x) + \ddot{u}(0, t) + x \ddot{u}(0, t). \quad (26)$$

From the initial and boundary conditions, we have the following equations as:

$$u(x, 0) = f_0(x), \quad u(x, 1) = f_1(x), \quad u(0, t) = g_0(t), \quad u(1, t) = g_1(t),$$

$$u(0, t_s) = g_0(t_s), \quad u(1, t_s) = g_1(t_s), \quad \dot{u}(0, t) = \dot{g}_0(t), \quad \dot{u}(1, t) = \dot{g}_1(t),$$

$$\ddot{u}(0, t) = \ddot{g}_0(t), \quad \ddot{u}(1, t) = \ddot{g}_1(t), \quad \ddot{u}(0, t_s) = \ddot{g}_0(t_s), \quad \ddot{u}(1, t_s) = \ddot{g}_1(t_s).$$

At $x = 1$ in the formulae (24), (25) and (26) and by using conditions, we have

$$u(0, t) - u(0, t_s) = -\frac{1}{2}(t - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(1) - (t - t_s) [\dot{g}_1(t_s) - \dot{g}_0(t_s) - \dot{u}(0, t_s)]$$

$$+ g_1(t) - g_0(t) + g_0(t_s) - g_1(t_s), \quad (27)$$

$$\dot{u}(0, t) - \dot{u}(0, t_s) = g_1(\dot{t}) - g_0(\dot{t}) + g_0(\dot{t}_s) - g_1(\dot{t}_s) - (t - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(1), \quad (28)$$

$$\ddot{u}(0, t) = - \sum_{i=1}^{2M} a_i P_{2,i}(1) - \ddot{g}_0(t) + \ddot{g}_1(t). \quad (29)$$

If the equations (27), (28) and (29) are substituted into equations (22) – (26), and the results are discretized by assuming $x \rightarrow x_l$ and $t \rightarrow t_{s+1}$, we obtain

$$u(x_l, t_{s+1}) = \frac{1}{2}(t_{s+1} - t_s)^2 \sum_{i=1}^{2M} a_i h_i(x_l) + (t_{s+1} - t_s) \dot{u}(x_l, t_s) + u(x_l, t_s), \quad (30)$$

$$\begin{aligned} u(x_l, t_{s+1}) &= \frac{1}{2}(t_{s+1} - t_s)^2 \sum_{i=1}^{2M} a_i P_{1,i}(x_l) + (t_{s+1} - t_s) \dot{u}(x_l, t_s) + u(x_l, t_s) \\ &\quad - \frac{1}{2}(t_{s+1} - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(1) - (t_{s+1} - t_s)[\dot{g}_1(t_s) - \dot{g}_0(t_s)] + g_1(t_{s+1}) \\ &\quad - g_1(t_s) + g_0(t_s) - g_0(t_{s+1}), \quad (31) \end{aligned}$$

$$\begin{aligned} u(x_l, t_{s+1}) &= \frac{1}{2}(t_{s+1} - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(x_l) + (t_{s+1} - t_s)[\dot{u}(x_l, t_s) - \dot{g}_0(t_s)] \\ &\quad + u(x_l, t_s) - u(0, t_s) - \frac{x_l}{2}(t_{s+1} - t_s)^2 \sum_{i=1}^{2M} a_i P_{2,i}(1) - x_l(t_{s+1} - t_s)[\dot{g}_1(t_s) - \dot{g}_0(t_s)] \\ &\quad - x_l[g_1(t_s) - g_0(t_s) + g_0(t_{s+1}) - g_1(t_{s+1})] + g_0(t_{s+1}), \quad (32) \end{aligned}$$

$$\begin{aligned} \dot{u}(x_l, t_{s+1}) &= (t_{s+1} - t_s) \sum_{i=1}^{2M} a_i P_{2,i}(x_l) + \dot{u}(x_l, t_s) - \dot{g}_0(t_s) \\ &\quad - x_l[\dot{g}_1(t_s) - \dot{g}_0(t_s)] - x_l(t_{s+1} - t_s) \sum_{i=1}^{2M} a_i P_{2,i}(1) - x_l[\dot{g}_0(t_{s+1}) - \dot{g}_1(t_{s+1})] \\ &\quad + \dot{g}_0(t_{s+1}), \quad (33) \end{aligned}$$

$$\ddot{u}(x_l, t_{s+1}) = \sum_{i=1}^{2M} a_i [P_{2,i}(x_l) - x_l P_{2,i}(1)] - x_l [\ddot{g}_0(t_{s+1}) - \ddot{g}_1(t_{s+1})] + \ddot{g}_0(t_{s+1}). \quad (34)$$

But, we know that

$$P_{2,i}(1) = \begin{cases} 0.5, & i = 1, \\ \frac{1}{4m^2}, & i > 1, \end{cases} \quad (35)$$

hence, in the given scheme

$$\ddot{u}(x_l, t_{s+1}) = a^2 u(x_l, t_{s+1}) + f(x_l, t_{s+1}), \quad (36)$$

which leads us, that the time layer t_s to t_{s+1} is used. From here, wavelet coefficients are calculated and solution of wave equation is obtained.

5. Numerical Examples

Example 1. Consider the wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0, \quad (37)$$

with initial conditions

$$u|_{t=0} = u_0 \sin(\pi x), \frac{\partial u}{\partial t}|_{t=0} = 0, 0 < x < 1, \quad (38)$$

and boundary conditions

$$u|_{x=0} = 0, u|_{x=1} = 0, t > 0. \quad (39)$$

The exact solution of the given problem is:

$$u(x, t) = u_0 \cos(\pi c t) \sin(\pi x). \quad (40)$$

The process is started with $u(x_l, 0) = u_0 \sin(\pi x_l)$, $u(x_l, 0) = -\pi^2 u_0 \sin(\pi x_l)$, $\dot{u}(x_l, 0) = 0$, $\dot{u}(x_l) = 0$. Numerical results are presented in the Table 1 for $t = 0.2$, $u_0 = 1$, $c = 3$, $J = 4$, $\Delta t = 0.00001$. Table 2, shows the comparison of absolute errors for both the Haar wavelet solutions.

Example 2. Consider the wave equation of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + x, 0 < x < 1, t > 0, \quad (41)$$

xL/64	Exact solution	Haar solution in Present Paper	Haar solution in [5]
1	-1.516e-002	-1.515e-002	-1.513e-002
3	-4.534e-002	-4.531e-002	-4.524e-002
5	-7.509e-002	-7.503e-002	-7.491e-002
7	-1.041e-001	-1.040e-001	-1.039e-001
29	-3.057e-001	-3.055e-001	-3.050e-001
31	-3.086e-001	-3.084e-001	-3.079e-001
33	-3.086e-001	-3.084e-001	-3.079e-001
35	-3.057e-001	-3.055e-001	-3.050e-001
57	-1.041e-001	-1.040e-001	-1.039e-001
59	-7.509e-002	-7.503e-002	-7.491e-002
61	-4.534e-002	-4.531e-002	-4.524e-002
63	-1.516e-002	-1.515e-002	-1.513e-002

Table 1: Comparison of Haar wavelet solutions.

xL/64	Absolute error in Present Paper	Absolute error in Paper [5]
1	9.900e-006	3.434e-005
3	2.960e-005	1.027e-004
5	4.902e-005	1.700e-004
7	6.797e-005	2.358e-004
29	1.995e-004	6.923e-004
31	2.015e-004	6.990e-004
33	2.015e-004	6.990e-004
35	1.995e-004	6.923e-004
57	6.797e-005	2.358e-004
59	4.902e-005	1.700e-004
61	2.960e-005	1.027e-004
63	9.900e-006	3.434e-005

Table 2: Comparison of absolute error in both Haar wavelet solutions.

with initial conditions

$$u|_{t=0} = 0, \frac{\partial u}{\partial t}|_{t=0} = 0, 0 < x < 1, \quad (42)$$

and boundary conditions

$$u|_{x=0} = 0, u|_{x=1} = 0, t > 0. \quad (43)$$

Figure 1: Numerical solution of Example 1 for $J = 4$.

xL/64	Exact solution	Haar solution in Present Paper	Haar solution in [5]
1	3.293e-005	3.253e-005	3.133e-005
3	9.772e-005	9.689e-005	9.345e-005
5	1.595e-004	1.586e-004	1.539e-004
7	2.170e-004	2.163e-004	2.118e-004
29	5.505e-004	5.503e-004	5.499e-004
31	5.545e-004	5.547e-004	5.549e-004
33	5.545e-004	5.547e-004	5.549e-004
35	5.505e-004	5.503e-004	5.503e-004
57	2.170e-004	2.164e-004	2.156e-004
59	1.595e-004	1.600e-004	1.597e-004
61	9.772e-005	9.925e-005	9.923e-005
63	3.293e-005	3.418e-005	3.389e-005

Table 3: Comparison of Haar wavelet solutions.

Figure 2: Numerical solution of Example 2 for $J = 4$.

The exact solution of the this problem is:

$$u(x, t) = \frac{2}{\pi^3 c^2} \sum_{i=1} \frac{(-1)^{k-1}}{k^3} [1 - \cos(k\pi ct)] \sin(k\pi x). \quad (44)$$

The process is started with $u(x_l, 0) = 0$, $u(x_l, 0) = 0$, $\dot{u}(x_l, 0) = 0$, $\dot{u}(x_l) = 0$. Numerical results are presented in the Table 3 for $t = 0.2$, $c = 15$, $J = 4$, $\Delta t = 0.00001$. Table 4 shows the comparison of absolute errors for both the Haar wavelet solutions.

Conclusion

It is concluded that Haar wavelet method is more accurate, simple, fast and computationally attractive than other known methods to solve wave equation. The above examples, demonstrate the simplicity of the Haar wavelet solution.

xL/64	Absolute error in Present Paper	Absolute error in Paper [5]
1	4.000e-007	1.592e-006
3	8.300e-007	4.270e-006
5	8.999e-007	5.607e-006
7	6.999e-007	5.224e-006
29	2.000e-007	6.201e-007
31	2.000e-007	3.300e-007
33	2.000e-007	4.021e-007
35	2.000e-007	1.698e-007
57	5.999e-007	1.399e-006
59	5.000e-007	1.656e-007
61	1.520e-006	1.503e-006
63	1.250e-006	9.588e-007

Table 4: Comparison of absolute error in both Haar wavelet solutions.

For getting the necessary accuracy the number of calculation points may be increased. Further, it is also clear from the above examples that numerical results obtained by the present method are better than that obtained in Shi [5].

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