A NEW ALGORITHM FOR IMPLICITIZING A PARAMETRIC ALGEBRAIC SURFACE

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Abstract: Given a parametric representation of an algebraic projective surface S of the ordinary space we give a new algorithm for finding the implicit cartesian equation of S . The algorithm is based on finding a suitable finite number of points on S and computing, by linear algebra, the equation of the surface of least degree that passes through the points.

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1. Introduction

The problem of determining an implicit representation of an irreducible parametric algebraic surface in $\mathbb{P}^3$ has been classically faced by elimination theory and can be solved by the computation of Gröbner bases or by the computation of resultants (see for example [2] and [4]). These methods have been implemented in various softwares of Computer Algebra. For doing these computations on the computer the surface has to be parametrically represented by polynomials with coefficients rational numbers (or more simply integers) and the result is an equation whose coefficients are rational numbers (or integers). Another possible

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representation of the coefficients of the parametric polynomials is in the set of integers modulo a prime number. In this paper, we introduce an alternative method that recondacts the implicitation of S to the computation of the equation of the surface that contains a suitable set of points on S . The algorithm is based on the fact that, if we find an appropriate set T of general points on S , a polynomial F vanishes on the polynomials that represent parametrically S if and only if F vanishes on T . This gives rise to a homogeneous linear system of equations. A non null solution of the system (which can be found very easily by Gauss reduction) gives the coefficients of the implicit equation of S .

Throughout this paper S will be an algebraic surface in the projective space $\mathbb{P}^3$ over an algebraically closed field k .

2. Computing the Implicit Equation of a Parametric Surface by Points

Definition 2.1. We say that an irreducible (algebraic) surface $S \subset \mathbb{P}^3$ is parametric if there exists a rational map $\Phi : \mathbb{P}^2 \rightarrow \mathbb{P}^3$, $\Phi(t_0, t_1, t_2) = (f_i(t_0, t_1, t_2))$ given by homogeneous polynomials $f_i(t_0, t_1, t_2)$, $i = 0, 1, 2, 3$ of the same degree r whose image is a dense subset of V . We set

$$S = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_i = f_i(t_0, t_1, t_2), i = 0, 1, 2, 3\}.$$

Since S is a surface of $\mathbb{P}^3$ it is also the set of solutions of a homogeneous polynomial $F \in K[X_0, \dots, X_3]$ in 4 variables. The corresponding equation $F = 0$ is said to implicitize the parametric surface S and F is said to be the polynomial defining S (F is unique modulo constant terms of k).

Lemma 2.2. *The degree of F is less or equal to r^2 .*

Proof. Let F be the polynomial defining S . The degree of F is the maximum number of points of intersection with a line $ax_0 + bx_1 + cx_2 + dx_3 = a x_0 + b x_1 + c x_2 + d x_3 = 0$. If we substitute the parametric equations of S in the equations of the line we get a system of two polynomials of degree r in the variables t_0, t_1, t_2, t_3 which has, as solutions, the points of intersection of two plane curves of degree r . By Bezout theorem these points are at most r^2 , and we have the claim. $\square$

Definition 2.3. [5] A set $S = \{P_1, \dots, P_N\} \subset \mathbb{P}^2$ of $N = \binom{n+2}{2}$ points in the plane is in generic position if it is not contained in a projective curve of degree $\leq n$.

Example 2.4. Six points that do not lie on a conic and ten points that do not lie on a cubic are in generic position.

Lemma 2.5. Let $S = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_i = f_i(t_0, t_1, t_2), i = 0, 1, 2, 3\}$ be a parametric surface given by homogeneous polynomials of the same degree r . Let d be a fixed integer, and $P_1, \dots, P_N$ be $N = \binom{rd+2}{2}$ points of $\mathbb{P}^2$ in generic position such that $Q_i = \Phi(P_i)$ form a set T of N points of $\mathbb{P}^3$. If F is a homogeneous polynomial of $K[X_0, \dots, X_3]$ with degree $d \leq d$ then F vanishes on S if and only if F vanishes on T . In particular S is not contained in a surface of degree $\leq d$ if and only if such is T .

Proof. Clearly, since $T \subset S$, a polynomial vanishing on S vanishes on T . We prove the converse by contradiction. Let F be a homogeneous polynomial of degree d vanishing on the set of points T and assume that there exists a point $P \in \mathbb{P}^2$ such that $F(\Phi(P)) \neq 0$. Then $F(f_0, f_1, f_2, f_3)$ is a non-zero polynomial of degree $d \leq dr$ vanishing on the $N = \binom{rd+2}{2}$ points P_i and this contradicts the assumption of generic position. $\square$

By Lemma 1.5 the computation of the equation that defines S is reduced to the computation of the equation that vanishes on a finite set of points. Before starting the computation one has to find a set of points in generic position. Almost all sets of $N = \binom{d+m}{m}$ points $S = \{P_1, \dots, P_N\} \subset \mathbb{P}^m$ are in generic position ([3]). It follows that a random choice for the coordinates of the P_i gives points in generic position. A systematic way of finding points in generic position in $\mathbb{P}^m$ is given by the following two results.

Proposition 2.6. Let d be a positive integer and $a_{ir} \in k$, $i = 0, \dots, d$, $r = 1, 2$. Suppose $a_{i_1 r} \neq a_{i_2 r}$, for any $i_1 \neq i_2$, and $r = 1, 2$. No curve of $\mathbb{A}^2$ with degree $\leq d$ contains the $N = \binom{d+2}{2}$ affine points of the set $S = \{(a_{i_1 1}, a_{i_2 2}) \in k^2 \mid i + j \leq d\}$. Hence the set of N projective points

$$S' = \{(a_{i_1 1}, a_{i_2 2}, 1) \in \mathbb{P}^2 \mid i_1 + i_2 \leq d\}$$

is in generic position.

Proof. We have to prove that any polynomial $F(X, Y)$, of degree less than d , vanishing on all the points of S is null. By the hypotheses the polynomial $G(X) = F(X, a_{02})$, of degree less than d , has $d + 1$ roots $a_{01}, \dots, a_{0d}$ thus it is null. Hence $F(X, Y) = (X - a_{02})F_1$ and F_1 has degree less than $d - 1$. By applying the same argument to the polynomial $F_1(X, a_{12})$ and so on, we get $F(X, Y) = \prod_{n=0}^d (X - a_{n2})F$ so that F has degree greater than d or it is null. $\square$

From Proposition 1.6 we get immediately that:

Corollary 2.7. . *Let d a positive integer and $\text{char}(k) = 0$ or $\text{char}(k) \geq d$. The set of $N = \binom{d+2}{2}$ projective points, where $i = i \cdot 1_k \in k$,*

$$S' = \{i_1, i_2, 1\} \in \mathbb{P}^2 \mid i_1 + i_2 \leq d\}$$

is in generic position.

3. The Algorithm

By the results of Section 1, the problem of determining the implicit equation of a parametric surface S is reconducted to the problem of finding a polynomial $F(X, Y, Z)$ that vanishes on a suitable set of points on S . In this section we construct an algorithm for solving this problem using only linear algebra.

Let $T = \{P_1, \dots, P_s\}$ be a set of points in $\mathbb{P}^2$ and d a positive integer. An homogeneous polynomial $F(X, Y, Z)$ of degree d vanishes on T if and only if, for any $i = 1, \dots, s$, $F(P_i) = 0$. This gives a homogeneous linear system with coefficients in k and indeterminates X, Y, Z . Denoting by $\mathcal{T}_i$, $i = 1, \dots, h$ the terms of degree d in the indeterminates X, Y, Z , ordered with respect to any term ordering, the set $B = \{\mathcal{T}_i, \dots, \mathcal{T}_h\}$ is a basis of the k vector space of the homogeneous polynomials $F(X, Y, Z)$ of degree d . Consider the matrix

$$M_d(T) = (\mathcal{T}_i(P_j))$$

whose generic element $(\mathcal{T}_i(P_j))$ is the evaluation of the term $\mathcal{T}_i$ at the point P_j .

If $F = a_1\mathcal{T}_1 + \dots + a_h\mathcal{T}_h$ we set $(F)_B = (a_1, \dots, a_h)$ and denote with $(F)_B^t$ the transpose of $(F)_B$.

Hence $F(X, Y, Z)$ vanishes on T if and only if $(F)_B^t$ is a vector of the null space of the matrix $M_d(T)$ and then there is a homogeneous polynomial vanishing on T if and only if the rank $\text{rk}(M_d(T))$ of the matrix $M_d(T)$ is less than $\binom{d+2}{2}$.

All the previous results allow to formulate the following algorithm. Let S be a surface:

$$S = \{(x_0, x_1, x_2, x_3) \in \mathbb{P}^3 \mid x_i = f_i(t_0, t_1, t_2), i = 0, 1, 2, 3\}$$

ALGORITHM

INPUT: degree d and coefficients of the polynomials f_i

OUTPUT: Implicit equation $F = 0$ of S

1. Set $d = 1$.
2. Computation, in degree d , of the set T of points of Proposition 1.6
3. Computation of the matrix $G_d(T)$
- 4 If $rk(G_d(T)) < \binom{d+3}{3}$ goto 5 else set $d = d + 1$ and goto 2.
5. Compute a solution $(F)_S$ of the null space of $G_d(T)$ and stop.

The vector $(F)_S$ gives the coefficients of the implicit equation $F = 0$ of S .

Our algorithm was implemented in C++ on an Intel Pentium running Linux. We have tested the algorithm for the computation of the implicit equation of the surface of the following example. The same example was considered in [2] and [4], who compute the implicit equation with other methods.

Example 3.1. Let S be the following parametric surface of $\mathbb{P}^3$, over a field of characteristic zero.

$$S = \{(x_0, \dots, x_4) \mid x_0 = t_1 t_2^2 - t_2 t_3^2, x_1 = t_1 t_2 t_3 + t_1 t_3^2, x_2 = 2t_1 t_3^2 - 2t_2 t_3^2, x_3 = t_1 t_2^2\}$$

By applying our algorithm we find that the implicit equation of S is

$$4x_0^2 + 8x_0x_1 - 4x_0x_2 - 4x_0x_3 + 4x_1^2 - 4x_1x_2 - 8x_1x_3 + x_2^2 + 2x_2x_3$$

which agrees with the results of [2] and [4].

Remark 3.2. The problem of implicitizing a parametric curve was tackled in [1]

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