**ON THE BEST APPROXIMATION FOR
A 2-HYPERPLANE IN 2-INNER PRODUCT SPACES**

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1. Introduction and Preliminaries

Recently, some results on best approximation theory in linear 2-normed spaces have been obtained by Y. J. Cho, S. Elumalai, S. S. Kim, Sh. Rezapour and others (see [2], [4], [6], [11], [1], [13]). These papers are based on the research works in normed linear spaces made by I. Singer ([12]), S. S. Dragomir ([3]), and others. The concept which is closely related to 2-normed linear space is 2-inner product space that introduced by Diminnie, Gähler and White in 1973

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and 1977 ([8, 9]), and has been developed extensively in different subjects by others ([5]). In [1] the authors get some results on best approximation for convex cones in 2-inner product spaces. In this paper we show that the 2-norm attaining and has the z -representer are equivalent and they are equivalent with z -proximality of a 2-hyperplane.

Definition 1. Let X be a linear space of dimension greater than 1 over field $\mathbb{R}$ of real numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a $\mathbb{R}$ -value function defined on $X \times X \times X$ satisfying the following conditions:

a) $\langle x, x | z \rangle \geq 0$ and $\langle x, x | z \rangle = 0$ if and only if x and z are linearly dependent,

$$\text{b) } \langle x, x | z \rangle = \langle z, z | x \rangle,$$

$$\text{c) } \langle y, x | z \rangle = \langle x, y | z \rangle,$$

$$\text{d) } \langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle \text{ for any scalar } \alpha \in \mathbb{R},$$

$$\text{e) } \langle x + x', y | z \rangle = \langle x, y | z \rangle + \langle x', y | z \rangle,$$

then $\langle \cdot, \cdot | \cdot \rangle$ is called a *2-inner product* and $(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a *2-inner product space* (or a *2-per-Hilbert space*).

A concept which is closely related to 2-inner product space and introduced by Gähler in 1965, is 2-normed space [7].

Definition 2. Let X be a linear space of dimension greater than 1 over field $\mathbb{R}$ of real numbers. Suppose $\| \cdot, \cdot \|$ is a real-valued function on $X \times X$ satisfying the following conditions:

a) $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

c) $\|\lambda x, y\| = |\lambda| \|x, y\|$ for all $\lambda \in \mathbb{R}$ and $x, y \in X$.

d) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Then $\| \cdot, \cdot \|$ is called a *2-norm* on X and $(X, \| \cdot, \cdot \|)$ is called a *linear 2-normed space*.

It is easy to show that the 2-norm $\| \cdot, \cdot \|$ is non-negative and $\|x, y + \alpha x\| = \|x, y\|$ for all $x, y \in X$ and $\alpha \in \mathbb{R}$. Every 2-normed space is a locally convex topological vector space. In fact for a fixed $b \in X$, $p_b(x) = \|x, b\|$; $x \in X$ is a semi-norm on X and the family $P = \{p_b : b \in X\}$ of seminorms generates a locally convex topology on X .

Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space, then we can define a 2-norm on $X \times X$ by $\|x, y\| = \sqrt{\langle x, x | y \rangle}$.

A standard example of a 2-normed space is $\mathbb{R}^2$ equipped with the following 2-norm

$$\|x, y\|_s = \text{the area of the triangle having vertices } 0, x \text{ and } y.$$

Let $(X, \|\cdot, \cdot\|)$ be a 2-normed space and W_1 and W_2 be two linear subspaces of X . A 2-functional $\varphi : W_1 \times W_2 \rightarrow \mathbb{R}$ is called a *bilinear 2-functional* on $W_1 \times W_2$, whenever for all $x_1, x_2 \in W_1, y_1, y_2 \in W_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$,

i) $\varphi(x_1 + x_2, y_1 + y_2) = \varphi(x_1, y_1) + \varphi(x_1, y_2) + \varphi(x_2, y_1) + \varphi(x_2, y_2)$,
 ii) $\varphi(\lambda_1 x_1, \lambda_2 y_1) = \lambda_1 \lambda_2 \varphi(x_1, y_1)$.

A bilinear 2-functional $\varphi : W_1 \times W_2 \rightarrow \mathbb{R}$ is said to be bounded if there exists a non-negative real number M (called a Lipschitz constant for φ) such that $|\varphi(x, y)| \leq M\|x, y\|$ for all $x \in W_1$ and $y \in W_2$. Also, the norm of a bilinear 2-functional φ is defined by

$$\|\varphi\| = \inf\{M \geq 0 : M \text{ is a Lipschitz constant for } \varphi\}.$$

It is known that

$$\begin{aligned} \|\varphi\| &= \sup\{|\varphi(x, y)| : (x, y) \in W_1 \times W_2, \|x, y\| \leq 1\} \\ &= \sup\{|\varphi(x, y)| : (x, y) \in W_1 \times W_2, \|x, y\| = 1\} \\ &= \sup\{|\varphi(x, y)|/\|x, y\| : (x, y) \in W_1 \times W_2, \|x, y\| > 0\}. \end{aligned}$$

Now we give some examples of linear 2-functionals as follows.

- Let g and h be two linear functional on X , then $\varphi(x, y) = g(x)h(y)$ is a linear 2-functional on $X \times X$.

- If $(X, \langle \cdot, \cdot | \cdot \rangle)$ is an inner product space, then the function

$$\langle x, y | z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle y, z \rangle & \langle z, z \rangle \end{vmatrix} = \|z\|^2 \langle x, y \rangle - \langle x, z \rangle \langle y, z \rangle$$

for all $x, y, z \in X$ defined a 2-inner product on $X \times X \times X$, which is also generated the standard 2-norm $\|\cdot, \cdot\|_s$. For fixed elements y_1, y_2 in X , define φ on $X \times X$ by

$$\varphi(x_1, x_2) = \begin{vmatrix} \langle x_1, y_1 \rangle & \langle x_1, y_2 \rangle \\ \langle x_2, y_1 \rangle & \langle x_2, y_2 \rangle \end{vmatrix}$$

Then φ is a bounded linear 2-functional on X and $\|\varphi\| = \|y_1, y_2\|_s$.

• Consider the 2-normed space $(\mathbb{R}^2, \|\cdot, \cdot\|_s)$ with the standard basis $\{e_1, e_2\}$. Define φ by

$$\varphi(x_1, x_2) = \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = \alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12}$$

where $x_i = \sum_{j=1}^2 \alpha_{ij}e_j$, $i = 1, 2$. Then φ is a bounded linear 2-functional on X and $\|\varphi\| \leq \|x_1, x_2\|_s$.

Let φ be a nonzero linear 2-functional on $X \times V(z)$. Then we define the *2-hyperplane* H through the origin by

$$H = \{x \in X; \varphi(x, z) = 0\}.$$

Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space. For a subspace G of X let $[x, G]$ be the subspace of X generated by x and G , where $x \in X \setminus \overline{G}$. Then for $z \in X \setminus [x, G]$, an element $g_0 \in G$ is called the *z-best approximation* element of x by G (with respect to z) if

$$\|x - g_0, z\| \leq \|x - g, z\|$$

for all $g \in G$. The set of all element of z -best approximation of x by G with respect to z is denoted by $P_{G,z}(x)$, that is,

$$P_{G,z}(x) = \{g_0 \in G; \|x - g_0, z\| \leq \|x - g, z\| \text{ for all } g \in G\}.$$

If for each $x \in X$, $P_{G,z}(x)$ is a nonempty set then G is called a *z-proximinal* set and if $P_{G,z}(x)$ is exactly singleton then G is called a *z-Chebyshev* set.

Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space and G be an arbitrary nonempty subset of X and $x \in X$. Then for every $x \in X$ and for every $z \in X \setminus [G, x]$ define

$$d_z(x, G) = \inf_{g \in G} \|x - g, z\|.$$

2. Main Results

In this section at first we give a formula for distance to a 2-hyperplane and then we show that the 2-norm attaining and has the z -representer are equivalent and they are equivalent with z -proximality of a 2-hyperplane.

Theorem 3. *Let $(X, \|\cdot, \cdot\|)$ be a linear 2-normed space, H a 2-hyperplane throughout the origin. Then*

$$d_z(x, G) = \frac{1}{\|\varphi\|} |\varphi(x, z)|.$$

Proof. Fix any $x \in X$. For any $y \in H$ we have

$$\frac{1}{\|\varphi\|} |\varphi(x, z)| = \frac{1}{\|\varphi\|} |\varphi(x - y, z)| \leq \|x - y, z\|$$

implies that

$$\frac{1}{\|\varphi\|} |\varphi(x, z)| \leq d_z(x, H). \quad (1)$$

Conversely, given any ε with $0 < \varepsilon < \|\varphi\|$, choose $y_0 \in X$ with $\|y_0, z\| = 1$ and $\varphi(y_0, z) > \|\varphi\| - \varepsilon$. Then $y = x - \frac{\varphi(x, z)}{\varphi(y_0, z)} y_0$ is in H and

$$\|x - y, z\| = \frac{|\varphi(x, z)|}{|\varphi(y_0, z)|} \leq \frac{|\varphi(x, z)|}{\|\varphi\| - \varepsilon}$$

implies that

$$d_z(x, H) \leq \frac{|\varphi(x, z)|}{\|\varphi\|}. \quad (2)$$

From (1) and (2) we get our consequence. $\square$

Corollary 4. *If $H = \{x \in X; \varphi(x, z) = c\}$ where $c \in \mathbb{R}$. Then we have*

$$d_z(x, G) = \frac{1}{\|\varphi\|} |\varphi(x, z) - c|.$$

Proof. Choose any $x_0 \in H$ and define

$$H_0 := H - x_0 = \{y \in X; \varphi(y, z) = 0\}.$$

Now by above theorem we have:

$$\begin{aligned} d_z(x, H) &= d_z(x - x_0, H - x_0) = d_z(x - x_0, H_0) \\ &= \frac{1}{\|\varphi\|} |\varphi(x - x_0, z)| = \frac{1}{\|\varphi\|} |\varphi(x, z) - c|. \end{aligned}$$

$\square$

Definition 5. A linear 2-functional φ is said to *attain its 2-norm* if there exists $x \in X$ with $\|x, z\| = 1$ and $\varphi(x, z) = \|\varphi\|$.

Definition 6. An element $x \in X$ is called a z -representer for a 2-functional φ if $\varphi(y, z) = \langle y, x|z \rangle$ for each $y \in X$.

Theorem 7. Let φ be a linear 2-functional. Then φ has a z -representer if and only if φ attains its 2-norm.

Proof. If $\varphi = 0$, then $x = 0$ is the z -representer for φ , and φ attains its 2-norm at any $y \in X$ with $\|y, z\| = 1$. Thus we assume $\varphi \neq 0$. By replacing φ with $\frac{\varphi}{\|\varphi\|}$, we may assume that $\|\varphi\| = 1$. Let φ have the z -representer x . Then $\|x, z\| = 1$ and $\varphi(x, z) = \langle x, x|z \rangle = \|x, z\| = 1 = \|\varphi\|$. Thus φ attains its 2-norm at x .

Conversely, suppose φ attains its 2-norm at x_1 , and let $M = \ker \varphi = \{x \in X | \varphi(x, z) = 0\}$. If $y_0 = x - \frac{\varphi(x, z)}{\|\varphi\|}x_1$, Since

$$\|x - y_0, z\| = \left\| \frac{\varphi(x, z)}{\|\varphi\|}x_1, z \right\| = \frac{|\varphi(x, z)|}{\|\varphi\|} \|x_1, z\| = \frac{|\varphi(x, z)|}{\|\varphi\|} = d_z(x, M)$$

so $y_0 = P_{M, z}(x)$ and $x_1 \in M^{\perp z}$, where $M^{\perp z} = \{y \in X | \langle y, x|z \rangle = 0 ; \forall x \in M\}$. Then we have

$$\begin{aligned} 0 &= \langle y_0, x_1|z \rangle = \left\langle x - \frac{\varphi(x, z)}{\|\varphi\|}x_1, x_1|z \right\rangle \\ &= \langle x, x_1|z \rangle - \frac{\varphi(x, z)}{\|\varphi\|} \langle x_1, x_1|z \rangle \\ &= \langle x, x_1|z \rangle - \frac{\varphi(x, z)}{\|\varphi\|} \end{aligned}$$

therefore

$$\varphi(x, z) = \langle x, x_1|z \rangle$$

and this show that φ has a z -representer. □

Theorem 8. Let X be a 2-inner product space, φ a non-zero linear 2-functional, $c \in \mathbb{R}$ and $H = \{y \in X | \varphi(y, z) = c\}$. Then the following statements are equivalent.

- (1) H is z -Chebyshev.
- (2) H is z -proximal.
- (3) Some $x \in X \setminus [H, z]$ has a z -best approximation in H .
- (4) φ attains its 2-norm.

(5) φ has a z -representer in X .

Moreover, if H is z -Chebyshev, then

$$P_{H,z}(x) = x - \frac{(\langle x, y|z \rangle - c)}{\|y, z\|^2} y$$

for every $x \in X$, where y is the z -representer for φ .

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) If some $x \in X \setminus [H, z]$ has a z -best approximation $y_0 \in H$, then by corollary 4,

$$\|x - y_0, z\| = \frac{1}{\|\varphi\|} |\varphi(x, z) - c| = \frac{1}{\|\varphi\|} |\varphi(x - y_0, z)|$$

setting

$$y = \frac{|\varphi(x - y_0, z)|}{\varphi(x - y_0, z) \|x - y_0, z\|} (x - y_0)$$

we see that $\|y, z\| = 1$ and $\varphi(y, z) = \|\varphi\|$. Thus φ attains its 2-norm at y .

(4) $\Rightarrow$ (5) It is consequence of theorem 7.

(5) $\Rightarrow$ (1) Let $y \in X$ be the z -representer of φ . Then $\varphi(x, z) = \langle x, y|z \rangle$, for all $x \in X$ and $\|y, z\| = \|\varphi\|$. Set

$$w = x - \frac{(\langle x, y|z \rangle - c)}{\|y, z\|^2} y = x - \frac{\varphi(x, z) - c}{\|\varphi\|^2} y.$$

Then $\varphi(w, z) = c$, so $w \in H$, and

$$\|x - w, z\| = \frac{|\varphi(x, z) - c|}{\|\varphi\|^2} \|y, z\| = \frac{|\varphi(x, z) - c|}{\|\varphi\|} = d_z(x, H).$$

Thus $w = P_{H,z}(x)$. □

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