ON THE SOLUTIONS OF NON-LINEAR TIME-FRACTIONAL GAS DYNAMIC EQUATIONS: AN ANALYTICAL APPROACH

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Abstract: We consider non-linear homogeneous and non-homogeneous gas dynamic equations of time-fractional type in this paper. The approximate solutions of these equations are calculated in the form of series obtained by q-Homotopy Analysis Method (q-HAM). Exact solution is obtained for time-fractional homogeneous case while for the case of time-fractional non-homogeneous, exact solution is possible for special case. This is due to the ability to control the auxiliary parameter h and the fraction factor present in this method. The presence of fraction-factor in this method gives it an edge over other existing analytical methods for non-linear differential equations. Comparisons are made with several other analytical methods.

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1. Introduction and Preliminaries

Calculus of non-integer order is increasingly being used to model physical systems. Caputo [2] for example used the modified form of the Darcys law to

incorporates the memory term in order to model transport through porous media. Other applications are: control theory of dynamical systems, electrical networks, ground water flow, astrophysics, meteorology, reactive flows and semiconductors see [12] and also [5], [7] for some detailed work on fractional differential equations.

Gas dynamics equations are mathematical expressions that are based on the physical laws of conservation such as the conservation of momentum, laws of conservation of mass, conservation of energy etc. The non-linear equations of ideal gas dynamics are applicable for three types of non-linear waves like refractions, shock fronts and contact discontinuities see [16].

Generally, in such related models, one has to solve a fractional PDE. Analytical methods commonly used to obtain solutions of these kind of non-linear equations have very restricted applications and the numerical techniques give rise to rounding of errors. Therefore, many analytical methods have been put to use successfully to obtain solutions of classical gas dynamics equations such as Adomian Decomposition Method (ADM) [15], Variational Iteration Method (VIM) [13], Homotopy Perturbation Method (HPM) [1], [15] and the time-fractional type was considered using differential transform method (DTM) in [3]. Recently, a modified HAM called q-Homotopy Analysis Method was introduced in [4], see also [6], [8], [9], [10]. It was proven that the presence of fraction factor in this method enables a fast convergence compared to the usual HAM. This makes the method more reliable.

In this paper, we apply the q-HAM to initial value problems of the time-fractional homogeneous and non-homogeneous gas dynamics equations. Our aim is to exploit the simple, natural and efficient nature of the so called relatively new analytical method (q-HAM) to obtain analytical solutions of the equations considered and exact solution where possible. Finally, we compare the applicability and performance of this method with the exact solutions for some special cases and the solutions obtained by other existing methods in the literature. Numerical results are obtained using Mathematica 9 and MATLAB R2012b.

Caputo's fractional derivative is adopted in this work.

Definition 1.1. The Riemann-Liouville's (RL) fractional integral operator of order $\alpha \geq 0$, of a function $f \in L^1(a, b)$ is given as [14]

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \quad \alpha > 0, \quad (1)$$

where Γ is the Gamma function and $I^0 f(t) = f(t)$.

Definition 1.2. The fractional derivative in the Caputo's sense is defined as [14],

$$\mathcal{D}^\alpha f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (2)$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $t > 0$.

Lemma 1.1. [14] Let $t \in (a, b]$. Then

$$\left[I^\alpha (t-a)^\beta \right] (t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} (t-a)^{\beta+\alpha}, \quad \alpha \geq 0, \quad \beta > 0. \quad (3)$$

Definition 1.3. [14] The generalized Mittag-Leffler function in two parameters α and β is given as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z, \beta \in \mathbb{C} \quad \text{and} \quad \operatorname{Re}(\alpha) > 0. \quad (4)$$

2. q-Homotopy Analysis Method (q-HAM)

Differential equation of the form

$$N[\mathcal{D}_t^\alpha u(x, t)] - f(x, t) = 0 \quad (5)$$

is considered, where N is a non-linear operator, $\mathcal{D}_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ denote the Caputo fractional derivative, (x, t) are independent variables, f is a known function and u is an unknown function. To generalize the original homotopy method, the zeroth-order deformation equation is constructed as

$$(1-nq)L(\phi(x, t; q) - u_0(x, t)) = qhH(x, t)(N[\mathcal{D}_t^\alpha \phi(x, t; q)] - f(x, t)), \quad (6)$$

where $n \geq 1$, $q \in [0, \frac{1}{n}]$ denotes the so-called embedded parameter, L is an auxiliary linear operator, $h \neq 0$ is an auxiliary parameter, $H(x, t)$ is a non-zero auxiliary function.

It is clearly seen that when $q = 0$ and $q = \frac{1}{n}$, equation (6) becomes

$$\phi(x, t; 0) = u_0(x, t) \quad \text{and} \quad \phi\left(x, t; \frac{1}{n}\right) = u(x, t) \quad (7)$$

respectively. So, as q increases from 0 to $\frac{1}{n}$, the solution $\phi(x, t; q)$ varies from the initial guess $u_0(x, t)$ to the solution $u(x, t)$.

If $u_0(x, t)$, L , h , $H(x, t)$ are chosen appropriately, solution $\phi(x, t; q)$ of equation (6) exists for $q \in [0, \frac{1}{n}]$.

Expansion of $\phi(x, t; q)$ in Taylor series gives

$$\phi(x, t; r) = u_0(x, t) + \sum_{m=1} u_m(x, t) q^m. \quad (8)$$

where

$$u_m(x, t) = \frac{1}{m!} \left. \frac{\partial^m \phi(x, t; q)}{\partial q^m} \right|_{q=0}. \quad (9)$$

Assume that the auxiliary linear operator L , the initial guess u_0 , the auxiliary parameter h and $H(x, t)$ are properly chosen such that the series 8 converges at $q = \frac{1}{n}$, then we have

$$u(x, t) = u_0(x, t) + \sum_{m=1} u_m(x, t) \left(\frac{1}{n} \right)^m. \quad (10)$$

Let the vector u_n be define as follows:

$$\vec{u}_n = \{u_0(x, t), u_1(x, t), \dots, u_n(x, t)\}. \quad (11)$$

Differentiating equation (6) m -times with respect to the (embedding) parameter q , then evaluating at $q = 0$ and finally dividing them by $m!$, we have what is known as the m^{th} -order deformation equation ([11]) as

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = hH(x, t) \mathcal{R}_m(\vec{u}_{m-1}). \quad (12)$$

with initial conditions

$$u_m^{(k)}(x, 0) = 0, \quad k = 0, 1, 2, \dots, m-1. \quad (13)$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} (N[\mathcal{D}_t^\alpha \phi(x, t; q)] - f(x, t))}{\partial q^{m-1}} \right|_{q=0} \quad (14)$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ n & otherwise, \end{cases} \quad (15)$$

3. The Time-Fractional Homogeneous Gas Dynamic Equation

We consider the time fractional homogeneous gas dynamic equation. Let

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{1}{2} \frac{\partial(u^2)}{\partial x} - u(1-u) = 0, \quad 0 \leq x \leq 1, \quad t > 0, \quad 0 < \alpha \leq 1 \quad (16)$$

subjected to the initial condition

$$u(x, 0) = ae^{-x}. \quad (17)$$

The exact solution to this problem, when $a = 1$ and $\alpha = 1$, is

$$u(x, t) = e^{t-x}. \quad (18)$$

Many authors have worked on this problem when $\alpha = 1$, using various methods see [1], [3], [13], [15].

3.1. Application of q-HAM

In order to use q-HAM to solve the problem considered in (16), we choose the linear operator

$$L[\phi(x, t; q)] = \mathcal{D}_t^\alpha \phi(x, t; q) \quad (19)$$

with property that $L[c_1] = 0$, c_1 is constant.

We use initial approximation $u_0(x, t) = ae^{-x}$. We can then define the non-linear operator as

$$N[\phi(x, t; q)] = \mathcal{D}_t^\alpha \phi(x, t; q) + \phi(x, t; q)\phi_x(x, t; q) - \phi(x, t; q) + (\phi(x, t; q))^2. \quad (20)$$

We construct the zeroth order deformation equation

$$(1 - nq)L[\phi(x, t; q) - u_0(x, t)] = qhH(x, t)N[\mathcal{D}_t^\alpha \phi(x, t; q)]. \quad (21)$$

We choose $H(x, t) = 1$ to obtain the mth-order deformation equation to be

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h\mathcal{R}_m(\vec{u}_{m-1}), \quad (22)$$

with initial condition for $m \geq 1$, $u_m(x, 0) = 0$, χ_m is as defined in (15) and

$$\mathcal{R}_m(\vec{u}_{m-1}) = \mathcal{D}_t^\alpha u_{m-1} + \sum_{k=0}^{m-1} u_k (u_{m-1-k})_x - u_{m-1} + \sum_{k=0}^{m-1} u_k u_{m-1-k}. \quad (23)$$

So, the solution to the equation (16) for $m \geq 1$ becomes

$$u_m(x, t) = \chi_m u_{m-1} + h I_t^\alpha [\mathcal{R}_m(\vec{u}_{m-1})]. \quad (24)$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned} u_1(x, t) &= \chi_1 u_0 + h I_t^\alpha [\mathcal{D}_t^\alpha u_0 + u_0(u_0)_x - u_0 + u_0^2] \\ &= -a h e^{-x} \frac{t^\alpha}{\Gamma(1 + \alpha)} \end{aligned} \quad (25)$$

$$\begin{aligned} u_2(x, t) &= \chi_2 u_1 + h I_t^\alpha [\mathcal{D}_t^\alpha u_1 + u_0(u_1)_x + u_1(u_0)_x - u_1 + 2u_0 u_1] \\ &= (n + h)u_1 + a h^2 e^{-x} \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \\ u_3(x, t) &= \chi_3 u_2 + h I_t^\alpha [\mathcal{D}_t^\alpha u_2 + u_0(u_2)_x + u_1(u_1)_x \\ &\quad + u_2(u_0)_x - u_2 + 2u_0 u_2 + u_1^2] \\ &= (n + h)u_2 + a(n + h)h^2 e^{-x} \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} \\ &\quad - a h^3 e^{-x} \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)}. \end{aligned} \quad (26)$$

In the same way, $u_m(x, t)$ for $m = 4, 5, \dots$ can be obtained Mathematica.

Then the series solution expression by q-HAM can be written in the form

$$\begin{aligned} u(x, t; n; h) &= a e^{-x} + \sum_{i=1} u_i(x, t; n; h) \left(\frac{1}{n}\right)^i \\ &= a e^{-x} - a h e^{-x} \frac{t^\alpha}{n \Gamma(1 + \alpha)} + \frac{(n + h)u_1}{n} + \frac{(n + h)u_2}{n^2} \\ &\quad + a h^2 e^{-x} \frac{t^{2\alpha}}{n^2 \Gamma(1 + 2\alpha)} + a(n + h)h^2 e^{-x} \frac{t^{2\alpha}}{n^3 \Gamma(1 + 2\alpha)} \\ &\quad - a h^3 e^{-x} \frac{t^{3\alpha}}{n^3 \Gamma(1 + 3\alpha)} + \dots \end{aligned} \quad (27)$$

Equation (27) is an appropriate solution to the problem (16) in terms of convergence parameter h and n .

Remark 3.1. When $n = 1$, we choose appropriate $h = -1$ to obtain

$$u(x, t) = a e^{-x} + a e^{-x} \frac{t^\alpha}{\Gamma(1 + \alpha)} + a e^{-x} \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + a e^{-x} \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)} + \dots$$

$$\begin{aligned}
&= ae^{-x} \left[1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \right] \\
&= ae^{-x} E_\alpha(t^\alpha).
\end{aligned} \tag{28}$$

Hence, we obtain exact solution of the time-fractional homogeneous gas dynamic equation (16). For $\alpha = 1$, we equally get the exact solution of the classical homogeneous gas dynamic equation which

$$u(x, t) = ae^{t-x}. \tag{29}$$

Remark 3.2. Observe that infinite sum is required in (27) to obtain the exact solution with $h = -1$ but Figure 1 displays a good approximation by taking $h = -1.2$ with just three terms of the series solution obtained in (27). Figures 2 and 3 give the effect of both h and n on the solution given by q-HAM with just few terms as well.

Figure 1: q-HAM solution plot of u for $h = -1.2$, $n = 1$, $a = 1$ and $\alpha = 1$ against exact solution

4. The Time-Fractional Non-Homogeneous Gas Dynamic Equation

We consider the time fractional non-homogeneous gas dynamics equation. Let

$$\frac{\partial^\beta u}{\partial t^\beta} + \frac{1}{2} \frac{\partial(u^2)}{\partial x} - u(1-u) = -e^{t-x}, \quad 0 \leq x \leq 1, \quad t > 0, \quad 0 < \beta \leq 1 \tag{30}$$

subjected to the initial condition

$$u(x, 0) = 1 - e^{-x}. \tag{31}$$

Figure 2: q-HAM solution plot of u for $h = -0.5$, $n = 1$, $a = 1$ and $\alpha = 1$ against exact solution

Figure 3: q-HAM solution plot of u for $h = -0.5$, $n = 3$, $a = 1$ and $\alpha = 1$ against exact solution

The exact solution to this problem, when $\beta = 1$, is

$$u(x, t) = 1 - e^{t-x}. \quad (32)$$

Many authors have worked on this problem when $\beta = 1$, using various methods see [1], [3], [13], [15].

4.1. Application of q-HAM

We follow the same procedure as in first case using the initial approximation to be $u_0(x, t) = 1 - e^{-x}$.

We construct the zeroth order deformation equation

$$(1 - nq)L[\phi(x, t; q) - u_0(x, t)] = qhH(x, t) \left(N \left[\mathcal{D}_t^\beta \phi(x, t; q) \right] + e^{t-x} \right). \quad (33)$$

We choose $H(x, t) = 1$ to obtain the m th-order deformation equation to be

$$L[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h\mathcal{R}_m(\vec{u}_{m-1}), \quad (34)$$

with initial condition for $m \geq 1$, $u_m(x, 0) = 0$, χ_m is as defined in (15) and

$$\mathcal{R}_m(\vec{u}_{m-1}) = \mathcal{D}_t^\beta u_{m-1} + \sum_{k=0}^{m-1} u_k (u_{m-1-k})_x - u_{m-1} + \sum_{k=0}^{m-1} u_k u_{m-1-k} + e^{t-x}. \quad (35)$$

So, the solution to the equation (30) for $m \geq 1$ becomes

$$u_m(x, t) = \chi_m u_{m-1} + hI_t^\beta [\mathcal{R}_m(\vec{u}_{m-1})]. \quad (36)$$

We therefore obtain components of the solution using q-HAM successively as follows

$$\begin{aligned} u_1(x, t) &= \chi_1 u_0 + hI_t^\beta \left[\mathcal{D}_t^\beta u_0 + u_0(u_0)_x - u_0 + u_0^2 + e^{t-x} \right] \\ &= he^{-x} t^\beta E_{1, \beta+1}(t) \end{aligned} \quad (37)$$

$$\begin{aligned} u_2(x, t) &= \chi_2 u_1 + hI_t^\beta \left[\mathcal{D}_t^\beta u_1 + u_0(u_1)_x + u_1(u_0)_x - u_1 + 2u_0 u_1 + e^{t-x} \right] \\ &= (n+h)u_1 + he^{-x} t^\beta E_{1, \beta+1}(t) \end{aligned} \quad (38)$$

$$\begin{aligned} u_3(x, t) &= \chi_3 u_2 + hI_t^\beta \left[\mathcal{D}_t^\beta u_2 + u_0(u_2)_x + u_1(u_1)_x + u_2(u_0)_x \right. \\ &\quad \left. - u_2 + 2u_0 u_2 + u_1^2 + e^{t-x} \right] \\ &= (n+h)u_2 + he^{-x} t^\beta E_{1, \beta+1}(t). \end{aligned} \quad (39)$$

In the same way, $u_m(x, t)$ for $m = 4, 5, \dots$ can be obtained using Mathematica 9.

Then the series solution expression by q-HAM can be written in the form

$$\begin{aligned} u(x, t; n; h) &= 1 - e^{-x} + \sum_{i=1} u_i(x, t; n; h) \left(\frac{1}{n} \right)^i \\ &= 1 - e^{-x} + \frac{he^{-x} t^\beta E_{1, \beta+1}(t)}{n} + \frac{(n+h)u_1}{n^2} + \frac{he^{-x} t^\beta E_{1, \beta+1}(t)}{n^2} \\ &\quad + \frac{(n+h)u_2}{n^3} + \frac{he^{-x} t^\beta E_{1, \beta+1}(t)}{n^3} + \dots \end{aligned} \quad (40)$$

Equation (40) is an appropriate solution to the problem (30) in terms of convergence parameter h and n .

Remark 4.1. Using the first two terms of the q-HAM series in (40), when $n = 1$ and $\alpha = 1$, we choose appropriate $h = -1$, we obtain

$$\begin{aligned} u(x, t) &= 1 - e^{-x} + te^{-x} E_{1,2}(t) \\ &= 1 - e^{-x} - e^{t-x} + e^{-x} \\ &= 1 - e^{t-x}. \end{aligned} \tag{41}$$

Hence, we obtain exact solution to the non-homogeneous gas dynamic equation (30) given by just two terms of the series.

Remark 4.2. Similar numerical comparisons can also be made with the exact solution as in the homogeneous case.

5. Conclusion

The major achievement of this paper is the demonstration of the successful application of the q-HAM to obtain analytical solutions of the non-linear gas dynamic equations of time-fractional type. Exact solution is obtained in the case of time-fractional homogeneous gas dynamic equation just by choosing an appropriate auxiliary parameter h . This choice of h is generally determined by what is called h -curve. Our results confirm that the method of solution used is really effective for handling solutions of a class of non-linear Partial Differential Equations of fractional order system. Considering the results obtained by other analytical methods such as HPM, VIM, ADM, DTM etc., the accuracy of Homotopy Analytical Method is clearly seeing in the sense that just two terms are needed in the case of the non-homogeneous gas dynamic equation unlike other methods.

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