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φ-2-ABSORBING IDEALS

A. Khaksari
Department of Mathematics
Payame Noor University
P.O. Box: 19395-3697, Tehran, IRAN

Abstract: Let R be a commutative ring with identity. 2-absorbing ideals have been studied by A. Badawi. A proper ideal I of R is 2-absorbing if $a, b, c \in R$ with $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. Let $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function where $I(R)$ is the set of ideals of R. We call a proper ideal I of R a φ-2-absorbing ideal if $a, b, c \in R$ with $abc \in I - \varphi(I)$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. So taking $\varphi_\emptyset(J) = \emptyset$ (resp., $\varphi_0(J) = 0, \varphi_2(J) = J^2$), a $\varphi_\emptyset$-2-absorbing ideal (resp., φ_0-2-absorbing ideal, φ_2- 2-absorbing ideal) is a 2-absorbing ideal (resp., weakly 2-absorbing ideal, almost 2-absorbing ideal). We show that φ-2-absorbing ideals enjoy analogs of many of the properties of 2-absorbing ideals.

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Throughout, R will be a commutative ring with identity. We denote the set of ideals of R by $I(R)$. By a proper ideal I of R we mean an ideal $I \in I(R)$ with $I \neq R$. We denote the set of proper ideals of R by $I^*(R)$.

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Badawi (2007) recently defined a proper ideal I of R to be 2-absorbing if for $a, b, c \in R$ with $abc \in I$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. With a 2-absorbing ideal, we make the following definitions. Let R be a commutative ring and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. We call a proper ideal I of R φ-2-absorbing if for $a, b, c \in R$, $abc \in I - \varphi(I)$ implies $ab \in I$ or $ac \in I$ or $bc \in I$. There is no loss of generality in assuming that $\varphi(I) \subseteq I$, because $I - \varphi(I) = I - (I \cap \varphi(I))$. We henceforth make this assumption. Given two functions $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in I(R)$.

Definition 1. Let $I(R)$ be the set of ideals of R, $I^*(R)$ the set of proper ideals of R and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a map.A *non*_zero proper ideal I of R is called a φ- 2-absorbing ideal if for all $a, b, c \in R$,$abc \in I-\ \varphi(I)$ implies $ab \in I$ or $ac \in I$ or $bc \in I$.

Definition 2.(1) A *non*_zero proper ideal I of R is called a φ-prime ideal if for all $a, b \in R$,$ab \in I-\ \varphi(I)$ implies $a \in I$ or $b \in I$.

Lemma 3. *(i) Every 2-absorbing ideal is φ- 2-absorbing ideal.*

(ii) Every φ-prime ideal is a φ-2-absorbing ideal.

Definition 4. Given two functions $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in I(R)$.

We maintain notation and terminology used in following example for the remainder of the article.

Definition 5. A *non*_zero proper ideal I of R is called a weakly 2-absorbing ideal if for all $a, b \in R$,$ab \in I - \{\ 0\}$ implies $a \in I$ or $b \in I$.

Definition 6. A *non*_zero proper ideal I of R is called an almost 2-absorbing ideal if for all $a, b \in R$,$ab \in I - \{\ I^2\}$ implies $a \in I$ or $b \in I$.

Example. Let R be a commutative ring. Define the following functions $\varphi_\alpha : I(R) \to I(R) \cup \{\emptyset\}$ and the corresponding $\varphi_\alpha-$2-absorbing ideals:

$\varphi_\emptyset$	$\varphi(J) = \emptyset$	2-absorbing ideal
φ_0	$\varphi(J) = 0$	weakly 2-absorbing ideal
φ_2	$\varphi(J) = J^2$	almost 2-absorbing ideal
φ_n	$\varphi(J) = J^n$	n-almost 2-absorbing ideal
φ_ω	$\varphi(J) = \cap J^n$	ω-2-absorbing ideal
φ_1	$\varphi(J) = J$	any ideal.

Observe that $\varphi_\emptyset \le \varphi_0 \le \varphi_\omega \le \cdots \le \varphi_{n+1} \le \varphi_n \le \cdots \le \varphi_2 \le \varphi_1$. Let A be an ideal of R. Define the function φ_A by $\varphi_A(J) = AJ$. So if (R, m) is quasilocal, $\varphi_2 \le \varphi_m \le \varphi_1$.

Example. Let (R, m) be a quasilocal ring and let $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function with $Im\varphi \subseteq I(R)$. If $m^3 = 0$, then every proper ideal of R is weakly 2-absorbing and hence φ-2-absorbing. More generally, if I is a proper ideal of R with $I \cap m^3 \subseteq \varphi(I)$, then I is φ-2-absorbing. For if $xyz \in I - \varphi(I)$, then $xyz \notin m^3$. So x or y or z is a unit and hence $xy \in I$ or $xz \in I$ or $yz \in I$. As an example, let $I = (\overline{x})$ in $R = k[[x, y]]/(x)(x, y), k$ a field. Then $I \cap m^3 = 0$ for $m = (\overline{x, y})$, so I is weakly 2-absorbing.

Proposition 7. *(1) Let R be a commutative ring and J a proper ideal of R. Let $\psi_1, \psi_2 : I(R) \to I(R) \cup \{\emptyset\}$ be functions with $\psi_1 \le \psi_2$. Then, if J is a ψ_1-2-absorbing ideal, then J is a ψ_2-2-absorbing ideal.*

(2) J 2-absorbing $\Longrightarrow J$ weakly 2-absorbing $\Longrightarrow$ J ω-2-absorbing $\Longrightarrow$ J (n+1)-almost 2-absorbing $\Longrightarrow$ J n-almost 2-absorbing ($n \ge 2$) $\Longrightarrow$ J almost 2-absorbing.

(3) J is ω-2-absorbing if only if J is n-almost 2-absorbing for all $n \ge 2$.

Proof. (1) Suppose that $abc \in J - \psi_2(J)$, where, $a, b, c \in R$.Since $\psi_1(J) \subseteq \psi_2(J)$, we have $abc \in J - \psi_1(J)$ and therefore $ab \in J$ or $ac \in J$ or $bc \in J$. Thus J is ψ_2-2-absorbing. (2) This follows from (1) and the ordering of the ϕ_α's given in Example. (3) Clear by (2).

Theorem 8. *Let R be a commutative ring with identity and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function, $J \in I^*(R)$ such that J is φ- 2-absorbing that is not 2-absorbing. Then $J^3 \subseteq \varphi(J)$.*

Proof. Suppose that $J^3 \not\subseteq \varphi(J)$, We show that J is 2-absorbing. Let $a, b, c \in R$ with $abc \in J$. If $abc \notin \varphi(J)$, then J, φ- 2-absorbing gives $ab \in J$ or $ac \in J$ or $bc \in J$. So assume that $abc \in \varphi(J)$. First, suppose that $abJ \not\subseteq \varphi(J)$; say $abj_0 \notin \varphi(J)$ where $J_0 \in J$. Then $ab(c + j_0) \in J - \varphi(J)$. So $ab \in J$ or $a(c + j_0) \in J$ or $b(c + j_0) \in J$; and hence $ab \in J$ or $ac \in J$ or $bc \in J$. So we can assume that $abJ \subseteq \varphi(J)$. Likewise, we can assume that $acJ \subseteq \varphi(J)$, $bcJ \subseteq \varphi(J)$. Since $J^3 \not\subseteq \varphi(J)$, there exist $j_1, j_2, j_3 \in J$ with $j_1j_2j_3 \notin \varphi(J)$. Then $(a+j_1)(b+j_2)(c+j_3) \in J-\varphi(J)$. So J φ- 2-absorbing gives $(a+j_1)(b+j_2) \in J$ or $(a+j_1)(c+j_3) \in J$ or $(b+j_2)(c+j_3) \in J$; hence $ab \in J$ or $ac \in J$ or $bc \in J$. So J is 2-absorbing.

Corollary 9. *Let J be φ- 2-absorbing and $J^3 \nsubseteq \varphi(J)$. Then J is a 2-absorbing ideal.*

Corollary 10. *Let J be a φ- 2-absorbing ideal, where $\varphi \leq \varphi_4$. Then J is ω- 2-absorbing.*

Proof. If J is 2-absorbing, then J is φ- 2-absorbing for each φ. Thus J is ω-2-absorbing. Suppose that J is not 2-absorbing. By Theorem 5, $J^3 \subseteq \varphi(J) \subseteq J^4$. Hence $\varphi(J) = J^n$ for each $n \geq 3$, so J is n-almost 2-absorbing for each $n \geq 3$ and thus w- 2-absorbing.

In following we give a way to construct φ-2-absorbing ideals J where, $\varphi_\omega \leq \varphi$.

Remark. If I is a weakly 2-absorbing ideal of T, then $J = I \times S$ need not be a weakly 2-absorbing ideal of $R = T \times S$, where T and S are commutative rings. Indeed, J is weakly 2-absorbing if and only if J is actually 2-absorbing. However, J is φ- 2-absorbing for each φ with $\varphi_\omega \leq \varphi$. If I is actually 2-absorbing, then J is 2-absorbing and hence is φ- 2-absorbing for all φ.

Theorem 11. *Suppose that T and S are commutative rings with identity,I an ideal of T and J an ideal of S. Then:*

(1) I is a 2-absorbing ideal of T if and only if $I \times S$ is a 2-absorbing ideal of $T \times S$.

(2) J is a 2-absorbing ideal of S if and only if $T \times J$ is a 2-absorbing ideal of $T \times S$.

Proof. Because the proof of (1) and (2) are similar, we only prove (1). Hence suppose that I is a 2-absorbing ideal of T and $(a,b)(c,d)(e,f) \in I \times S$. Thus $ace \in I$, and hence $ac \in I$ or $ae \in I$ or $ce \in I$. Therefore $(a,b)(c,d) \in I \times S$ or $(a,b)(e,f) \in I \times S$ or $(c,d)(e,f) \in I \times S$. Conversely, suppose that $abc \in I$. Hence $(abc,1) \in I \times S$ and hence $(a,1)(b,1)(c,1) \in I \times S$. Since $I \times S$ is 2-absorbing $(a,1)(b,1) \in I \times S$ or $(a,1)(c,1) \in I \times S$ or $(b,1)(c,1) \in I \times S$, and hence $ab \in I$ or $ac \in I$or $bc \in I$. Hence I is a 2-absorbing ideal.

Theorem 12. *Let R and S be commutative rings and let $\psi_1 : I(R) \to I(R) \cup \{\emptyset\}, \psi_2 : I(S) \to I(S) \cup \{\emptyset\}$ be functions. Let $\varphi = \psi_1 \times \psi_2$. Then the following are φ- 2-absorbing ideals of $R \times S$. Also if $I \times J$ is φ-2-absorbing, then I is ψ_1- 2-absorbing and J is ψ_2-2-absorbing.*

(1) $I \times J$, where I is a proper ideal of R and J is a proper ideal of S with $\psi_1(I) = I$ and $\psi_2(J) = J$.

(2) $I \times S$, where I is ψ_1- 2-absorbing in R which must be 2-absorbing if $\psi_2(S) \neq S$.

(3) $R \times J$, where J is ψ_2- 2-absorbing in S which must be 2-absorbing if $\psi_1(R) \neq R$.

Proof. Case(1) is clear since $I \times J - \varphi(I \times J) = I \times J - \psi_1(I) \times \psi_2(J) = I \times J - I \times J = \emptyset$. If I is 2-absorbing, certainly, $I \times S$ is 2-absorbing and hence φ- 2-absorbing. So suppose that I is ψ_1- 2-absorbing and $\psi_2(S) = S$. Suppose that $(a_1a_2a_3, b_1b_2b_3) = (a_1, b_1)(a_2, b_2)(a_3, b_3) \in I \times S - \psi_1(I) \times \psi_2(S) = I \times S - \psi_1(I) \times S = (I - \psi_1(I)) \times S$. Then $a_1a_2a_3 \in I - \psi_1(I) \Rightarrow a_1a_2 \in I$ or $a_1a_3 \in I$ or $a_2a_3 \in I$, so $(a_1a_2, b_1b_2) \in I \times S$ or $(a_1a_3, b_1b_3) \in I \times S$ or $(a_2a_3, b_2b_3) \in I \times S$. So $I \times S$ is φ- 2-absorbing. The proof for case (3) is similar. Next suppose that $I \times J$ is φ- 2-absorbing. Let $abc \in I - \psi_1(I)$. Then $(a, 0)(b, 0)(c, 0) = (abc, 0) \in I \times J - \varphi(I \times J)$, so $(ab, 0) \in I \times J$ or $(ac, 0) \in I \times J$ or $(bc, 0) \in I \times J$, *i.e.*, $ab \in I$ or $ac \in I$ or $bc \in I$. So I is ψ_1- 2-absorbing. Likewise, J is ψ_2- 2-absorbing.

Theorem 13. *Suppose that R and S are commutative rings, and I is an ideal in R. Then $I \times S$ is a weakly 2-absorbing ideal of $R \times S$ if and only if I is a 2-absorbing ideal in R.*

Proof. Suppose that I is a 2-absorbing ideal of R, then $I \times S$ is a 2-absorbing ideal in $R \times S$, hence $I \times S$ is a weakly 2-absorbing ideal of $R \times S$. Conversely, suppose that $I \times S$ is a weakly 2-absorbing ideal in $R \times S$ and $abc \in I$ for some $a, b, c \in R$, then $(abc, 1) \in I \times S$, hence $(abc, 1) \in I \times S - 0$. Since $I \times S$ is weakly 2-absorbing and we have $(a, 1)(b, 1)(c, 1) = (abc, 1) \in I \times S - 0$, hence $ab \in I$ or $ac \in I$ or $bc \in I$.

Note. If I is a weakly 2-absorbing ideal of T, then $I \times S$ in $R \times S$ is φ-2-absorbing for each φ with $\varphi_w \leq \varphi$. If I is actually 2−absorbing, then $I \times S$ is 2-absorbing and hence is φ- 2-absorbing for all φ. Suppose that I is not 2-absorbing. Then $I^3 = 0$. So $(I \times S)^3 = I^3 \times S = 0 \times S$ and hence $\varphi_\omega(I \times S) = 0 \times S$. Hence $I \times S - \varphi_\omega(I \times S) = I \times S - 0 \times S = (I - \{0\}) \times S$. Then $(x_1, x_2)(y_1, y_2)(z_1, z_2) = (x_1y_1z_1, x_2y_2z_2) \in I \times S - \varphi_\omega(I \times S)$ and hence $x_1y_1z_1 \in I - 0 \Rightarrow x_1y_1 \in I$ or $x_1z_1 \in I$ or $y_1z_1 \in I \Rightarrow (x_1, x_2)(y_1, y_2) \in I \times S$ or $(x_1, x_2)(z_1, z_2) \in I \times S$ or $(y_1, y_2)(z_1, z_2) \in I \times S$. So $I \times S$ is φ_ω-2-absorbing and hence φ-2-absorbing.

Theorem 14. *(1) Let T and S be commutative rings and let I be a weakly 2-absorbing ideal of T. Then $J = I \times S$ is a φ-2-absorbing ideal of $R = T \times S$ for each φ with $\varphi_w \leq \varphi \leq \varphi_1$.*

(2) Let R be a commutative ring and let J be a finitely generated proper ideal of R. Suppose that J is φ-2-absorbing where $\varphi \leq \varphi_4$. Then either J is weakly 2-absorbing or $J^3 \neq 0$ is idempotent and R decomposed as $T \times S$ where $S = J^3$ and $J = I \times S$ where I is weakly 2-absorbing. Hence J is φ- 2-absorbing for each φ with $\varphi_\omega \leq \varphi \leq \varphi_1$.

Proof. (1) is proved in the previous paragraph. (2) If J is 2-absorbing, then J is weakly 2-absorbing. So we can assume that J is not 2-absorbing. Then $J^3 \subseteq \varphi(J)$; and hence $J^3 \subseteq \varphi(J) \subseteq \varphi_4(J) = J^4$. So $J^3 = J^4$. Hence J^3 is idempotent. Since J^3 is finitely generated, $J^3 = (e)$ for some idempotent $e \in R$. Suppose $J^3 = 0$. Then $\varphi(J) \subseteq J^4 = 0$. So $\varphi(J) = 0$ and hence J is weakly 2-absorbing. So assume $J^3 \neq 0$. Put $S = J^3 = Re$ and $T = R(1-e)$; So R decomposes as $T \times S$ where $S = J^3$. Let $I = J(1-e)$, so $J = I \times S$ where $I^3 = (J(1-e))^3 = J^3(1-e)^3 = (e)(1-e) = 0$. We show that I is weakly 2-absorbing. Let $xyz \in I^3 - \{0\}$; so $(x,1)(y,1)(z,1) = (xyz,1) \in I \times S - (I \times S)^3 = I \times S - 0 \times S \subseteq J - \varphi(J)$ since $\varphi \leq \varphi_4$ implies $\varphi(J) \subseteq J^4 = (I \times S)^4 = 0 \times S$. Hence $(xy,1) \in J$ or $(xz,1) \in J$ or $(yz,1) \in J$ so $xy \in I$ or $xz \in I$ or $yz \in I$. Hence I is weakly 2-absorbing.

Corollary 15. *Let R be an indecomposable commutative ring and J a finitely generated φ-2-absorbing ideal of R, where $\varphi \leq \varphi_4$. Then J is weakly 2-absorbing. If further R is an integral, J is actually 2-absorbing.*

Corollary 16. *Let R be a Noetherian integral domain. A proper ideal J of R is 2-absorbing if and only if $xyz \in J - J^4$ implies $xy \in J$ or $xz \in J$ or $yz \in J$.*

Definition 17. A non-zero nonunit element a in a commutative ring R is irreducible if $a = bc$ implies $b \in (a)$ or $c \in (a)$.

Theorem 18. *Let R be a commutative ring and let $a \in R$ be a nonunit.*

(1) Suppose that $(0 : a) \subseteq (a)$. Then (a) is φ- 2-absorbing for some φ with $\varphi \leq \varphi_2$ if and only if (a) is 2-absorbing.

(2) Suppose that (R, M) is quasi-local.

(C_1) The ideal (a) is φ- 2-absorbing for some φ with $\varphi \leq \varphi_4$ if and only if (a) is weakly 2-absorbing.

(C_2) *The ideal* (a) *is* φ_M*- 2-absorbing if and only if* a *is irreducible.*

Proof. (1) ($\Leftarrow$) A 2-absorbing ideal is φ-2-absorbing for every φ.
($\Rightarrow$) we may assume that (a) is φ_2-2-absorbing. Let $xyz \in (a)$. If $xyz \notin (a)^2$, then $xy \in (a)$ or $xz \in (a)$ or $yz \in (a)$. So suppose that $xyz \in (a)^2$.Now $(x+a)yz \in (a)$. If $(x+a)yz \notin (a)^2$, then $(x+a)y \in (a)$ or $(x+a)z \in (a)$ or $yz \in (a)$ and hence $xy \in (a)$ or $xz \in (a)$ or $xz \in (a)$. So assume that $(x+a)yz \in (a)^2$. Then $xyz \in (a)^2$ gives $ayz \in (a)^2$ and hence $yz \in (a) + (0 : a) \subseteq (a)$.

(2)(C_1) If (a) is weakly 2-absorbing, then (a) is φ-2-absorbing for each φ with $\varphi_0 \leq \varphi \leq \varphi_4$. Conversely, suppose that (a) is φ- 2-absorbing for some φ with $\varphi \leq \varphi_4$. Since a quasilocal ring has no nontrivial idempotents, (a) is weakly 2-absorbing by Theorem 11(2).

(C_2) Note that a is irreducible if and only if $a = bcd$.
Implies one of the following types:
1) $b \in (a)$ or $cd \in (a)$
2) $c \in (a)$ or $bd \in (a)$
3) $d \in (a)$ or $bc \in (a)$
But (a) is φ_M- 2-absorbing if and only if $bcd \in (a) - M(a)$ implies $bc \in (a)$ or $bd \in (a)$ or $cd \in (a)$. But $bcd \in (a) - M(a)$if and only if $bcd = ua$ for some unit $u \in R$. Thus (a) φ_M- 2-absorbing if and only if $a = bcd$ implies $bc \in (a)$ or $bd \in (a)$ or $cd \in (a)$.

Theorem 19. *Let* I *be a proper ideal of a commutative ring* R *and let* $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ *be a function. Then the following are equivalent:*

1) I *is* φ*- 2-absorbing;*

2) For $x, y \in R$ *such that* $xy \in R - I$,$(I : xy) = (I : x) \cup (I : y) \cup (\varphi(I) : xy)$;

3) For $x, y \in R$ *such that* $xy \in R - I$,$(I : xy) = (I : x)$ *or* $(I : xy) = (I : y)$ *or* $(I : xy) = (\varphi(I) : xy)$;

4) For ideals A, B, C *of* R, $ABC \subseteq I$,$ABC \not\subseteq \varphi(I)$ *implies* $AB \subseteq I$ *or* $AC \subseteq I$ *or* $BC \subseteq I$.

Proof. ($1 \Rightarrow 2$) let $xy \in R - I$. Let $z \in (I : xy)$; so $xyz \in I$. If $xyz \notin \varphi(I)$ then $yz \in I$ or $xz \in I$; so $z \in (I : y)$ or $z \in (I : x)$. If $xyz \in \varphi(I)$ then $z \in (\varphi(I) : xy)$. So $(I : xy) \subseteq (I : x) \cup (I : y) \cup (\varphi(I) : xy)$, the other containment always holds(remember we are assuming $\varphi(I) \subseteq I$).

($2 \Rightarrow 3$) If an ideal is a union of two ideals, it is equal to one of them.

$(3 \Rightarrow 4)$ Let A, B, C be ideals of R with $ABC \subseteq I$. Suppose that $AB \nsubseteq I$ and $AC \nsubseteq I$ and $BC \nsubseteq I$, we show that $ABC \subseteq \varphi(I)$. Let $ab \in AB$. First, suppose that $ab \notin I$. Then $abC \subseteq I$ gives $C \subseteq (I : ab)$. Now $AC \nsubseteq I$, $BC \nsubseteq I$; so $(I : ab) = (\varphi(I) : ab)$. Hence $abC \subseteq \varphi(I)$. Next, let $ab \in I \cap AB$. Choose $a'b' \in AB - I$. Then $(ab + a'b')C \subseteq \varphi(I)$. Let $c \in C$ then $abc = (ab + a'b')c - a'b'c \in \varphi(I)$. Thus $ABC \subseteq \varphi(I)$.

$(4 \Rightarrow 1)$ Let $abc \in I - \varphi(I)$. Then $(a)(b)(c) \subseteq I$, $(a)(b)(c) \nsubseteq \varphi(I)$. So $(a)(b) \subseteq I$ or $(b)(c) \subseteq I$ or $(a)(c) \subseteq I$; i. e., $ab \in I$ or $bc \in I$ or $ac \in I$.

Lemma 20. *Suppose that I is a 2-absorbing ideal of a ring R and let S be a multiplicatively closed subset of R. Then IR_S is a 2-absorbing ideal of R_S.*

Proof. Suppose that $xyz \in IR_S$ for some $x, y, z \in R_S$. Then there are elements $s \in S_1$ and $x_1, x_2, x_3 \in R$ such that $xyz = (\frac{x_1}{s})(\frac{x_2}{s})(\frac{x_3}{s}) = (\frac{x_1x_2x_3}{s^3}) \in IR_S$. Thus, $x_1x_2x_3 \in I$. Since I is a 2-absorbing ideal of R. We have $x_1x_2 \in I$ or $x_1x_3 \in I$ or $x_2x_3 \in I$, and thus $xy \in IR_S$ or $xz \in IR_S$ or $yz \in IR_S$.

Note. Given a function $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ we define $\varphi_S : I(R_S) \to I(R_S) \cup \{\emptyset\}$ by $\varphi_S(J) = (\varphi(J \cap R))_S$ (and $\varphi_S(J) = \emptyset$ if $\varphi(J \cap R) = \emptyset$). Note that $\varphi_S(J) \subseteq J$ and $(\varphi_\alpha)_S = \varphi_\alpha$ for $\alpha \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$. We show that if $(\varphi(I))_S \subseteq \varphi_S(I_S)$(which is the case for φ_α for $\alpha \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$), then I φ_S-2-absorbing. Given an ideal J of R, define $\varphi_J : I(\frac{R}{J}) \to I(\frac{R}{J}) \cup \{\emptyset\}$ by $\varphi_J(\frac{I}{J}) = \frac{\varphi(I)+J}{J}$ for $I \supseteq J$ (and $\varphi_J(\frac{I}{J}) = \emptyset$ if $\varphi(I) = \emptyset$). Note that $\varphi_J(\frac{I}{J}) \subseteq \frac{I}{J}$ and $(\varphi_\alpha)_J = \varphi_\alpha$ for $\alpha \in \{\emptyset\} \cup \{0\} \cup \mathbb{N}$. If $I \supseteq J$ are ideals of R and I is 2-absorbing (resp., weakly 2-absorbing, n-almost 2-absorbing), then so is $\frac{I}{J}$. we show that if I is φ-2-absorbing, then $\frac{I}{J}$ is φ_J-2-absorbing in $\frac{R}{J}$.

Proposition 21. *Let R be a commutative ring and let $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ be a function. Let I be a φ- 2-absorbing ideal of R.*

(1) If I is an ideal of R with $J \subseteq I$. Then $\frac{I}{J}$ is a φ_J- 2-absorbing ideal of $\frac{R}{J}$.

(2) Suppose that S is a multiplicatively closed subset of R with $I \cap S = \emptyset$ and $\varphi(I)_S \subseteq \varphi_S(I_S)$. Then I_S is a φ_S- 2-absorbing ideal of R_S.

Proof. (1) Let $a, b, c \in R$. Suppose that $\overline{a}\overline{b}\overline{c} \in \frac{I}{J} - \varphi_J(\frac{I}{J}) = \frac{I}{J} - \frac{(\varphi(I)+J)}{J}$. Hence $abc \in I - (\varphi(I) + J)$. Thus $abc \in I - \varphi(I)$; so $ab \in I$ or $ac \in I$ or $bc \in I$. Therefore $\overline{ab} \in \frac{I}{J}$ or $\overline{ac} \in \frac{I}{J}$ or $\overline{bc} \in \frac{I}{J}$; so $\frac{I}{J}$ is φ_J-2-absorbing.

(2)Let $\frac{x}{s}.\frac{y}{t}.\frac{z}{q} \in I_S - \varphi_S(I_S)$. So $xyzu \in I$ for some $u \in S$; but $xyzw \notin \varphi_S(I_S) \cap R$ for every $w \in S$. Now if $xyzw \in \varphi(I)$, then $\frac{x}{s}\frac{y}{t}\frac{z}{q} \in (\varphi(I)_S) \subseteq \varphi_S(I_S)$;

a contradiction. So $xy(zu) \in I - \varphi(I)$ and hence I φ-2-absorbing gives $xy \in I$ or $x(zu) \in I$ or $y(zu) \in I$. Hence $\frac{x}{s}\frac{y}{t} \in I_S$ or $\frac{x}{s}\frac{z}{q} \in I_S$ or $\frac{y}{t}\frac{z}{q} \in I_S$. Thus I_S is φ-2−absorbing.

Theorem 22. *A commutative ring R has every proper(principal) ideal almost 2-absorbing if and only if R is von Neumann regular or (R, M) is quasilocal with $M^3 = 0$.*

Proof. ($\Leftarrow$) Suppose that R is von Neumann regular. Then each proper ideal of R is idempotent and hence almost 2-absorbing. If (R, M) is quasilocal with $M^3 = 0$, then every proper ideal of R is weakly 2-absorbing and hence almost 2-absorbing.

($\Rightarrow$) Suppose that every proper principal ideal of R is almost 2-absorbing. Let M be a maximal ideal of R. Then every proper principal ideal of R_M is almost 2-absorbing. Let $a, b, c \in M_M$. Then (abc) is almost 2-absorbing. Now $abc \in (abc)$. So either $ab \in (abc), ac \in (abc), bc \in (abc)$ or $abc \in (abc)^2$. Hence $(ab) = (ab)(c), (ac) = (ac)(b), (bc) = (bc)(a)$ or $(abc) = (abc)(abc)$. By Nakayama's lemma $(abc) = 0$. Hence $M_M^3 = 0$. Let $x \in R$. Then in R_M, $\frac{X^3}{1}$ is either a unit or 0. So $(x^3) = (x^4)$ locally and hence globally. Thus $(x^3) = (e)$ for some idempotent e. First suppose that R is indecomposable. Then $x^3 = 0$ for each nonunit $x \in R$. So R is quasilocal with unique maximal ideal M. Now suppose that R is not indecomposable; say $R = R_1 \times R_2$. Suppose that R_1 is not von Neumann regular. Then R_1 has a nonidempotent ideal I. By hypothesis $I \times 0$ is almost 2-absorbing. But this contradicts Theorem 9. So R_1 is von Neumann regular. Likewise,R_2 is von Neumann regular. Hence their product $R = R_1 \times R_2$ is also von Neumann regular.

Corollary 23. *Let R be a commutative ring and $\varphi : I(R) \to I(R) \cup \{\emptyset\}$ a function with $\varphi_w \leq \varphi \leq \varphi_2$. Then every proper (principal) ideal of R φ-2-absorbing if and only if R is von Neumann regular.*

Proof. ($\Rightarrow$) Since a φ- 2-absorbing ideal with $\varphi \leq \varphi_2$ is almost 2-absorbing.

($\Leftarrow$) Suppose that R is von Neumann regular. Let I be an ideal of R. Since I is idempotent, $I = \bigcap_{n=1}^{\infty} I^n \subseteq \varphi(I) \subseteq I^2 = I$; so $\varphi(I) = I$. Hence every proper ideal of R is φ- 2-absorbing.

Lemma 24. *Let R be a commutative local ring with maximal ideal M such that $M^3 = 0$ (in particular $M^2 = 0$). Then every proper ideal is weakly 2-absorbing.*

Proof. Suppose that I is a proper ideal and $abc \in I - 0 = I - M^3$. Hence $abc \notin M^3$; so at least one of a or b or c does not belong to M. So a or b or c is a unit and hence ab or ac or bc is in I. So I is weakly 2-absorbing.

Lemma 25. *Let R be a commutative local ring with maximal ideal M and I is a proper ideal in R such that $I^3 = M^3$ then I is φ- 2-absorbing for some φ with $\varphi \geq \varphi_3$.*

Proof. Let $abc \in I - \varphi(I)$ for some $a, b, c \in R$. Since $\varphi \geq \varphi_3$, we have $abc \in I - \varphi_3(I)$. Then $abc \in I - I^3 = I - M^3$. So a or b or c is a unit and hence ab or ac or bc is in I. So I is ϕ-2-absorbing.

Corollary 26. *If (R, M) is a commutative local ring and I is a proper ideal such that $M^3 \subseteq I \subseteq M$, then I is almost 2-absorbing if and only if $M^3 = I^3$.*

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