HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY S-CONVEX FUNCTIONS

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Abstract: In this paper, the author give some new Hermite-Hadamard type inequality, which estimate the difference between the middle and the leftmost terms in the ordinary Hermite-Hadamard type inequality, for harmonically s-convex functions in the second sense by setting up an integral identity for differentiable functions.

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1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard’s inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \rightarrow R$ be
a convex function and \( a, b \in I \) with \( a < b \). Then following double inequalities hold:

\[
\frac{a + b}{2} \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]  

(1)

Hermite-Hadamard’s inequalities for convex, \((\alpha, m)\)-convex, \(GA\)-convex and geometric convex functions and have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1, 2, 3, 7, 8, 9, ?, 10, 12, 13] and references therein.

Let us recall some definitions of several kinds of convex functions:

**Definition 1.** Let \( I \) be an interval in \( R \). Then \( f : I \rightarrow R \) is said to be convex on \( I \) if the inequality

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds, for all \( x, y \in I \) and \( t \in [0, 1] \).

**Definition 2.** Let \( I \) be an interval in \( R_+ = (0, \infty) \). A function \( f : I \rightarrow R \) is said to be harmonically convex on \( I \) if the inequality

\[
f \left( \frac{xy}{tx + (1 - t)y} \right) \leq tf(y) + (1 - t)f(x)
\]

holds, for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (2) is reversed, then \( f \) is said to be harmonically concave.

In [14], Zhang et al. defined the harmonically quasi-convex functions and supplied several properties of this kind of functions.

**Definition 3.** Let \( I \) be an interval in \( R_+ = (0, \infty) \). A function \( f : I \rightarrow R \) is said to be harmonically quasi-convex on \( I \) if the inequality

\[
f \left( \frac{xy}{tx + (1 - t)y} \right) \leq \sup \left\{ f(x) : f(y) \right\}
\]

holds, for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (3) is reversed, then \( f \) is said to be harmonically quasi-concave.

In [4], İmdat Işcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

**Theorem 1.1.** Let \( f : I \subseteq R_+ = (0, \infty) \rightarrow R \) be a harmonically convex function on an interval \( I \) and \( f \in L[a, b] \), where \( a, b \in I \) with \( a < b \).

\[
f \left( \frac{2ab}{a + b} \right) \leq \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2}dx \leq \frac{f(a) + f(b)}{2}.
\]  

(3)
Also, in [4, 5, 6], Imdat Işcan established some new Hermite-Hadamard type and Ostrowski type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

**Theorem 1.2.** Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $| f' |^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b-a)}{2} \lambda_1^\frac{1}{1-q} [\lambda_2 | f'(a) |^q + \lambda_3 | f'(b) |^q]^\frac{1}{q},$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left( \frac{(a+b)^2}{4ab} \right),$$

$$\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right),$$

$$\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left( \frac{(a+b)^2}{4ab} \right) = \lambda_1 - \lambda_2.$$

In [4, 5, 6], Imdat Işcan introduced the harmonically $s$-convex functions and established some new supplied Hermite-Hadamard type inequalities:

**Definition 4.** Let $I$ be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically $s$-convex in the second sense on $I$ if the inequality

$$f \left( \frac{xy}{tx + (1-t)y} \right) \leq t^s f(y) + (1 - t)^s f(x)$$

holds, for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. If the inequality in (4) is reversed, then $f$ is said to be harmonically $s$-concave in the second sense.

**Theorem 1.3.** Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically $s$-convex function in the second sense and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following double inequalities hold:

$$2^{s-1} \left( \frac{2ab}{a+b} \right) \leq \frac{1}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{s + 1}.$$
In this article we consider the following special functions:

**Definition 5.** The hypergeometric function $2F_1[a, b, c, x]$ is defined for $|x| < 1$ by the power series

$$2F_1[a, b, c, x] = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}. \quad (6)$$

Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)\cdots(q+n-1), & n > 0. \end{cases}$$

**Definition 6.** The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$ 

In this paper, we give some new Hermite-Hadamard type inequalities, which gives an upper bound for the approximation of the integral average

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx$$

by the value $\frac{f(a)+f(b)}{2}$ for harmonically $s$-convex functions in the second sense by setting up an integral identity for differentiable functions.

### 2. Main Results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the rightmost and and middle parts of (1) for functions whose derivatives are harmonically $s$-convex in the second sense, we need the following lemma [11]:

**Lemma 2.1.** Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior $I^0$ of an interval $I$ such that $f^\prime \in L([a, b])$, where $a, b \in I$ with $a < b$. Then the following identity

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx = \frac{ab(a-b)}{2} \left[ \int_0^1 a(t-1) A_t^2(a, b) f^\prime \left( \frac{ab}{A_t(a, b)} \right)dt \right. + \left. \int_0^1 \frac{bt}{A_t^2(a, b)} f^\prime \left( \frac{ab}{A_t(a, b)} \right)tdt \right] \quad (7)$$

holds for $t \in [0, 1]$, where $A_t(a, b) = (1-t)a + tb$. 

Now we turn our attention to establish the Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in (1), for harmonically $s$-convex functions in the second sense by using the above lemma.

**Theorem 2.1.** Let $f : I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^0$, the interior of an interval $I$, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically $s$-convex in the second sense on $[a, b]$ for $q \geq 1$, then for all $t \in [0, 1]$ the following inequality holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b-a)}{2} \left[ a \left\{ \mu_{11}(a, b) |f'(a)|^q + \mu_{12}(a, b) |f'(b)|^q \right\}^{\frac{1}{q}} \\
+ b \left\{ \mu_{12}(b, a) |f'(a)|^q + \mu_{11}(b, a) |f'(b)|^q \right\}^{\frac{1}{q}} \right].
\]

where

\[
\mu_{11}(a, b) = \frac{\beta(1 + q, 1 + s)}{a^{3q}} \, _2F_1[3q, 1 + s, 2 + q + s, 1 - \frac{b}{a}],
\]

\[
\mu_{12}(a, b) = \frac{1}{a^{3q}(1 + q + s)} \, _2F_1[1, 3q, 2 + q + s, 1 - \frac{b}{a}].
\]

**Proof.** From Lemma 1, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b-a)}{2} \left[ a \left\{ \int_0^1 \frac{1-t}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) | \, dt \right\}^{\frac{1}{q}} \\
+ b \left\{ \int_0^1 \frac{t}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) | \, dt \right\}^{\frac{1}{q}} \right]
\]

\[
\leq \frac{ab(b-a)}{2} \left[ a \left( \int_0^1 (1-t)^{\frac{1}{q}} \left( \int_0^1 \frac{(1-t)^q}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) |^q \, dt \right)^{\frac{1}{q}} \right] \\
+ b \left( \int_0^1 (1-t)^{\frac{1}{q}} \left( \int_0^1 \frac{t^q}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) |^q \, dt \right)^{\frac{1}{q}} \right]
\]

\[
= \frac{ab(b-a)}{2} \left[ a \left( \int_0^1 (1-t)^{\frac{1}{q}} \left( \int_0^1 \frac{(1-t)^q}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) |^q \, dt \right)^{\frac{1}{q}} \right] \\
+ b \left( \int_0^1 t^{\frac{1}{q}} \left( \int_0^1 \frac{t^q}{A_t^q(a, b)} |f' \left( \frac{ab}{A_t(a, b)} \right) |^q \, dt \right)^{\frac{1}{q}} \right]
\]
\[ + b \left( \int_0^1 \frac{t^q}{A_t^q(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q \right) ^{\frac{1}{q}}. \]

Since \(|f'|^q\) is harmonically s-convex in the second sense on \([a,b]\), we have

\[(a) \int_0^1 (1-t)^q \frac{t^q}{A_t^q(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \]
\[\leq \int_0^1 (1-t)^q \left\{ t^s |f'(a)|^q + (1-t)^s |f'(b)|^q \right\} dt \]
\[= \mu_{11}(a,b)|f'(a)|^q + \mu_{12}(a,b)|f'(b)|^q, \quad (9)\]

\[(b) \int_0^1 \frac{t^q}{A_t^q(a,b)} \left| f' \left( \frac{ab}{A_t(a,b)} \right) \right|^q dt \]
\[\leq \int_0^1 \frac{t^q}{A_t^q(a,b)} \left\{ t^s |f'(a)|^q + (1-t)^s |f'(b)|^q \right\} dt \]
\[= \mu_{12}(b,a)|f'(a)|^q + \mu_{11}(b,a)|f'(b)|^q. \quad \] (10)

By substituting (10) and (11) in (9), we get the desired result.

Therefore, we can deduce the following results:

**Corollary 2.1.** In Theorem 2.1, additionally, if \(|f'(x)| \leq M\) for \(x \in [a,b]\), then the following inequality holds.

\[
\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \]
\[\leq \frac{ab(b-a)M}{2} \left[ \mu_{11}(a,b) + \mu_{12}(a,b) \right] ^{\frac{1}{q}} \]
\[+ b \left\{ \mu_{12}(a,b) + \mu_{11}(a,b) \right\} ^{\frac{1}{q}}. \quad (11)\]

**Theorem 2.2.** Let \(f : I \subseteq R_+ = (0, \infty) \to R\) be a differentiable function on \(I_0\), the interior of an interval \(I\), such that \(f' \in L([a,b])\), where \(a,b \in I\) with \(a < b\). If \(|f|^q\) is harmonically s-convex in the second sense on \([a,b]\) for \(q \geq 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then for all \(t \in [0,1]\) the following inequality holds:

\[
\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \]
\[\leq \frac{ab(b-a)}{2^{1+\frac{1}{p}}} \left[ a \left\{ \mu_{21}(a,b)|f'(a)|^q + \mu_{22}(a,b)|f'(b)|^q \right\} ^{\frac{1}{q}} \right]. \]
\[ + b \left\{ \mu_{22}(b, a) \left| f'(a) \right|^q + \mu_{21}(b, a) \left| f'(b) \right|^q \right\}^{\frac{1}{q}}, \]  

where
\[
\mu_{21}(a, b) = \frac{1}{a^{3q}} \left\{ \frac{2 F_1[3q, 1 + s, 2 + s, 1 - \frac{b}{a}]}{1 + s} - \frac{2 F_1[3q, 2 + s, 3 + s, 1 - \frac{b}{a}]}{2 + s} \right\},
\]
\[
\mu_{22}(a, b) = \frac{1}{a^{3q}(2 + s)} 2 F_1[1, 3q, 3 + s, 1 - \frac{b}{a}].
\]

**Proof.** From Lemma 1, we have
\[
\left| \frac{f(a) + f(b)}{2} \right| - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} dx \\
\leq \frac{ab(b - a)}{2} \left[ a \int_0^1 \frac{1 - t}{A_t^3(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt + b \int_0^1 \frac{t}{A_t^3(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right| dt \right]
\]
\[
= \frac{ab(b - a)}{2^{1 + \frac{1}{p}}} \left[ a \left( \int_0^1 \frac{1 - t}{A_t^{3q}(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} + b \left( \int_0^1 \frac{t}{A_t^{3q}(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q dt \right)^{\frac{1}{q}} \right]
\]

Since \( |f'|^q \) is harmonically \( s \)-convex in the second sense on \([a, b]\), we have
\[
(a) \int_0^1 \frac{1 - t}{A_t^{3q}(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q dt
\]
\[
\leq \int_0^1 \frac{1 - t}{A_t^{3q}(a, b)} \left\{ t^s |f'(a)|^q + (1 - t)^s |f'(b)|^q \right\} dt
\]
= \mu_{21}(a, b) |f'(a)|^q + \mu_{22}(a, b) |f'(b)|^q, \quad (14)

(b) \int_0^1 \frac{t}{A_t^3(q, a, b)} |f\left(\frac{ab}{A_t(a, b)}\right)|^q dt
\leq \int_0^1 \frac{t}{A_t^3(q, a, b)} \left\{ t^s |f'(a)|^q + (1-t)^s |f'(b)|^q \right\} dt
= \mu_{22}(b, a) |f'(a)|^q + \mu_{21}(b, a) |f'(b)|^q. \quad (15)

By substituting (15) and (16) in (14), we get the desired result.

**Theorem 2.3.** Let \( f : I \subseteq \mathbb{R}^+ = (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^0 \), the interior of an interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is harmonically \( s \)-convex in the second sense on \([a, b]\) for \( q \geq 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), then for all \( t \in [0, 1] \) the following inequality holds:

\[ \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \]

\leq \frac{a^\frac{1}{q} b^\frac{1}{q} (b-a)}{2^{3-\frac{1}{q}}} \left[ a^\frac{1}{q} \left\{ \mu_{31}(a, b) |f'(a)|^q + \mu_{32}(a, b) |f'(b)|^q \right\} \right]^{\frac{1}{q}}
+ b^\frac{1}{q} \left\{ \mu_{32}(b, a) |f'(a)|^q + \mu_{31}(b, a) |f'(b)|^q \right\}^{\frac{1}{q}}, \quad (16)

where

\[ \mu_{31}(a, b) = \frac{1}{a^{3q}} \left\{ \frac{2F_1[3q, 2 + s, 3 + s, 1 - \frac{b}{a}]}{2 + s} - \frac{2F_1[3q, 1 + s, 2 + s, 1 - \frac{b}{a}]}{1 + s} \right\}, \]
\[ \mu_{32}(a, b) = \frac{1}{a^{3q}(2 + s)} 2F_1[1, 3q, 3 + s, 1 - \frac{b}{a}]. \]

**Proof.** From Lemma 1, we have

\[ \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \]
\[ \leq \frac{ab(b-a)}{2} \left[ a \int_0^1 \frac{1-t}{A_t^3(a, b)} |f\left(\frac{ab}{A_t(a, b)}\right)| dt \right] \]
Theorem 2.4. Let \( f : I \subseteq R_+ = (0, \infty) \to R \) be a differentiable function on \( I^0, \) the interior of an interval \( I, \) such that \( f' \in L([a, b]), \) where \( a, b \in I \) with \( a < b. \) If \( |f'|^q \) is harmonically \( s \)-convex in the second sense on \([a, b],\) for \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1, \) then for all \( t \in [0, 1] \) the following inequality

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right|
\]
\[
\frac{ab(b^2 - a^2)}{2} \left( \frac{1}{1 + p} \right)^\frac{1}{p} \left\{ \mu_4(a, b) \left| f'(a) \right|^q + \mu_4(b, a) \left| f'(b) \right|^q \right\}^{\frac{1}{q}} \quad (20)
\]
holds, where
\[
\mu_4(a, b) = \frac{1}{a^{3q}(1 + s)} \quad _2F_1 \left[ 3q, 1 + s, 2 + s, 1 - \frac{b}{a} \right].
\]

**Proof.** From Lemma 1, we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{ab(b - a)}{2} \left[ a \left( \int_0^1 \frac{1 - t}{A^3_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q \, dt \right) \right. \\
+ b \left( \int_0^1 \frac{t}{A^3_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q \, dt \right] \\
= \frac{ab(b^2 - a^2)}{2} \left( \frac{1}{1 + p} \right)^\frac{1}{p} \left( \int_0^1 \frac{1}{A^3_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q \, dt \right)^\frac{1}{q}. \quad (21)
\]
Since \(|f'|^q\) is harmonically \(s\)-convex in the second sense on \([a, b]\), we have
\[
\int_0^1 \frac{1}{A^3_t(a, b)} \left| f' \left( \frac{ab}{A_t(a, b)} \right) \right|^q \, dt \\
\leq \int_0^1 \frac{1}{A^3_t(a, b)} \left\{ t^s \left| f'(a) \right|^q + (1 - t)^s \left| f'(b) \right|^q \right\} \, dt \\
= \mu_4(a, b) \left| f'(a) \right|^q + \mu_4(b, a) \left| f'(b) \right|^q. \quad (22)
\]
By substituting (22) in (21), we get the desired result (20).

**Theorem 2.5.** Let \( f : I \subseteq R_+ = (0, \infty) \to R \) be a differentiable function on \( I_0 \), the interior of an interval \( I \), such that \( f' \in L([a, b]) \), where \( a, b \in I \) with
If \(|f'|^q\) is harmonically \(s\)-convex in the second sense on \([a, b]\) for \(q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then for all \(t \in [0, 1]\) the following inequality holds,

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b - a)}{2} \left\{ a \mu_5^a(a, b) + b \mu_5^b(b, a) \right\} \left\{ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right\} \left\{ \frac{1}{1 + s} \right\} \frac{1}{q}
\]

(23)

holds, where

\[
\mu_5(a, b) = \frac{1}{a^{3p}(1 + p)} F_1[1, 3p, 2 + p, 1 - \frac{b}{a}].
\]

**Proof.** From Lemma 1, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{ab(b - a)}{2} \left\{ a \mu_5^a(a, b) + b \mu_5^b(b, a) \right\} \left\{ \left| f'(a) \right|^q + \left| f'(b) \right|^q \right\} \left\{ \frac{1}{1 + s} \right\} \frac{1}{q}
\]

(24)

Since \(|f'|^q\) is harmonically \(s\)-convex in the second sense on \([a, b]\), we have

\[
\int_0^1 \left| \frac{ab}{A_t(a, b)} \right|^q \, dt \\
\leq \int_0^1 \left\{ t^s \left| f'(a) \right|^q + (1 - t)^s \left| f'(b) \right|^q \right\} \, dt \\
= \frac{\left| f'(a) \right|^q + \left| f'(b) \right|^q}{1 + s}
\]

(25)

By substituting (25) in (24), we get the desired result (23).
References


