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HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY S-CONVEX FUNCTIONS

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Abstract: In this paper, the author give some new Hermite-Hadamard type inequality, which estimate the difference between the middle and the leftmost terms in the ordinary Hermite-Hadamard type inequality, for harmonically s-convex functions in the second sense by setting up an integral identity for differentiable functions.

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1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be

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a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard's inequalities for convex, (α, m)-convex, GA-convex and geometric convex functions and have received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [1, 2, 3, 7, 8, 9, **?**, 10, 12, 13] and references therein.

Let us recall some definitions of several kinds of convex functions:

Definition 1. Let I be an interval in R. Then $f : I \to R$ is said to be convex on I if the inequality

$$f\left(tx + (1-t)\, y\right) \leq tf\left(x\right) + \left(1-t\right) f\left(y\right)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf\left(y\right) + \left(1-t\right) f\left(x\right)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

In [14], Zhang *etal.* defined the harmonically quasi-convex functions and supplied several properties of this kind of functions.

Definition 3. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically quasi-convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \sup\Big\{f\left(x\right), f\left(y\right)\Big\} \tag{2}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (3) is reversed, then f is said to be harmonically quasi-concave.

In [4], Imdat Işcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

Theorem 1.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically convex function on an interval I and $f \in L[a, b]$, where $a, b \in I$ with $a < b$.*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

Also, in [4, 5, 6], Imdat Işcan established some new Hermite-Hadamard type and Ostrowski type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

Theorem 1.2. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $| f' |^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\right| \\ &\leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2 \mid f'(a) \mid^q + \lambda_3 \mid f'(b) \mid^q\right]^{\frac{1}{q}}, \end{aligned}$$

where

$$\begin{aligned} \lambda_1 &= \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_2 &= -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right), \\ \lambda_3 &= \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right) \\ &= \lambda_1 - \lambda_2. \end{aligned}$$

In [4, 5, 6], Imdat Işcan introduced the harmonically s-convex functions and established some new supplied Hermite-Hadamard type inequalities:

Definition 4. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically s-convex in the second sense on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq t^s f(y) + (1-t)^s f(x) \tag{4}$$

holds, for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$. If the inequality in (4) is reversed, then f is said to be harmonically s-concave in the second sense.

Theorem 1.3. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically s-convex function in the second sense and $a, b \in I$ with $a < b$. If $f \in L[a, b]$, then following double inequalities hold:*

$$2^{s-1}\left(\frac{2ab}{a+b}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{s+1}. \tag{5}$$

In this article we consider the following special functions:

Definition 5. The hypergeometric function ${}_2F_1[a,b,c,x]$ is defined for $|x|<1$ by the power series

$$ {}_2F_1[a,b,c,x] = \Sigma_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \tag{6}$$

Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$(q)_n = \begin{cases} 1, & n=0 \\ q(q+1)\cdots(q+n-1), & n>0. \end{cases}$$

Definition 6. The beta function, also called the Euler integral of the first kind, is a special function defined by

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

In this paper, we give some new Hermite-Hadamard type inequalities, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u$ by the value $\frac{f(a)+f(b)}{2}$ for harmonically s-convex functions in the second sense by setting up an integral identity for differentiable functions.

2. Main Results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the rightmost and and middle parts of (1) for functions whose derivatives are harmonically s-convex in the second sense, we need the following lemma [11]:

Lemma 2.1. *Let $f: I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a,b])$, where $a,b \in I$ with $a<b$. Then the following identity*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &= \frac{ab(a-b)}{2}\Big[\int_0^1 \frac{a(t-1)}{A_t^3(a,b)} f'\Big(\frac{ab}{A_t(a,b)}\Big)\mathrm{d}t \\ &\quad + \int_0^1 \frac{bt}{A_t^3(a,b)} f'\Big(\frac{ab}{A_t(a,b)}\Big)\mathrm{d}t\Big] \end{aligned} \tag{7}$$

holds for $t \in [0,1]$, where $A_t(a,b) = (1-t)a + tb$.

Now we turn our attention to establish the Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in (1), for harmonically s-convex functions in the second sense by using the above lemma.

Theorem 2.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically s-convex in the second sense on $[a, b]$ for $q \geq 1$, then for all $t \in [0, 1]$ the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\leq \frac{ab(b-a)}{2}\Big[a\Big\{\mu_{11}(a,b)\big|f'(a)\big|^q + \mu_{12}(a,b)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}} \\
&\quad + b\Big\{\mu_{12}(b,a)\big|f'(a)\big|^q + \mu_{11}(b,a)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big],
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\mu_{11}(a,b) &= \frac{\beta(1+q,1+s)}{a^{3q}}\,{}_2F_1[3q, 1+s, 2+q+s, 1-\frac{b}{a}], \\
\mu_{12}(a,b) &= \frac{1}{a^{3q}(1+q+s)}\,{}_2F_1[1, 3q, 2+q+s, 1-\frac{b}{a}].
\end{aligned}$$

Proof. From Lemma 1, we have

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\leq \frac{ab(b-a)}{2}\Big[a\int_0^1 \frac{1-t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t \\
&\quad + b\int_0^1 \frac{t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big] \\
&\leq \frac{ab(b-a)}{2} \\
&\quad \times \Big[a\Big(\int_0^1 1\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \frac{(1-t)^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad + b\Big(\int_0^1 1\mathrm{d}t\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \frac{t^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big] \\
&= \frac{ab(b-a)}{2}\Big[a\Big(\int_0^1 \frac{(1-t)^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}
\end{aligned}$$

$$+ b\Big(\int_0^1 \frac{t^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q \mathrm{d}t\Big)^{\frac{1}{q}}\Big].$$

Since $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$, we have

$$\begin{aligned}
(a) &\int_0^1 \frac{(1-t)^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q \mathrm{d}t \\
&\le \int_0^1 \frac{(1-t)^q}{A_t^{3q}(a,b)}\Big\{t^s|f'(a)\Big|^q + (1-t)^s|f'(b)\Big|^q\Big\}\mathrm{d}t \\
&= \mu_{11}(a,b)\big|f'(a)\big|^q + \mu_{12}(a,b)\big|f'(b)\big|^q, \qquad (9)\\
(b) &\int_0^1 \frac{t^q}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q \mathrm{d}t \\
&\le \int_0^1 \frac{t^q}{A_t^{3q}(a,b)}\Big\{t^s|f'(a)\Big|^q + (1-t)^s|f'(b)\Big|^q\Big\}\mathrm{d}t \\
&= \mu_{12}(b,a)\big|f'(a)\big|^q + \mu_{11}(b,a)\big|f'(b)\big|^q. \qquad (10)
\end{aligned}$$

By substituting (10) and (11) in (9), we get the desired result.

Therefore, we can deduce the following results:

Corollary 2.1. *In Theorem 2.1, additionally, if $|f'(x)| \le M$ for $x \in [a,b]$, then the following inequality holds.*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\le \frac{ab(b-a)M}{2}\Big[a\Big\{\mu_{11}(a,b)+\mu_{12}(a,b)\Big\}^{\frac{1}{q}} \\
&\quad + b\Big\{\mu_{12}(a,b)+\mu_{11}(a,b)\Big\}^{\frac{1}{q}}\Big]. \qquad (11)
\end{aligned}$$

Theorem 2.2. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$ for $q \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for all $t \in [0,1]$ the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\le \frac{ab(b-a)}{2^{1+\frac{1}{p}}}\Big[a\Big\{\mu_{21}(a,b)\big|f'(a)\big|^q + \mu_{22}(a,b)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}
\end{aligned}$$

$$+\,b\Big\{\mu_{22}(b,a)\big|f'(a)\big|^q+\mu_{21}(b,a)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big], \tag{12}$$

where

$$\mu_{21}(a,b)=\frac{1}{a^{3q}}\Big\{\frac{{}_2F_1[3q,1+s,2+s,1-\frac{b}{a}]}{1+s}-\frac{{}_2F_1[3q,2+s,3+s,1-\frac{b}{a}]}{2+s}\Big\},$$

$$\mu_{22}(a,b)=\frac{1}{a^{3q}(2+s)}\quad {}_2F_1[1,3q,3+s,1-\frac{b}{a}]. \tag{13}$$

Proof. From Lemma 1, we have

$$\Big|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big|$$
$$\le\frac{ab(b-a)}{2}\Big[a\int_0^1\frac{1-t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t+b\int_0^1\frac{t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big]$$

$$\le\frac{ab(b-a)}{2}\times\Big[a\Big(\int_0^1(1-t)\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{(1-t)}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+\,b\Big(\int_0^1 t\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]$$
$$=\frac{ab(b-a)}{2^{1+\frac{1}{p}}}\Big[a\Big(\int_0^1\frac{1-t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+\,b\Big(\int_0^1\frac{t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big].$$

Since $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$, we have

$$(a)\int_0^1\frac{1-t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t$$
$$\le\int_0^1\frac{1-t}{A_t^{3q}(a,b)}\Big\{t^s|f'(a)\Big|^q+(1-t)^s|f'(b)\Big|^q\Big\}\mathrm{d}t$$

$$= \mu_{21}(a,b)\big|f'(a)\big|^q + \mu_{22}(a,b)\big|f'(b)\big|^q, \tag{14}$$

$$(b) \int_0^1 \frac{t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q dt$$
$$\le \int_0^1 \frac{t}{A_t^{3q}(a,b)}\Big\{t^s\big|f'(a)\big|^q + (1-t)^s\big|f'(b)\big|^q\Big\}dt$$
$$= \mu_{22}(b,a)\big|f'(a)\big|^q + \mu_{21}(b,a)\big|f'(b)\big|^q. \tag{15}$$

By substituting (15) and (16) in (14), we get the desired result.

Theorem 2.3. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$ for $q \ge 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then for all $t \in [0,1]$ the following inequality holds:*

$$\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\Big|$$

$$\le \frac{a^{\frac{1}{q}}b^{\frac{1}{q}}(b-a)}{2^{3-\frac{1}{q}}}\Big[a^{\frac{1}{q}}\Big\{\mu_{31}(a,b)\big|f'(a)\big|^q + \mu_{32}(a,b)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}$$
$$+ b^{\frac{1}{q}}\Big\{\mu_{32}(b,a)\big|f'(a)\big|^q + \mu_{31}(b,a)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}}\Big], \tag{16}$$

where

$$\mu_{31}(a,b) = \frac{1}{a^{3q}}\Big\{\frac{{}_2F_1[3q, 2+s, 3+s, 1-\frac{b}{a}]}{2+s} - \frac{{}_2F_1[3q, 1+s, 2+s, 1-\frac{b}{a}]}{1+s}\Big\},$$
$$\mu_{32}(a,b) = \frac{1}{a^{3q}(2+s)}\ {}_2F_1[1, 3q, 3+s, 1-\frac{b}{a}].$$

Proof. From Lemma 1, we have

$$\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\Big|$$
$$\le \frac{ab(b-a)}{2}\Big[a\int_0^1 \frac{1-t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|dt$$

$$
\begin{aligned}
&+ b\int_0^1 \frac{t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big] \\
\le\ & \frac{ab(b-a)}{2} \\
&\times \Big[a\Big(\int_0^1 \frac{1-t}{A_t^3(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \frac{(1-t)}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ b\Big(\int_0^1 \frac{t}{A_t^{3q}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \frac{t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big] \\
=\ & \frac{a^{\frac{1}{q}}b^{\frac{1}{q}}(b-a)}{2^{1+\frac{1}{p}}}\Big[a^{\frac{1}{q}}\Big(\int_0^1 \frac{1-t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ b^{\frac{1}{q}}\Big(\int_0^1 \frac{t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]. \qquad (17)
\end{aligned}
$$

Since $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$, we have

$$
(a)\int_0^1 \frac{1-t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t
$$

$$
\begin{aligned}
&\le \int_0^1 \frac{1-t}{A_t^{3q}(a,b)}\Big\{t^s\big|f'(a)\big|^q + (1-t)^s\big|f'(b)\big|^q\Big\}\mathrm{d}t \\
&= \mu_{31}(a,b)\big|f'(a)\big|^q + \mu_{32}(a,b)\big|f'(b)\big|^q, \qquad (18) \\
(b)&\int_0^1 \frac{t}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t \\
&\le \int_0^1 \frac{t}{A_t^{3q}(a,b)}\Big\{t^s\big|f'(a)\big|^q + (1-t)^s\big|f'(b)\big|^q\Big\}\mathrm{d}t \\
&= \mu_{32}(b,a)\big|f'(a)\big|^q + \mu_{31}(b,a)\big|f'(b)\big|^q. \qquad (19)
\end{aligned}
$$

By substituting (18) and (19) in (17), we get the desired result (16).

Theorem 2.4. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $t \in [0,1]$ the following inequality*

$$
\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|
$$

$$\leq \frac{ab(b^2-a^2)}{2}\Big(\frac{1}{1+p}\Big)^{\frac{1}{p}}\Big\{\mu_4(a,b)\big|f'(a)\big|^q+\mu_4(b,a)\big|f'(b)\big|^q\Big\}^{\frac{1}{q}} \tag{20}$$

holds, where

$$\mu_4(a,b)=\frac{1}{a^{3q}(1+s)}\ \ {}_2F_1[3q,1+s,2+s,1-\frac{b}{a}].$$

Proof. From Lemma 1, we have

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big|\\&\leq\frac{ab(b-a)}{2}\Big[a\int_0^1\frac{1-t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\\&\qquad+b\int_0^1\frac{t}{A_t^3(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big]\end{aligned}$$

$$\begin{aligned}&\leq\frac{ab(b-a)}{2}\\&\quad\times\Big[a\Big(\int_0^1(1-t)^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{1}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\\&\qquad+b\Big(\int_0^1t^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{1}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\\&=\frac{ab(b^2-a^2)}{2}\Big(\frac{1}{1+p}\Big)^{\frac{1}{p}}\Big(\int_0^1\frac{1}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}.\end{aligned}\tag{21}$$

Since $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$, we have

$$\begin{aligned}&\int_0^1\frac{1}{A_t^{3q}(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\\&\leq\int_0^1\frac{1}{A_t^{3q}(a,b)}\Big\{t^s|f'(a)\Big|^q+(1-t)^s|f'(b)\Big|^q\Big\}\mathrm{d}t\\&=\mu_4(a,b)\big|f'(a)\big|^q+\mu_4(b,a)\big|f'(b)\big|^q.\end{aligned}\tag{22}$$

By substituting (22) in (21), we get the desired result (20).

Theorem 2.5. *Let $f:I\subseteq R_+=(0,\infty)\to R$ be a differentiable function on I^0, the interior of an interval I, such that $f'\in L([a,b])$, where $a,b\in I$ with*

$a < b$. If $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $t \in [0,1]$ the following inequality

$$\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)}{2}\Big\{a\mu_5^{\frac{1}{p}}(a,b) + b\mu_5^{\frac{1}{p}}(b,a)\Big\}\Big\{\frac{\big|f'(a)\big|^q + \big|f'(b)\big|^q}{1+s}\Big\}^{\frac{1}{q}} \tag{23}$$

holds, where

$$\mu_5(a,b) = \frac{1}{a^{3p}(1+p)}{}_2F_1[1, 3p, 2+p, 1-\frac{b}{a}].$$

Proof. From Lemma 1, we have

$$\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|$$

$$\leq \frac{ab(b-a)}{2}\Big[a\Big(\int_0^1 \frac{(1-t)^p}{A_t^{3p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} + b\Big(\int_0^1 \frac{t^p}{A_t^{3p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^1 \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\Big]$$
$$= \frac{ab(b-a)}{2}\Big\{a\mu_5^{\frac{1}{p}}(a,b) + b\mu_5^{\frac{1}{p}}(b,a)\Big\}\Big(\int_0^1 \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}. \tag{24}$$

Since $|f'|^q$ is harmonically s-convex in the second sense on $[a,b]$, we have

$$\int_0^1 \Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|^q\mathrm{d}t \leq \int_0^1 \Big\{t^s\Big|f'(a)\Big|^q + (1-t)^s\Big|f'(b)\Big|^q\Big\}\mathrm{d}t = \frac{\Big|f'(a)\Big|^q + \Big|f'(b)\Big|^q}{1+s} \tag{25}$$

By substituting (25) in (24), we get the desired result (23).

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