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GENERALIZATIONS OF THE HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY UASI-CONVEX FUNCTIONS

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Abstract: In this paper, some new results related to the right-hand side of the Hermite-Hadamard type inequality for the class of functions whose derivatives at certain powers are harmonically quasi-convex functions are obtained.

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1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be

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a convex function and $a, b \in I$ with $a < b$. Then following double inequalities hold:

$$\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Hermite-Hadamard's inequalities for convex, (α, m)-convex, GA-convex and geometric convex functions have received renewed attention in recent years and a remarkable variety of refinements and generalizations for them has been found in [1, 2, 3, 7, 8, 9, 10, 11, 13, 14] and references therein.

Let us recall some definitions of several kinds of convex functions:

Definition 1. Let I be an interval in R. Then $f : I \to R$ is said to be convex on I if the inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right) \tag{2}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically convex on I if the inequality

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf\left(y\right) + (1-t)\,f\left(x\right) \tag{3}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

In [4], Imdat Işcan established the following result of the Hermite-Hadamard type for harmonically convex functions:

Theorem 1.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a harmonically convex function on an interval I and $f \in L[a, b]$, where $a, b \in I$ with $a < b$.*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2}\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

Also, in [4, 5, 6], Imdat Işcan established some new Hermite-Hadamard type and Ostrowski type inequalities, which estimate the difference between the middle and the rightmost terms in (3), for harmonically convex functions:

Theorem 1.2. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I in $R_+ = (0, \infty)$ and $f' \in L[a, b]$, where $a, b \in I$*

with $a < b$. If $| f' |^q$ is harmonically convex function on $[a, b]$ for $q \geq 1$, then the following inequality holds:

$$\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|$$
$$\leq \frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2 \mid f'(a) \mid^q + \lambda_3 \mid f'(b) \mid^q\right]^{\frac{1}{q}},$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right)$$
$$= \lambda_1 - \lambda_2.$$

In [15], Zhang et. al defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 3. Let I be an interval in $R_+ = (0, \infty)$. A function $f : I \to R$ is said to be harmonically quasi-convex on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \sup\Big\{f(x), f(y)\Big\}$$

holds, for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (2) is reversed, then f is said to be harmonically quasi-concave.

In this article we consider the following special functions:

Definition 4. The hypergeometric function ${}_2F_1[a, b, c, x]$ is defined for $| x |< 1$ by the power series

$${}_2F_1[a,b,c,x] = \Sigma_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}. \tag{5}$$

Here $(q)_n$ is the Pochhammer symbol, which is defined by

$$(q)_n = \begin{cases} 1, & n = 0 \\ q(q+1)\cdots(q+n-1), & n > 0. \end{cases}$$

In this paper, we give some new Hermite-Hadamard type inequalities, which gives an upper bound for the approximation of the integral average $\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u$ by the value $\frac{f(a)+f(b)}{2}$, that is, estimate the difference between the middle and the rightmost terms in (1), for harmonically s-convex functions in the second sense by setting up an integral identity for differentiable functions.

2. Main Results

In order to find some new inequalities of Hermite-Hadamard-like type inequalities connected with the rightmost and and middle parts of (1) for functions whose derivatives are harmonically s-convex in the second sense, we need the following lemma [12]:

Lemma 2.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. Then the following identity*

$$\begin{aligned} &\frac{rf(a)+f(b)}{r+1} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &= \frac{ab(a-b)}{r+1}\int_0^1 \frac{1-(r+1)t}{A_t^2(a,b)} f'\Big(\frac{ab}{A_t(a,b)}\Big)\mathrm{d}t \end{aligned} \quad (6)$$

holds for $r \in [0, 1]$, where $A_t(a, b) = (1 - t)a + tb$.

Proof By the integration by parts, we have

$$\begin{aligned} &\int_0^1 \frac{1-(r+1)t}{A_t^2(a,b)} f'\Big(\frac{ab}{A_t(a,b)}\Big)\mathrm{d}t \\ &= \frac{1}{ab(b-a)}\Big[rf(a)+f(b)-(r+1)\int_0^1 f\Big(\frac{ab}{A_t(a,b)}\Big)\mathrm{d}t\Big] \\ &= \frac{1}{ab(b-a)}\Big[rf(a)+f(b)-(r+1)\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big], \end{aligned}$$

which implies that the identity (6) holds.

Now we turn our attention to establish the Hermite-Hadamard type inequalities, which estimate the difference between the middle and the leftmost terms in (1), for harmonically quasi-convex functions in the second sense by using the above lemma.

Theorem 2.1. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is harmonically quasi-convex on $[a,b]$, then for all $t \in [0,1]$ the following inequality holds:*

$$\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \Big\{\frac{b-ra}{r+1} + \frac{ab}{b-a}\ln\big[\frac{ab(r+1)^2}{(ra+b)^2}\big]\Big\}\sup\Big\{\big|f'(a)\big|,\big|f'(b)\big|\Big\}. \tag{7}$$

Proof From Lemma 1, we have

$$\begin{aligned}&\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\ &\leq \frac{ab(b-a)}{r+1}\int_0^1 \frac{|\,1-(r+1)t\,|}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t \\ &= \frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t \\ &\quad + \int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big].\end{aligned}$$

Since $|f'|^q$ is harmonically quasi-convex on $[a,b]$, we have

$$\begin{aligned}&\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\ &\leq \frac{ab(b-a)}{r+1}\Big\{\int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A_t^2(a,b)}\mathrm{d}t + \int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\mathrm{d}t\Big\} \\ &\quad \times \sup\Big\{\big|f'(a)\big|,\big|f'(b)\big|\Big\} \\ &= \Big\{\frac{b-ra}{r+1} + \frac{ab}{b-a}\ln\big[\frac{ab(r+1)^2}{(ra+b)^2}\big]\Big\}\sup\Big\{\big|f'(a)\big|,\big|f'(b)\big|\Big\},\end{aligned}$$

where we have used the facts that

$$(i)\ \int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A_t^2(a,b)}\mathrm{d}t = \frac{1}{a(b-a)} + \frac{r+1}{(b-a)^2}\ln\big[\frac{a(r+1)}{ra+b}\big],$$

$$(ii)\ \int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\mathrm{d}t = -\frac{r}{b(b-a)} + \frac{r+1}{(b-a)^2}\ln\big[\frac{b(r+1)}{ra+b}\big].$$

Therefore, we can deduce the following results:

Corollary 2.1. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. Assume $|f'|$ is harmonically quasi-convex on $[a, b]$.*

(1) If $r = 1$ in (7), then the following inequality holds:

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\ &\le \Big\{\frac{b-a}{2} + \frac{ab}{b-a}\ln\big[\frac{4ab}{(a+b)^2}\big]\Big\}\sup\Big\{\big|f'(a)\big|, \big|f'(b)\big|\Big\}.\end{aligned}$$

(2) If $r = 0$ in (7), then the following inequality holds:

$$\begin{aligned}&\Big|f(b) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\ &\le \Big\{b + \frac{ab}{b-a}\ln\big[\frac{a}{b}\big]\Big\}\sup\Big\{\big|f'(a)\big|, \big|f'(b)\big|\Big\}.\end{aligned}$$

Theorem 2.2. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q \ge 1$, then fot all $x \in [a, b]$ the following inequality holds:*

$$\begin{aligned}&\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\ &\le \frac{ab(b-a)}{(r+1)^{1+\frac{1}{p}}}\Big\{\mu_{21}^{\frac{1}{q}}(a,b,r,q) + r^{\frac{1}{p}}\mu_{22}^{\frac{1}{q}}(a,b,r,q)\Big\} \\ &\quad \times\Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}, \qquad (8)\end{aligned}$$

where

$$\begin{aligned}\mu_{21}(a,b,r,q) &= \frac{1}{(1+r)(1+q)a^{2q}}\ {}_2F_1[1, 2q, 2+q, -\frac{b-a}{(1+r)a}], \\ \mu_{22}(a,b,r,q) &= \frac{r^{1+q}(1+r)^{2q-1}}{(1+q)(ra+b)^{2q}}\ {}_2F_1[1+q, 2q, 2+q, -\frac{r(b-a)}{ra+b}].\end{aligned}$$

Proof From Lemma 1, we have

$$\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|$$

$$\leq \frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t + \int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big].$$

By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|$$
$$\leq \frac{ab(b-a)}{r+1}\Big[\Big(\int_0^{\frac{1}{r+1}} 1\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{r+1}} \frac{\{1-(r+1)t\}^q}{A_t^{2q}(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}} + \Big(\int_{\frac{1}{r+1}}^1 1\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{r+1}}^1 \frac{\{(r+1)t-1\}^q}{A_t^{2q}(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}}\Big]$$
$$\times\Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}$$
$$= \frac{ab(b-a)}{(r+1)^{1+\frac{1}{p}}}\Big[\Big(\int_0^{\frac{1}{r+1}} \frac{\{1-(r+1)t\}^q}{A_t^{2q}(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}} + r^{\frac{1}{p}}\Big(\int_{\frac{1}{r+1}}^1 \frac{\{(r+1)t-1\}^q}{A_t^{2q}(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}}\Big]\Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}$$
$$= \frac{ab(b-a)}{(r+1)^{1+\frac{1}{p}}}\Big\{\mu_{21}^{\frac{1}{q}}(a,b,r,q) + r^{\frac{1}{p}}\mu_{22}^{\frac{1}{q}}(a,b,r,q)\Big\} \times\Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}.$$

Corollary 2.2. *In the inequality (8) in Theorem 2.2, additionally, if* $|f'(x)| \leq M$ *for* $x \in [a,b]$, *then the following inequality hold:*

$$\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)M}{(r+1)^{1+\frac{1}{p}}}\Big\{\mu_{21}^{\frac{1}{q}}(a,b,r,q) + r^{\frac{1}{p}}\mu_{22}^{\frac{1}{q}}(a,b,r,q)\Big\}.$$

Theorem 2.3. *Let* $f : I \subseteq R_+ = (0,\infty) \to R$ *be a differentiable function on* I^0, *the interior of an interval* I, *such that* $f' \in L([a,b])$, *where* $a,b \in I$ *with* $a < b$. *If* $|f'|^q$ *is harmonically quasi-convex on* $[a,b]$ *for* $q \geq 1$ *with* $\frac{1}{p}+\frac{1}{q} = 1$, *then for all* $x \in [a,b]$ *the following inequality holds:*

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)}{r+1}\Big\{\frac{b-ra}{ab(b-a)}+\frac{r+1}{(b-a)^2}\ln\big[\frac{ab(r+1)^2}{(ra+b)^2}\big]\Big\} \times\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}.$$

Proof From Lemma 1, we have

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t +\int_{\frac{1}{r+1}}^1\frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big].$$

By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)}{r+1} \times\Big[\Big(\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}} +\Big(\int_{\frac{1}{r+1}}^1\frac{(r+1)t-1}{A_t^2(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{r+1}}^1\frac{(r+1)t-1}{A_t^2(a,b)}\mathrm{d}t\Big)^{\frac{1}{q}}\Big] \times\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}$$

$$=\frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\mathrm{d}t+\int_{\frac{1}{r+1}}^1\frac{(r+1)t-1}{A_t^2(a,b)}\mathrm{d}t\Big] \times\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}$$

$$= \frac{ab(b-a)}{r+1}\Big\{\frac{b-ra}{ab(b-a)} + \frac{r+1}{(b-a)^2}\ln\big[\frac{ab(r+1)^2}{(ra+b)^2}\big]\Big\} \times \Big(\sup\Big\{|f'(a)|^q, |f'(b)|^q\Big\}\Big)^{\frac{1}{q}},$$

which completes the proof.

Theorem 2.4. *Let $f : I \subseteq R_+ = (0, \infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [a, b]$ the following inequality holds:*

$$\Big|\frac{rf(a) + f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \le ab(b-a)^{\frac{1}{p}}\big(\frac{1}{1+p}\big)^{\frac{1}{p}}\big(\frac{1}{2q-1}\big)^{\frac{1}{q}} \times \Big[\mu_{41}^{\frac{1}{q}}(a, b, r, q) + r^{1+\frac{1}{p}}\mu_{42}^{\frac{1}{q}}(a, b, r, q)\Big] \times \Big(\sup\Big\{|f'(a)|^q, |f'(b)|^q\Big\}\Big)^{\frac{1}{q}},$$

where

$$\mu_{41}(a, b, r, q) = \{a(1+r)\}^{1-2q} - (ra+b)^{1-2q},$$
$$\mu_{42}(a, b, r, q) = (ra+b)^{1-2q} - \{b(1+r)\}^{1-2q}.$$

Proof From Lemma 1 and the the Hölder integral inequality, we have

$$\Big|\frac{rf(a) + f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \le \frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}} \frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t + \int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big].$$

By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$\Big|\frac{rf(a) + f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \le \frac{ab(b-a)}{r+1}$$

$$
\begin{aligned}
&\times \Big[\Big(\int_0^{\frac{1}{r+1}} \{1-(r+1)t\}^p \mathrm{d}t\Big)^{\frac{1}{p}} \Big(\int_0^{\frac{1}{r+1}} \frac{1}{A_t^{2q}(a,b)} \mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \Big(\int_{\frac{1}{r+1}}^1 \{(r+1)t-1\}^p \mathrm{d}t\Big)^{\frac{1}{p}} \Big(\int_{\frac{1}{r+1}}^1 \frac{1}{A_t^2(a,b)} \mathrm{d}t\Big)^{\frac{1}{q}}\Big] \\
&\quad \times \Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}} \\
&= \frac{ab(b-a)}{r+1}\Big[\Big(\frac{1}{(1+p)(1+r)}\Big)^{\frac{1}{p}} \Big(\int_0^{\frac{1}{r+1}} \frac{1}{A_t^{2q}(a,b)} \mathrm{d}t\Big)^{\frac{1}{q}} \\
&+ \Big(\frac{r^{1+p}}{(1+p)(1+r)}\Big)^{\frac{1}{p}} \Big(\int_{\frac{1}{r+1}}^1 \frac{1}{A_t^2(a,b)} \mathrm{d}t\Big)^{\frac{1}{q}}\Big] \\
&\quad \times \Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}} \\
&\le ab(b-a)^{\frac{1}{p}}\Big(\frac{1}{1+p}\Big)^{\frac{1}{p}}\Big(\frac{1}{2q-1}\Big)^{\frac{1}{q}} \\
&\times \Big[\mu_{41}^{\frac{1}{q}}(a,b,r,q) + r^{1+\frac{1}{p}} \mu_{42}^{\frac{1}{q}}(a,b,r,q)\Big] \\
&\quad \times \Big(\sup\Big\{\big|f'(a)\big|^q, \big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\int_0^{\frac{1}{r+1}} \frac{1}{A_t^{2q}(a,b)} \mathrm{d}t = \frac{\{a(1+r)\}^{1-2q} - (ra+b)^{1-2q}}{(2q-1)(1+r)^{1-2q}(b-a)}, \\
&\int_{\frac{1}{r+1}}^1 \frac{1}{A_t^2(a,b)} \mathrm{d}t = \frac{(ra+b)^{1-2q} - \{b(1+r)\}^{1-2q}}{(2q-1)(1+r)^{1-2q}(b-a)}, \\
&\int_0^{\frac{1}{r+1}} \big(1-(r+1)t\big)^p \mathrm{d}t = \frac{1}{(1+p)(1+r)}, \\
&\int_{\frac{1}{r+1}}^1 \big((r+1)t-1\big)^p \mathrm{d}t = \frac{1}{(1+p)(1+r)}.
\end{aligned}
$$

Theorem 2.5. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for all $x \in [a,b]$ the following inequality holds:*

$$
\Big|\frac{rf(a)+f(b)}{r+1} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big|
$$

$$\leq ab(b-a)^{\frac{1}{q}}(\frac{1}{1+q})^{\frac{1}{q}}(\frac{1}{2p-1})^{\frac{1}{p}}$$
$$\times\left[\mu_{41}^{\frac{1}{p}}(a,b,r,p)+r^{1+\frac{1}{q}}\mu_{42}^{\frac{1}{q}}(a,b,r,p)\right]$$
$$\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

where $\mu_{4i}(i=1,2)$ are defined in Theorem 2.4.

Proof From Lemma 1 and the the Hölder integral inequality, we have

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big|$$
$$\leq\frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t$$
$$+\int_{\frac{1}{r+1}}^1\frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big].$$

By the harmonically quasi-convexity of $|f'|^q$ and using the Hölder integral inequality, we have

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\Big|$$
$$\leq\frac{ab(b-a)}{r+1}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$\times\Big[\Big(\int_0^{\frac{1}{r+1}}\frac{1}{A_t^{2p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{r+1}}\{1-(r+1)t\}^q\mathrm{d}t\Big)^{\frac{1}{q}}$$
$$+\Big(\int_{\frac{1}{r+1}}^1\frac{1}{A_t^{2p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big]\Big(\int_{\frac{1}{r+1}}^1\{(r+1)t-1\}^q\mathrm{d}t\Big)^{\frac{1}{q}}$$

$$=ab(b-a)^{\frac{1}{q}}(\frac{1}{1+q})^{\frac{1}{q}}(\frac{1}{2p-1})^{\frac{1}{p}}$$
$$\times\left\{\mu_{41}^{\frac{1}{p}}(a,b,r,p)+r^{1+\frac{1}{q}}\mu_{42}^{\frac{1}{p}}(a,b,r,p)\right\}$$
$$\times\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 2.6. *Let $f : I \subseteq R_+ = (0,\infty) \to R$ be a differentiable function on I^0, the interior of an interval I, such that $f' \in L([a,b])$, where $a,b \in I$ with $a < b$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for all $x \in [a,b]$ the following inequality holds:*

$$\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \leq \frac{ab(b-a)}{(r+1)^{1+\frac{1}{q}}}\Big\{\mu_{21}^{\frac{1}{p}}(a,b,r,p)+r^{\frac{1}{q}}\mu_{22}^{\frac{1}{p}}(a,b,r,p)\Big\} \times \Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}.$$

Proof From Lemma 1, Hölder integral inequality and the harmonically quasi-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\Big|\frac{rf(a)+f(b)}{r+1}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x\Big| \\
&\leq \frac{ab(b-a)}{r+1}\Big[\int_0^{\frac{1}{r+1}}\frac{1-(r+1)t}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t \\
&\quad+\int_{\frac{1}{r+1}}^1 \frac{(r+1)t-1}{A_t^2(a,b)}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big] \\
&\leq \frac{ab(b-a)}{r+1} \\
&\quad\times\Big[\Big(\int_0^{\frac{1}{r+1}}\frac{\{1-(r+1)t\}^p}{A_t^{2p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{r+1}}\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big)^{\frac{1}{q}} \\
&\quad+\Big(\int_{\frac{1}{r+1}}^1\frac{\{(r+1)t-1\}^p}{A_t^{2p}(a,b)}\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{r+1}}^1\Big|f'\Big(\frac{ab}{A_t(a,b)}\Big)\Big|\mathrm{d}t\Big)^{\frac{1}{q}}\Big]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{ab(b-a)}{r+1}\Big[\mu_{21}^{\frac{1}{p}}(a,b,r,p)\Big(\frac{1}{r+1}\Big)^{\frac{1}{q}}\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}} \\
&\quad+\mu_{22}^{\frac{1}{p}}(a,b,r,p)\Big(\frac{r}{r+1}\Big)^{\frac{1}{q}}\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}\Big] \\
&= \frac{ab(b-a)}{(r+1)^{1+\frac{1}{q}}}\Big\{\mu_{21}^{\frac{1}{p}}(a,b,r,p)+r^{\frac{1}{q}}\mu_{22}^{\frac{1}{p}}(a,b,r,p)\Big\} \\
&\quad\times\Big(\sup\Big\{\big|f'(a)\big|^q,\big|f'(b)\big|^q\Big\}\Big)^{\frac{1}{q}}.
\end{aligned}$$

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