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ORTHOGONAL GENERALIZED DERIVATIONS ON SEMIRINGS

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Abstract: In [6] we discussed the notion of orthogonal derivations on semirings. In this paper, we introduce the notion of orthogonal generalized derivations on semirings and prove some results on semirpime semirings.

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1. Introduction

A semiring $(S, +, \cdot)$ is an algebraic structure such that $(S, +)$ and $(S, \cdot)$ are semigroups connected by ring like distributive laws. The study of semiring dates back to H.S.Vandiver[8]. In 1957, Posner proved two striking results on derivations on prime rings [5]. In [4], Jonathan Golan introduced the term derivation on semirings but nothing has been discussed on it. In [2],[3] Chandramouleeswaran and Thiruveni studied the notion of derivations on semirings in detail.

In [1], Bresar and Vukuman introduced the notion of orthogonality for a pair d,g of derivations on a semiprime ring and they gave several necessary and sufficient conditions for d and g to be orthogonal. In [7], the author discussed the notion of orthogonal derivations and orthogonal generalized derivation on ΓM modules. Motivated by these works, in [6] we introduced the notion of orthogonal derivations on semirings. In this paper, we define and discuss the notion of orthogonal generalized derivation on semirings and prove some simple but elegant results on semiprime semirings.

2. Preliminaries

Definition 5. A semiring S is said to be 2-torsion free if $2a = 0 \implies a = 0, a \in S$.

Definition 6. A semiring $(S, +, \cdot)$ is said to be a semiring with zero, if it has an element 0 in S such that

$$x + 0 = 0 = 0 + x \text{ and } x \cdot 0 = 0 = 0 \cdot x \ \ \forall\, x \ \in S.$$

Definition 7. An additive mapping $d : S \to S$ is called a derivation if

$$d(xy) = d(x)y + xd(y) \ \forall x, y \in S.$$

Definition 8. Let $(S, +, \cdot)$ be a semiring and let d be a nontrival derivation on S. An additive mapping $D : S \to S$ is called a generalized derivation on S, associated with the derivation d, if

$$D(xy) = D(x)y + xd(y) \quad \forall\, x, y \in S.$$

Lemma 2.1. *[6] Let S be a 2- torsion free semiprime semiring, a and b the elements of S, then the following are equivalant.*

1. $asb = 0$
2. $bsa = 0$
3. $asb + bsa = 0$.

If one of these conditions are fulfilled then ab = ba = 0

Lemma 2.2. *[6] Let S be a 2 torsion free semiprime semiring. Suppose that additive mappings f and g of a semiring S into S satisfying $f(x)sg(x) = 0$, $\forall x \in S$ then $f(x)sg(y) = 0 \ \forall x, y \in S$.*

Lemma 2.3. *[6] Let S be a 2 torsion free semiprime Semiring and let d and g be derivations of S in to S. Derivations d and g are orthogonal iff $d(x)g(y) + g(x)d(y) = 0 \ \forall x, y \in S$.*

Lemma 2.4. *[6] Let S be a 2 torsion free semiprime semiring. Suppose d and g are derivations of S in to S. Then d and g are orthogonal iff $dg = 0$.*

Lemma 2.5. *[6] Let S be a 2 torsion free semiprime Semiring.Suppose d and g are derivations of S in toS. Thend andg are orthogonal iff $dg + gd = 0$.*

Lemma 2.6. *[6] Le tS be a 2 torsionfree semiprime semiring. Supposed and g are derivations of S into S. Then d and g are orthogonal iff dg is a derivation.*

Notation Throughout this paper we shall assume that S is a 2-torsion free semiprime semiring with 0 and 1.

3. Orthogonal Generalized Derivation on Semirings

In this section we introduce the notion of generalized orthogonal derivations and prove some results.

Definition 9. Let S be a semiring. Let d and g be derivations on S. Let D and G be generalized derivations on S associated with d and g respectively. D and G are said to be orthogonal generalized derivations if $D(x)SG(y) = G(x)SD(y)$ $\forall$ x,y $\in$ S

Example 1. Let $S = \{0, a, b, c\}$.Define the operations $+$ *and* $\cdot$ on S as follows

$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

Then $(S, +, \cdot)$ is a semiring.
Define $d : S \to S$ such that $d(0) = 0$ *and* $d(x) = c$ *when* $x = a, b, c$
$g : S \to S$ such that $g(0) = 0$, $g(a) = b$ *and* $g(x) = c$ *when* $x = b, c$
clearly d and g are derivations on the semiring S.
$D = d + b_L$ *and* $G = g + c_L$ *where* b_L *and* c_L *denotes the left multiplication*
Let $S_1 = S \bigoplus S$.
Define d_1: $S_1 \to S_1$ by $d_1(x, y)) = (d(x), 0)$
$g_1 : S_1 \to S_1$ by $g_1(x, y) = (0, g(y))$,
d_1 *and* g_1 are orthogonal derivations.
Define $D_1 : S_1 \to S_1$ by $D_1(x, y) = ((D(x), 0)$
Define $G_1 : S_1 \to S_1$ by $G_1(x, y) = (0, G(y))$
D_1 *and* G_1 are orthogonal generalized derivations with derivations d_1 *and* g_1

Theorem 3.1. *Let D and G be orthogonal generalized derivations on S associated with derivations d, g respectively then the following relations hold*

1. $D(x)G(y) = G(x)D(y) = 0$. *Hence* $D(x)G(y) + G(x)D(y) = 0 \ \forall\ x, y \in S.$
2. *d and G are orthogonal and* $d(x)G(y) = G(x)d(y) = 0$
3. *g and D are orthogonal and* $g(x)D(y) = D(x)g(y) = 0$
4. *d and g are orthogonal derivations*

Proof. 1) Since D and G are orthogonal $D(x)zG(y) = G(x)zD(y) = 0$.

By lemma 2.9 $D(x)G(y) = G(x)d(y) = 0$.
Hence $D(x)G(y) + G(x)D(y) = 0 \quad \forall\ x, y \in S$.

2) From (1) $D(x)G(y) = 0$.
Replacing x by zx, $z \in S$ we get

$$\begin{aligned} 0 &= D(zx)G(y) \\ &= D(z)xG(y) + zd(x)G(y) \\ &= zd(x)G(y) \ \ (Since\ D\ and\ G\ are\ orthogonal) \end{aligned}$$

That is $zd(x)G(y) = 0$
premultiplying by $d(x)G(y)$ we get
$(d(x)G(y))z(d(x)G(y)) = 0, \ \forall\ z \in S$
That is $(d(x)G(y))S(d(x)G(y)) = 0\ \forall\ x, y, z \in S$
Since S is semiprime,

$$d(x)G(y) = 0 \tag{3.1}$$

Replacing x by xz in 3.1 we have

$$\begin{aligned} 0 &= d(xz)G(y) \\ &= d(x)zG(y) + xd(z)G(y) \\ &= d(x)zG(y)\ \ By\ 3.1 \end{aligned}$$

That is

$$d(x)zG(y) = 0 \tag{3.2}$$

From result (1) $0 = G(x)D(y)$
Replacing y by zy, $z \in S$, we get

$$\begin{aligned} 0 &= G(x)D(zy) \\ &= G(x)D(z)y + G(x)zd(y) \\ &= G(x)zd(y)\ \ By\ result\ (1) \end{aligned}$$

$$G(x)zd(y) = 0 \tag{3.3}$$

By lemma 2.9 we have $G(x)d(y) = 0$.
Also from 3.2 and 3.3

$$d(x)zG(y) = 0 = G(x)zd(y)\ \forall\ x, y, z \ \in\ S$$

Hence d and G are orthogonal.

Analogously we can prove result (3).

(4) Suppose $0 = D(x)G(y)$.

Replacing x by xz and y by yu we have

$$\begin{aligned}
0 &= D(xz)G(yu) \\
&= (D(x)z + xd(z))(G(y)u + yg(u)) \\
&= D(x)zG(y)u + D(x)(zy)g(u) + xd(z)G(y)u + xd(z)yg(u) \\
&= xd(z)yg(u) \text{ by } \textit{result } (2) \textit{ and } (3)
\end{aligned}$$

That is $xd(z)yg(u) = 0$.

Pre multiplying by $d(z)yg(u)$ we have

$$(d(z)yg(u))x(d(z)yg(u)) = 0 \; \forall \; u, x, y, z \in S.$$

That is,

$$(d(z)yg(u))S(d(z)yg(u)) = 0 \; \forall \; u, x, y, z \in S.$$

By semiprimeness of S, we have

$$d(z)yg(u) = 0 \; \forall \; u, y, z \in S.$$

Similarly, from $G(x)D(y) = 0$ we can prove that $g(z)yd(u) = 0 \; \forall \; u, y, z \in S$.

This completes the proof that d and g are orthogonal.

Lemma 3.1. *Let S be a 2- torsion semiprime semiring and let D be a generalized derivation on S associated with a derivation d. If $D(x)D(y) = 0 \; \forall \; x, y \in S$, then $D = d = 0$.*

Proof. Replacing y by yz in the given condition $D(x)D(y) = 0 \; \forall \; x, y \in S$ we get

$$\begin{aligned}
0 &= D(x)D(yz) \\
&= D(x)((D(y)z + yd(z)) \\
&= D(x)D(y)z + D(x)yd(z) \\
&= D(x)yd(z)
\end{aligned}$$

By lemma 2.9,

$$D(x)d(z) = d(z)D(x) = 0 \; \forall \; x, z \; \in S \tag{3.4}$$

Replacing x by xz in $d(z)D(x) = 0$ we get

$$0 = d(z)D(xz)$$

$$\begin{aligned} &= d(z)((D(x)z + xd(z)) \\ &= d(z)D(x)z + d(z)xd(z) \\ &= d(z)yd(z) \ By\ 3.4 \end{aligned}$$

Since S is semiprime $d(z) = 0, \ \forall \ z \in S$
Replace x by yx in $D(x)D(y) = 0$ we get

$$\begin{aligned} 0 &= D(yx)D(y) \\ &= (D(y)x + yd(x))D(y) \\ &= D(y)xD(y) + yd(x)D(y) \\ &= D(y)xD(y) \ \ By\ 3.4 \end{aligned}$$

Since S is semiprime, $D(y) = 0, \ \forall \ y \in S$
Hence $D = d = 0$.

Lemma 3.2. *Let D and G be generalized derivations on a semiring S with derivations d and g respectively. If the following relations hold*

(i) $D(x)G(y) + G(x)D(y) = 0$

(ii) $d(x)G(y) + g(x)D(y) = 0$

then$D$ and G are orthogonal.

Proof. Replacing x by xz in (i) we have

$$\begin{aligned} 0 &= D(xz)G(y) + G(xz)D(y) \\ &= (D(x)z + xd(z))G(y) + (G(x)z + xg(z))D(y) \\ &= D(x)zG(y) + xd(z)G(y) + G(x)zD(y) + xg(z)D(y) \\ &= D(x)zG(y) + x(d(z)G(y) + g(z)D(y)) + G(x)zD(y) \\ &= D(x)zG(y) + G(x)zD(y) \ By\ (ii) \end{aligned}$$

That is

$$D(x)zG(y) + G(x)zD(y) = 0, \ \forall \ \ x, y, z \ \in S \tag{3.5}$$

Replacing y by x we have
$D(x)zG(x) + G(x)zD(x) = 0$
by lemma 2.9 $D(x)zG(x) = 0, \ \forall \ x, z \in S$
By lemma 2.10 $D(x)zG(y) = 0$
similarly replacing x by y in 3.5 we have $G(y)zD(y) = 0$
by lemma 2.10 $G(y)zD(x) = 0 \ \forall \ x, y, z \in S.$
D and G are orthogonal

Lemma 3.3. *Let S be a 2 - torsion free semiprime Semiring. Suppose D and G are generalized derivations on S with derivations d and g respectively such that $D(x)G(y) = d(x)G(y) = 0\ \forall\ x, y \in S$, then D and G are orthogonal.*

Proof. Replacing x byxz in $D(x)G(y) = 0$, we get

$$\begin{aligned} 0 &= D(xz)G(y) \\ &= (D(x)z + xd(z))G(y) \\ &= D(x)zG(y) + xd(z)G(y) \\ &= D(x)zG(y) \end{aligned}$$

That is$D(x)zG(y) = 0,\ \forall\ x, y, z \in S$.

By lemma 2.9 $G(y)zD(x) = 0$.

Replacing y by x we get $G(x)zD(x) = 0$.

By lemma 2.10 $G(x)zD(y) = 0,\ \forall\ x, y, z\ \in S$.

Thus we have proved that D and G are orthogonal.

Lemma 3.4. *Let S be a 2 - torsion free semiprime Semiring. Suppose D and G are generalized derivations on S with derivations d and g respectively such that $D(x)G(y) = 0,\ \ dG = dg = 0$. Then D and G are orthogonal.*

Proof. Suppose $dg = 0$. By lemma 2.12, d and g are orthogonal.

Therefore $d(x)zg(y) = 0$ and$g(x)zd(y) = 0\ \forall\ x, y, z\ \in S$.

By lemma 2.9

$$d(x)g(y) = 0 = g(x)d(y) \tag{3.6}$$

Suppose $dG = 0$. That is, $dG(x) = 0\ \forall\ x \in S$.

Replacing x by xy, we get

$$\begin{aligned} 0 = dG(xy) &= d(G(xy) \\ &= d(G(x)y + xg(y)) \\ &= dG(x)y + G(x)d(y) + d(x)g(y) + xdg(y) \\ &= G(x)d(y) \quad By\ 3.6 \end{aligned}$$

Replacing x by xz, we have $G(xz)d(y) = 0$.

That is, $G(x)zd(y) + xg(z)d(y) = 0$.

By 3.6 $G(x)zd(y) = 0\ \forall\ x, y, z \in S$.

By lemma 2.9 $d(y)zG(x) = 0\ and\ d(y)G(x) = 0$.

By lemma 3.6 D and G are orthogonal.

Lemma 3.5. *Let S be a 2-torsion free semiprime Semiring. Suppose D and G are Generalized derivations on S associated with derivations d and g*

respectively such that DG is a generalized derivation on S with derivation dg and $D(x)G(y) = 0 \ \forall \ x, y \in S$, then D and G are orthogonal.

Proof.

$$\begin{aligned} DG(xy) &= D(G(x)y + xg(y)) \\ &= DG(x)y + G(x)d(y) + D(x)g(y) + xdg(y) \end{aligned}$$

But $DG(xy) = DG(x)y + xdg(y)$. Comparing

$$G(x)d(y) + D(x)g(y) = 0 \tag{3.7}$$

Replacing y by yz in $D(x)G(y) = 0$ we have $D(x)yg(z) = 0$.
By lemma 2.9 $g(z)D(x) = 0$.
Replacing z by yz, we have

$$\begin{aligned} 0 &= g(yz)D(x) \\ &= g(y)zD(x) + yg(z)D(x) \\ &= g(y)zD(x) \end{aligned}$$

By lemma 2.9 $D(x)g(y) = 0 \ \ \forall \ x, y \ \in S$.
Using in 3.7 $G(x)d(y) = 0$
Replacing y by yz we have $G(x)d(yz) = 0$.
$G(x)d(y)z + G(x)yd(z) = 0$
using $G(x)d(y) = 0$ we have $G(x)yd(z) = 0 \ \forall \ x, y, z \ \in S$.
By lemma 3.3 $d(z)G(x) = 0 \ \forall \ x, z \in S$.
By lemma 3.6 D and G are orthogonal.

4. Products of Generalized Derivation on Semirings

In this section we prove a necessary and sufficient condition for the product of two generalized derivations to be a generalized derivation in terms of the orthogonal derivations.

Lemma 4.1. *Let D and G be generalized derivations on S associated with derivations d and g respectively. Then DG is a generalized derivation associated with the derivation dg if and only if D and g and G and d are orthogonal derivations.*

Proof. Assume DG is a generalized derivation with derivation dg.
By lemma 2.14 d and g are orthogonal derivations.

Also by equation 3.7, $G(x)d(y) + D(x)g(y) = 0$.

Replacing y by yz we have

$$\begin{aligned} 0 &= G(x)d(yz) + D(x)g(yz) \\ &= G(x)d(y)z + G(x)yd(z) + D(x)g(y)z + D(x)yg(z) \\ &= (G(x)d(y) + D(x)g(y))z + G(x)yd(z) + D(x)yg(z) \ By\ 3.7 \end{aligned}$$

That is,

$$G(x)yd(z) + D(x)yg(z) = 0 \ \forall\ x, y \in S. \tag{4.1}$$

Replacing x by $xg(z)$ we have

$$\begin{aligned} 0 &= G(xg(z))yd(z) + D(xg(z))yg(z) \\ &= G(x)g(z)yd(z) + xg(g(z))yd(z) + D(x)g(z)yg(z) + xd(g(z))yg(z) \\ &= D(x)(g(z)y)g(z) = 0 \ \forall\ x, y, z \ \in S \end{aligned}$$

By Lemma 2.9 $D(x)g(z) = 0$.

Replacing z by yz we get and using the above condition, we have $D(x)yg(z) = 0$.

Now, equation 4.1 implies that $G(x)yd(z) = 0$. By lemma 2.9

$$d(z)G(x) = 0. \tag{4.2}$$

Replacing z by xy and x by zy, we get

$$\begin{aligned} 0 &= d(xy)G(zy) \\ &= (d(x)y + xd(y))(G(z)y + zg(y)) \\ &= d(x)yG(z)y + d(x)(yz)g(y) + xd(y)G(z)y + xd(y)zg(y) \\ &= d(x)yG(z)y \ \textit{By equation } 4.2 \textit{ and orthogonality of } d \textit{ and } g \end{aligned}$$

Postmultiplying by $d(x)yG(z)$ we have $(d(x)yG(z))y(d(x)yG(z)) = 0$.

Since S is semiprime $d(x)yG(z) = 0 \ \forall\ x, y, z \ \in S$.

Therefore G and d are orthogonal.

Analogously we can prove that D and g are orthogonal.

Conversely assume that both the pairs D and g and G and d are orthogonal.

Now by the orthogonality of D and g we have

$D(x)yg(z) = 0 = g(x)yD(z) \ \forall\ x, y, z \ \in S$.

By lemma 2.9 $D(x)g(z) = 0 \ \forall\ x, z \in S$.

Replacing x by rx in $D(x)yg(z) = 0$

$$0 = D(rx)yg(z)$$

$$= D(r)xg(z) + rd(x)yg(z)$$
$$= rd(x)yg(z)$$

Premultiplying by $d(x)yg(z)$ we have $(d(x)yg(z))r(d(x)yg(z)) = 0$.

By semiprimeness of $S, d(x)yg(z) = 0 \ \forall \ x, y, z \ \in S$.

By the orthogonality of G and d we have

$G(x)yd(z) = 0 = d(x)yG(z) \ \forall \ x, y, z \ \in S$.

By lemma 2.9 $G(x)d(z) = 0 \ \forall \ x, z \in S$.

Replacing x by zx in $G(x)yd(z) = 0$ we have $g(x)yd(z) = 0 \ \forall \ x, y, z \ \in S$.

Then d and g are orthogonal derivations.

By lemma 2.14 dg is a derivation

Using $d(x)yg(z) = 0,\ D(x)g(y) = 0,\ G(x)d(y) = 0$ in
$DG(xy) = DG(x)y + G(x)d(y) + D(x)g(y) + xdg(y)$
we have $DG(xy) = Dg(x)y + xdg(y)$.

Thus DG is a generalized derivation with derivation dg.

Corollary 4.1. *Let D and G be generalized derivations on S associated with derivations d and g,then GD is generalized derivation on S with derivation gd iff D and g , G and d are orthogonal.*

Lemma 4.2. *Let D be a generalized derivation on S associated with derivation d. If D^2 is a generalized derivation with derivation d^2 then $d = 0$.*

Proof. Since d^2 is a derivation, d and d orthogonal by lemma 2.14.

Hence $d(x)yd(z) = 0 \ \forall \ x, y, z \in S$. Replacing z by $x, d(x)yd(x) = 0$
Since S is semiprime $d(x) = 0 \ \forall \ x \in S$.

Lemma 4.3. *Let S be a 2 - torsion free prime semiring. If D and G are generalized derivations on S associated with derivations d and g respectively such that D and g, G and d are orthogonal, then $D = d = 0$ or $G = g = 0$.*

Proof. Since D and g are orthogonal we have

$$D(x)yg(z) = 0 \ \ and\ g(x)yD(z) = 0 \ \forall \ x, y, z \in S.$$

Since G and d are orthogonal we have

$$G(x)yd(z) = 0 \ \ and\ d(x)yG(z) = 0 \ \forall \ x, y, z \in S.$$

Since S is prime and $D(x)yg(z) = 0$ we have $D(x) = 0$ *or* $g(z) = 0 \ \forall \ x, z \in S$.
From $d(x)yG(z) = 0$ we have $d(x) = 0$ or $G(z) = 0$
Therefore $D = d = 0$ or $G = g = 0$

Lemma 4.4. *If D and G are orthogonal generalized derivations on S associated with derivations d and g respectively, then $dG = Gd = 0$ and $gD = Dg = 0, DG = GD = 0$.*

Proof. By result (2),(4) of theorem 3.3 d and G ,d and g are orthogonal. Hence $d(x)zG(y) = 0$ and $d(x)zg(y) = 0, d(x)g(y) = 0 \ \forall \ x, y, z \ \in S$.

$$\begin{aligned} 0 &= G(d(x)zG(y)) \\ &= Gd(x)zG(y) + d(x)g(zG(y)) \\ &= Gd(x)zG(y) + d(x)g(z)G(y) + d(x)zg(G(y)) \\ &= Gd(x)zG(y) \ \forall \ x, y, z \in \ S \end{aligned}$$

That is $G(d(x))zG(y) = 0 \ \forall \ x, y, z \in \ S$
Replacing y by $d(x)$ we have $G(d(x))zG(d(x)) = 0$
Since S is semiprime, $Gd = 0$
Similarly from $d(G(x)zd(y)) = 0$, $D(g(x)zD(y)) = 0$, $g(D(x)zg(y)) = 0$ and $G(D(x)zG(y)) = 0$
we have $dG = Dg = gD = DG = GD = 0$.

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