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# SOME COMMON FIXED POINT THEOREMS IN FUZZY METRIC SPACE

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**Abstract:** In this paper we prove some common fixed point theorems (generalizing the result in [9]), using the condition for continuous self mapping $A, B, C, S, T$ and $U$ of $(X, M, \star)$, where the pairs $\{A, S\}$ $\{B, T\}$ and $\{C, U\}$ are owc and have unique common fixed point in $X$.

**AMS Subject Classification:** 47H10, 54H25
**Key Words:** fuzzy metric space, common fixed point, occasionally weakly compatible mapping, weakly compatible maps

## 1. Introduction

In 1965 Zadeh in [7] intorudced the concept of fuzzy set as a new way to represent vagueness in our everday life. In the next decade Kramosil and Michalek introduced the concept of fuzzy metric space in 1975.

Consequently, some metric fixed point results were generalized to FM space by George and Veeramani.

Vasuki in [12] proved fixed point theorem for R-weakly commutating mappins. In [13], Pant intorudced the new concept reciprocally continuous mappings and proved some common fixed point theorems.

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This paper presents some common fixed point theorem which is occasionally weakly compatible mapping (OWC) in fuzzy metric space and generalization of results in [9] the condition for continuous self mapping $A, B, C, S, T$ and $U$ of complete fuzzy metric space $(X, M, \star)$ have a unique common fixed point.

## 2. Preliminary

Definition 2.1. A 3-tuple $(X, M, \star)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $\star$ is a continuous t-norm and $M$ is a fuzzy set of $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$

**1)** $M(x, y, t) > 0$;

**2)** $M(x, y, t) = 1$ if and only if $x = y$;

**3)** $M(x, y, t) = M(y, x, t)$;

**4)** $M(x, y, :) : (0, \infty) \to (0, 1]$ is continuous.

Then $M$ is called a fuzzy metric on $X$, and $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

**Definition 2.2.** A binary operation $\star : [0, 1]X[0, 1] \to [0, 1]$ is a continuous t-norm if $\star$ is satisfying conditions:

**(i)** $\star$ is an commutative and associative;

**(ii)** $\star$ is continuous;

**(iii)** $a \star 1 = a$ for all $a \in [0, 1]$;

**(iv)** $a \star b \leq c \star d$ whenever a $a \leq c, b \leq d$ and $a, b, c, d, \epsilon \in [0, 1]$.

**Definition 2.3.** A pair of self mappings $(f, g)$ of a fuzzy metric space $(X, M, \star)$ is said to be

**(i)** Weakly commuting if $M(fgx, gfx, t) \geq M(fx, gx, t)$ for all $x \in X\, ad\, t > 0$;

**(ii)** R-weakly commuting if there exist some $R > 0$ such that

$$M(fgx, gfx, t) \geq M(fx, gx, t/R) \quad \text{for all } x \in X\, ad\, t > 0.$$

**Definition 2.4.** The self mapping $f$ and $g$ of a fuzzy metric space $(X, M, \star)$ are called reciprocally continuous on $X$ if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$, whenever $\{x_n\}$ is a sequence in $X$ (such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x$ in $X$.)

**Definition 2.5.** Two self mapping $f$ and $g$ of a fuzzy metric space $(X, M, \star)$ are called compatible if

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1,$$

where $\{x_n\}$ is a sequence in $X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$$

for some $x \in X$.

**Definition 2.6.** Two self mapping $f$ and $g$ of a set $X$ are occasionally weakly compatile ($owc$) if there is a point $x$ in $X$ which is coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

**Definition 2.7.** A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \star)$ is called cauchy sequence if for every $\in> 0$ and each $t > 0$ there exist $n_0 \in N$ such that $M(x_n, x_{n+p}, t) > 1- \in$ for all $n \geqslant n_0$ and $t > 0$.

**Definition 2.8.** A fuzzy metric space in which every cauchy sequence is convergent is said to be complete. A sequence $\{x_n\}$in a fuzzy metric space $(X, M, \star)$ is called cauchy sequence if for each $\in> 0$ there exist $n_0 \in N$ such that

$$M(x_n, x_m, t) > 1- \in \quad \text{for all } n, m \geqslant n_0.$$

## 3. Main Results

**Theorem 3.1.** *Let $(X, M, \star)$be a complete fuzzy metric space and let $A, B, C, S, T$ and $U$ be self mapping of $X$.*

*Let the pairs $\{A, S\}, \{B, T\}$ and $\{C, U\}$ be owc. If there exist $q \in (0, 1)$ such that:*

$$M(Ax, By, Cz, qt) \geq \phi[\min\{M(Sx, Ty, Uz, t), M(Sx, Ax, t)M(Ty, By, t),$$

$$M(Uz, Cz, T) \star \min\{M(By, Ty, t)M(Ax, Ty, t), M(By, Sx, t)\} \\ \star \min\{M(Cz, Uz, t), M(By, Uz, t)M(Cz, Ty, t)\}, \quad (1)$$

*for all* $x, y, z \in X$ *and* $\phi : [0, 1] \to [0, 1]$, *such that* $\phi(t) > t$ *for all* $0 < t < 1$.

*Then there exist a unique common fixed point of* $A, B, C, S, T, U,$.

*Proof.* Let the pairs $\{A, S\}, \{B, T\}$ *and* $\{C, U\}$ be *owc*. So there are points $x, y, z \in X$ such that $Ax = Sx, By = Ty$ and $Cz = Uz$ we claim that $Ax = By = Cz$

If not, by inequality (1):

$$\begin{aligned} M(Ax, By, Cz, qt) \geq & \phi[\min\{M\{Sx, Ty, Uz, t), M(Sx, Ax, t), \\ & M(Ty, By, t), M(Uz, Cz, T) \\ & \star \min\{M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\} \\ & \star \min\{M(Cz, Uz, t), M(By, Uz, t), M(Cz, Ty, t)\} \\ = & \phi[\min\{M(Ax, By, Cz, t), M(Ax, Ax, t), M(By, By, t), \\ & M(Cz, Cz, t)\} \\ & \star \min\{M(By, By, t), M(Ax, By, t), M(By, Ax, t)\} \\ & \star \min\{M(Cz, Cz, t), M(By, Cz, t), M(Cz, By, t)\}] \\ = & \phi[M(Ax, By, Cz, t)] \\ > & M(Ax, By, Cz, t). \end{aligned}$$

Therefore $Ax = By = Cz$, i.e. $Ax = Sx = By = Ty = Cz = Uz$.

Suppose that there is another point $z_1$ such that $Bz_1 = Tz_1$. Then by inequality (1) we have $Bz_1 = Tz_1 = Cz_1 = Uz_1 = Az_1 = Sz_1$. So $By = Bz_1$ and $v = By = Ty$ is the unique point of coincidence of $B$ and $T$.

Using the results in [4] $v$ is the only common fixed point of $B$ and $T$.

Similarly there is a unique point $z_1 \in X$ such that $Cz_1 = Uz_1 = Az_1 = Sz_1$.

Assume that $v \neq z_1$. Then we have

$$\begin{aligned} M(v, z_1, z, qt) = & M(Av, Bz_1, Cz, qt) \\ \geqslant & \phi[\min\{M(Sv, Tz_1, Uz, t), M(Sv, Av, t)M(Tz_1, \\ & Bz_1, t), M(Uz, Cz, t)\} \\ & \star \min\{M(Bz_1, Tz_1, t), M(Av, Tz_1, t), M(Bz_1, Sv, t)\} \\ & \star \min\{M(Cz, Uz, t), M(Bz_1, Uz, t), M(Cz, Tz_1, t)\}] \\ = & \phi[\min\{M(v, z_1, z, t), M(v, v, t), M(z_1, z_1, t), M(z, z, t)\} \\ & \star \min\{M(z_1, z_1, t), M(v, z_1, t), M(z_1, v, t)\} \end{aligned}$$

$$
\begin{aligned}
&\star \min\{M(z,z,t), M(z_1,z,t), M(z,z_1,t)\}] \\
=&\phi[M(v,z_1,z,t)] \\
>&M(v,z_1,z,t).
\end{aligned}
$$

Therefore we have $v = z_1 = z$ then (see [4]) $z_1$ is a common fixed point of $A, B, C, S, T$and $U$.

The uniqueness of the fixed point holds from inequality (1). ☐

**Theorem 3.2.** *Let $(X, M, \star)$ be a complete fuzzy metric space and let $A, B, C, S, T$ and $U$ be self mappings of $X$.*

*Let the pair $\{A, S\}, \{B, T\}$ and $\{C, U\}$ be owc. If there exist $q \in (0, 1)$ such that*

$$
\begin{aligned}
M(Ax, By, Cz, qt) \geqslant &[\min\{M(Sx, Ty, Uz, t), M(Sx, Ax, t), \\
&M(By, Ty, t), M(Cz, Uz, t)\} \\
&\star \min\{M(Ax, Ty, t), M(By, Uz, t), M(By, Sx, t), \\
&M(Cz, Ty, t)\},
\end{aligned}
$$

*for all $x, y, z \in X$ and for all $t > 0$, then there exist a unique point $z_1 \in X$, such that*

$$Az_1 = Sz_1 = Bz_1 = Tz_1 = Cz_1 = Uz_1 = z_1.$$

*Then $v \in X$ is a unique point fixed such that $Av = Sv = Bv = Tv = Cv = Uv = v$, moreover $z_1 = v$ so that there is a unique common fixed point of $A, B, C, S, T$ and $U$,.*

*Proof.* Let the pairs $\{A, S\}, \{B, T\}$ *and* $\{C, U\}$ be *owc*. So there are points $x, y, z \in X$ such that $Ax = Sx, By = Ty$ and $Cz = Uz$ we claim that $By = Cz$.

If not, using inequality (2), we obtain:

$$
\begin{aligned}
M(Ax, By, Cz, qt) \geqslant &[\min\{M\{Sx, Ty, Uz, t), M(Sx, Ax, t), \\
&M(By, Ty, t), M(Cz, Uz, t)\} \\
&\star \min\{M(Ax, Ty, t), M(By, Uz, t), \\
&M(By, Sx, t)M(Cz, Ty, t)\}] \\
=&[\min\{M(Ax, By, Cz, t), M(Ax, Ax, t)M(By, By, t), \\
&M(Cz, Cz, t)\} \\
&\star \min\{M(Ax, By, t), M(By, Ax, t), M(By, Cz, t), \\
&M(Cz, By, t)\}]
\end{aligned}
$$

$$=[M(Ax, By, Cz, t)].$$

Therefore $Ax = By = Cz$, i.e. $Ax = Sx = By = Ty = Cz = Uz$. Suppose that there is another point $z_1$ such that $Bz_1 = Tz_1$.

Then by inequality (2) we have $Az_1 = Sz_1 = Bz_1 = Tz_1 = Cz_1 = Uz_1$. So $By = Bz_1$ and $v = By = Ty$ is the unique point of coincidence of $B$ and $T$ then (see [9]) $v$ is the only common fixed point of $B$ and $T$.

Similarly there is a unique point $z_1 \in X$ such that $z_1 = Az_1 = Sz_1 = Cz_1 = Uz_1$.

Assume that $v \neq z_1$ we have

$$\begin{aligned}
M(v, z_1, z, qt) =& M(Av, Bz_1, Cz, qt)\\
\geqslant& [\min\{M(Sx, Tz_1, Uz, t), M(Sx, Az_1, t), M(Bz_1, Tz_1, t),\\
& M(Cz, Uz, t)\}\\
& \star \min\{M(Ax, Tz_1, t), M(Bz_1, Uz, t), M(Bz_1, Sv, t),\\
& M(Cz, Tz_1, t)\}]\\
=& [\min\{M(v, z_1, z, t), M(v, z_1, t), M(z_1, z_1, t)M(z_1, z_1, t\}\\
& \star \min\{M(v, z_1, t), M(z_1, z, t), M(z_1, v, t), M(z, z_1, t)\}]\\
=& M(v, z_1, z, t) \quad \Rightarrow v = z = z_1.
\end{aligned}$$

**Theorem 3.3.** *Let $(X, M, \star)$ be a complete fuzzy metric space and let $A, B, C, S, T$ and $U$ be self mappings of $X$.*

*Let the pair $\{A, S\}, \{B, T\}$ and $\{C, U\}$ be owc. If there exist $q \in (0, 1)$ such that:*

$$\begin{aligned}
M(Ax, By, Cz, qt) \geqslant \phi\{& M(Sx, Ty, Uz, t), M(Sx, By, t), M(Ty, Cz, t), \}\\
& M(By, Ty, t), M(Cz, Uz, t)M(Ax, Ty, t)M(By, Uz, t)\}, \quad (2)
\end{aligned}$$

*for all $x, y, z \in X$ and $\phi : [0, 1]^4 \to [0, 1]$ such that $\phi(t, t, 1, t) > t$ for all $0 < t < 1$.*

*Then there exist a unique common fixed point of $A, B, C, S, T$, and $U$.*

*Proof.* Let the pairs $\{A, S\}, \{B, T\}$ *and* $\{C, U\}$ be *owc*.

So there are points $x, y, z \in X$ such that $Ax = Sx, By = Ty$ and $Cz = Uz$. We claim that $By = Cz$

By inequality (3) we have

$$\begin{aligned}
M(Ax, By, Cz, qt) \geqslant& \phi\{M\{Sx, Ty, Uz, t), M(Sx, By, t), M(Ty, Cz, t),\\
& M(By, Ty, t), M(Cz, Uz, t), M(Ax, Ty, t)M(By, Uz, t)\}
\end{aligned}$$

$$
\begin{aligned}
&=\phi\{M(Ax,By,Cz,t),M(Ax,By,t),M(By,Cz,t),\\
&\quad M(By,By,t),M(Cz,Cz,t),M(Ax,By,t)M(By,Cz,t)\}\\
&=\phi\{M(Ax,By,Cz,t),M(Ax,By,t)\\
&\quad M(By,Cz,t),1,1,M(Ax,By,t),M(By,Cz,t)\}\\
&>[M(Ax,By,Cz,t)].
\end{aligned}
$$

Therefore $Ax = By = Cz$, i.e. $Ax = Sx = By = Ty = Cz = Uz$. Suppose that there is another point $z_1$ such that $Bz_1 = Tz_1$.

Then by inequality (3) we have $Bz_1 = Tz_1 = Cz = Uz = Az_1 = Sz_1$. So $Cz = Cz_1$ and $v = Cz = Tz$ is the unique point of coincidence of $C$ and $T$ then (see [9]) $v$ is a unique common fixed point of $C$ and $U$.

Similarly there is a unique point $z_1 \in X$ such that $z_1 = Bz_1 = Tz_1$Thus $z_1$ is a common fixed point of $A, B, C, S, T$ and $U$.

The uniqueness of the fixed point holds from (3):

$$
\begin{aligned}
M(v,z_1,z,qt) =&M(Av,Bz_1,Cz,qt) \geqslant \phi\{M(Sv,Tz_1,Uz,t),\\
&M(Sv,Bz_1,t),M(Tz_1,Cz,t)\\
&M(Bz_1,Tz_1,t),M(Cz,Uz,t),M(Av,Tz_1,t),M(Bz_1,Uz,t)\}\\
=&\phi\{M(v,z_1,z,t),M(v,z_1,t),M(z_1,z,t)M(z_1,z,t),M(z,z,t),\\
&M(v,z_1,t),M(z_1,z,t)\}\\
=&M(v,z_1,z,t) \quad \Rightarrow v = z = z_1.
\end{aligned}
$$

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