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# DOUBLE HOPF BIFURCATION FOR AN HOPFIELD EURAL NETWORK MODEL WITH TIME DELAYED FEEDBACK

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**Abstract:** In this paper, we consider a system of delay differential equations which represents the general model of a Hopfield neural networks type. We focus on the case that the corresponding linear system has two pairs of purely imaginary eigenvalues at the trivial equilibrium, giving rise to double Hopf bifurcations. An analytical approach is used to find the explicit expressions for the critical values of the system parameters at which nonresonant or resonant double Hopf bifurcations may occur. We also investigate the occurrence of an double Hopf bifurcation about the trivial equilibrium.

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**Key Words:** Hopfield neural networks, Delayed differential equation, Double Hopf bifurcation

## 1. Introduction

The vast applications of Hopfield neural networks (HNNs) in many areas such as classification, associative memory, pattern recognition, parallel computations,

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and optimization [1-4,12], have been the focus of detailed studies by researchers. Time delay has important influences on the dynamical behavior of neural networks. It is well known that neural system may lose its stability by making the equilibrium point unstable even for very small delays. However, effect of time delay is an interesting problem. Marcus and Westervelt [6] first considered the effect of including discrete time delays in the connection terms to represent the time of propagation between neurons. They found out that the delay can destabilize the network as a whole and create oscillatory behavior.

The study of the local asymptotic stability of neural network models with multiple time delays are complex. In order to reach a deep and clear understanding of the dynamics of such models, most researchers have limited their study to the of models with a single delay [5,11]. In some papers, multiple delays are considered but there are no self-connection terms and moreover the systems with two delays have been generally investigated [8-9]. For example, Song, and Xu [8], investigated the stability of a two-neuron system with different time delay as follows:

$$\begin{cases} \frac{dx_1(t)}{dt} = -x_1 + a_1 S(x(t-\tau_1)) + a_2 S((t-\tau_2)) + P, \\ \frac{dx_2(t)}{dt} = -x_2 + a_3 S(x(t-\tau_2)) + a_4 S((t-\tau_1)) + Q. \end{cases}$$

They showed that multiple delays can lead the system dynamic behavior to exhibit stability switches and this system may undergo some bifurcations as double Hopf bifurcationat at certain values of the parameters. The model considered here is more general than the one in Song's studies and also the model considered in [10]. In fact, we have considered a Hopfield neural network with arbitrary neurons in which each neuron is bidirectionally connected to all others. Foe such a system, we have previously studied the occurrence of a Hopf bifurcation about the trivial equilibrium [7]. In this paper, the particular attention is focused on dynamics of the system in the vicinity of the critical point at which double Hopf bifurcation may occur. The critical parameter values are obtained explicitly using an analytical approach.

## 2. Local Analysis and Double Hopf Bifurcation

Consider the following delayed neural network described as:

$$\dot{u}_i(t) = -c_i u_i(t) + \sum_{j=1}^{n} a_{ij} f\left(u_j(t-\tau_j)\right), i = 1, 2, \ldots, n, \tag{1}$$

where $u_i(t)$ represents the activation state of $i$-th neuron $(i = 1, 2, \ldots, n)$ at time $t$, $a_{ij}$ is the weight of synaptic connections from $i$-th neuron to $j$-th neuron and $\tau_j \geq 0$ is the time delay. In system (1), each neuron is connected not only to itself but also to the other neuron via a non linear sigmoidal function $f$ which is a typical transmitting function among neurons. The initial value is assumed to be

$$u_i(\theta) = \varphi_i(\theta) \ for\ \theta \in [-k, 0],$$

where $\varphi_i(\theta) \in C([-k,0],\mathbb{R})$, $i = 1,2,\ldots,n$ and $k = max_{1\leq j\leq n}\tau_j$. The natural phase space for (1) is the the Banach space $C = C([-k, 0], \mathbb{R}^n)$ of continuous functions defined on $[-k, 0]$ equipped with the supremum norm

$$\|\varphi\| = sup_{-k\leq s\leq 0}\|\varphi(s)\|.$$

Suppose $f:\mathbb{R} \to \mathbb{R}$ is continuous, then the solutions of equation (1) define the continuous semiflow

$$\begin{array}{l}\Phi:\ \mathbb{R}^+ \times C \mapsto C \\ (t,\varphi) \mapsto x_t^{\varphi}\end{array}.$$

A function $\widehat{\xi} \in C$ is an equilibrium point (or stationary point) of $\Phi$ if $\widehat{\xi}(s) = (\xi_1, \xi_2, \ldots, \xi_n)$ for all $-k \leq s \leq 0$, satisfying $-c_i\xi_i + \sum_{j=1}^{n} a_{ij} f(\xi_j) = 0$, $i = 1,2,\ldots,n$.

Suppose that $f \in C^1(\mathbb{R})$, $f(0) = 0$ and $uf(u) > 0$ for $u \neq 0$. It is clear that the origin of the state space is a stationary point of system (1). For stability analysis, the system (1) has been linearized about the origin of state space and the following system of linearized equations is obtained:

$$\dot{u}_i(t) = -c_i u_{ii}(t) + \sum_{j=1}^{n} \alpha_{ij} u_j(t - \tau_j), \quad i = 1,2,\ldots,n. \tag{2}$$

where $\alpha_{ij} = a_{ij} f'(0), i, j = 1,2,\ldots,n$. The associated characteristic equation of system (2) is as follows:

$$\begin{vmatrix} \lambda + c_1 - \alpha_{11}e^{-\lambda\tau_1} & -\alpha_{12}e^{-\lambda\tau_2} & \cdots & -\alpha_{1n}e^{-\lambda\tau_n} \\ -\alpha_{21}e^{-\lambda\tau_1} & \lambda + c_2 - \alpha_{22}e^{-\lambda\tau_2} & \cdots & -\alpha_{2n}e^{-\lambda\tau_n} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1}e^{-\lambda\tau_1} & -\alpha_{n2}e^{-\lambda\tau_2} & \cdots & \lambda + c_n - \alpha_{nn}e^{-\lambda\tau_n} \end{vmatrix} = 0. \tag{3}$$

The zero solution of system (1)is stable if and only if all roots $\lambda$ of characteristic equation (1) have negative real parts. In this paper, we shall find some conditions which ensure that all roots of characteristic equation (1) have negative real parts. The characteristic equation of the linearized system (1) about the origin of state space is a transcendental equation involving exponential functions and it is difficult to find all value of parameter $\tau$ such that all the characteristic roots have negative real parts. If $c_3 = c_4 = \ldots c_n = 1$ and $A, B, \Lambda$ are defined as

$$A = \begin{pmatrix} G & E \\ 0 & H \end{pmatrix}_{n\times n}, B= \begin{pmatrix} e^{-\lambda\tau_3} & 0 & 0 & 0 \\ 0 & e^{-\lambda\tau_4} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda\tau_n} \end{pmatrix}_{(n-2)\times(n-2)}, \Lambda= \lambda+1.$$

Here

$$G = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}_{2\times 2}, \ H= \begin{pmatrix} \alpha_{33} & \alpha_{34} & \cdots & \alpha_{3n} \\ \alpha_{43} & \alpha_{44} & \cdots & \alpha_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}_{(n-2)\times(n-2)}$$

then, the characteristic equation (3) can be written as the following equation:

$$det \begin{pmatrix} \lambda+c_1-\alpha_{11}e^{-\lambda\tau_1} & -\alpha_{12}e^{-\lambda\tau_2} \\ -\alpha_{21}e^{-\lambda\tau_1} & \lambda+c_2-\alpha_{22}e^{-\lambda\tau_2} \end{pmatrix} det\left(\Lambda I_{(n-2)\times(n-2)}-HB\right) = 0. \quad (4)$$

Now, motivating of Leverrier's method, we propose the following formula:

$$det\,(\Lambda I_{m\times m}-HB) \ = \ \Lambda^m+h_1\Lambda^{m-1}+h_2\text{m}+\ldots+h_{m-1}\Lambda+h_m= 0, \qquad (5)$$

where $h_k=-\frac{1}{k}\left(S_k+S_{k-1}h_1+\ldots+S_1h_{k-1}\right), S_k=tr((HB)^k)$; for $k= 1, 2,\ldots, m$ and $m = n - 2$

Suppose that $\tau_k=\tau$, $\overline{S}_k=tr\left(H^k\right)$ and $\overline{h}_k=-\frac{1}{k}\left(\overline{S}_k+\overline{S}_{k-1}\overline{h}_1+\ldots+\overline{S}_1\overline{h}_{k-1}\right)$ hold for $k= 1, 2,\ldots, m$, then it is clear that matrix $B$ will have the following form:

$$B= \begin{pmatrix} e^{-\lambda\tau} & 0 & 0 & 0 \\ 0 & e^{-\lambda\tau} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\lambda\tau} \end{pmatrix} =e^{-\lambda\tau}I_{m\times m}$$

and

$$S_k=tr\left((HB)^k\right)=tr\left(\left(e^{-\lambda\tau}H\right)^k\right)=e^{-k\lambda\tau}\overline{S}_k,$$

$$h_k = -\frac{1}{k} e^{-k\lambda\tau}\left(\overline{S}_k + \overline{S}_{k-1}\overline{h}_1 + \ldots + \overline{S}_1\overline{h}_{k-1}\right) = e^{-k\lambda\tau}\overline{h}_k, k = 1, 2, \ldots, m.$$

Therefore, if $\tau_k = \tau$ for $k = 1, 2, \ldots, m$ it is easy to verify that the equation (5) can be simplified to

$$\begin{aligned} det\left(\Lambda I_{m\times m} - HB\right) &= \Lambda^{mn} + \overline{h}_1 e^{-\lambda\tau}\Lambda^{m-1} + \overline{h}_2 e^{-2\lambda\tau}\Lambda^{m-2} + \ldots + \\ &\qquad \overline{h}_{m-1} e^{-(m-1)\lambda\tau}\Lambda + e^{-m\lambda\tau}\overline{h}_m = 0 \quad . \qquad (6) \end{aligned}$$

Suppose that $\beta_1, \beta_2$ , $, \beta_m$ are the eigenvalues of the matrix $H$ and $\mathcal{P}_j(\lambda, \tau) = \lambda + 1 - \beta_j e^{-\lambda\tau}$, then it is easy to see that Formula (6) can be rewritten as follows:

$$\begin{aligned} \mathcal{P}(\lambda, \tau) = det\left(\Lambda I - AB\right) &= \left(\Lambda - \beta_1 e^{-\lambda\tau}\right)\left(\Lambda - \beta_2 e^{-\lambda\tau}\right)\ldots\left(\Lambda - \beta_m e^{-\lambda\tau}\right) \\ &= \left(\lambda + 1 - \beta_1 e^{-\lambda\tau}\right)\left(\lambda + 1 - \beta_2 e^{-\lambda\tau}\right)\ldots\left(\lambda + 1 - \beta_m e^{-\lambda\tau}\right) \\ &= \mathcal{P}_1(\lambda, \tau)\mathcal{P}_2(\lambda, \tau)\ldots\mathcal{P}_m(\lambda, \tau) = 0. \qquad (7) \end{aligned}$$

Having applied Formula (7), the sufficient conditions for local stability of system (1) is obtained.

**Theorem 2.1.** *Suppose that the eigenvalues $\beta_1, \beta_2$ , $, \beta_m$ of the matrix $A$ be real, i.e.*

$$\beta_j = a_j \quad , \quad a_j \in \mathbb{R} \quad , \qquad j = 1, 2, \ldots, m.$$

*If* $\max_{j=1,2,\ldots,n}|a_j| < 1$, *then for any arbitrary amount of* $\tau$, ($\tau \geq 0$), *all the roots of equation* (7) *have negative real part.*

*Proof.* We suppose that $\lambda = \mu + i\omega$ be a root of characteristic equation (7). It is easy to see that $\lambda$ is a root of (7) if and only if $\lambda$ satisfies

$$\mathcal{P}(\lambda, \tau) = \mathcal{P}(\mu + i\omega, \tau) = \mathcal{P}_1(\mu + i\omega, \tau)\mathcal{P}_2(\mu + i\omega, \tau)\ldots\mathcal{P}_n(\mu + i\omega, \tau) = 0.$$

Therefore, there must exist some $j$ $(1 \leq j \leq n)$ , such that

$$\mathcal{P}_j(\mu + i\omega, \tau) = \mu + i\omega + 1 - a_j e^{-(\mu + i\omega\ \ )\tau} = \ \ 0. \qquad (8)$$

Let $R_j(\mu, \tau)$ and $I_j(\mu, \tau)$ be the real and imaginary parts of (8) respectively, we have

$$R_j(\mu, \tau) = \mu + 1 - a_j e^{-\mu\tau} cos(\omega\tau) = 0. \qquad (9)$$

$$I_j(\mu,\tau) = \omega + a_j e^{-\mu\tau} sin(\omega\tau) = 0. \tag{10}$$

To prove the theorem, see [7].

**Theorem 2.2.** *Let $H_{m\times m} = (\alpha_{ij})$ be a real matrix consisting of the coefficient of system* (2) *with $p$ real eigenvalues $\beta_j = a_j \;\; , j = 1, \ldots, p$ and $2q$ complex eigenvalues ${\beta_j}^{\pm} = a_j \pm ib_j$, $j = p+1, \ldots, m-q$.*

*If $max_{j=1,2,\ldots,p} |a_j| < 1$ and $max_{j=p+1,\ldots,m-q} \{|a_j| + |b_j|\} < 1$, then for any arbitrary amount of $\tau$, ($\tau \geq 0$), all the roots of characteristic equation* (7) *have negative real part.*

*Proof.* In this case, Formula (7) will be transformed to the following form:

$$\mathcal{P}(\lambda,\tau) = \mathcal{P}_1(\lambda,\tau)\ldots\mathcal{P}_p(\lambda,\tau)\,{\mathcal{P}_{p+1}}^{\pm}(\lambda,\tau)\ldots{\mathcal{P}_{m-q}}^{\pm}(\lambda,\tau) = 0. \tag{11}$$

Such that

$$\mathcal{P}_j(\lambda,\tau) = \lambda + 1 - a_j e^{-\lambda\tau} = \;\; 0, \quad j = 1, \ldots, p,$$

$${\mathcal{P}_j}^{\pm}(\lambda,\tau) = \lambda + 1 - (a \pm ib)\, e^{-\lambda\tau} = 0, \quad j = p+1, \ldots, m-q.$$

Let $\lambda = \mu + i\omega$ be a root of characteristic equation (11). It is easy to see that $\lambda$ is a root of (11) if and only if $\lambda$ satisfies

$$\mathcal{P}(\lambda,\tau) = \mathcal{P}_1(\mu+i\omega,\tau)\ldots\mathcal{P}_p(\mu+i\omega,\tau)\,\mathcal{P}^{\pm}_{p+1}(\mu+i\omega,\tau)\ldots\mathcal{P}^{\pm}_{m-q}(\mu+i\omega,\tau) = 0.$$

Therefore, there must exist some $1 \leq j \leq p$, such that

$$\mathcal{P}_j(\mu+i\omega,\tau) = \mu + i\omega + 1 - \beta_j e^{-(\mu+i\omega\;\;)\tau} = 0$$

or there must exist some $p+1 \leq j \leq n-q$ such that

$${\mathcal{P}_j}^{\pm}(\lambda,\tau) = \mu + i\omega + 1 - (a_j \pm ib_j)\, e^{-(\mu+i\omega\;\;)\tau} = 0.$$

To prove the theorem, see [7].

Let $\tau_k = \tau$ for $k = 1, 2, \ldots, n$, and the conditions of theorem 2.2 are satisfied for the second part of characteristic equation (4). The first part of characteristic equation (4) can be rewritten in the form

$$det\begin{pmatrix} \lambda + c_1 - \alpha_{11}e^{-\lambda\tau} & -\alpha_{12}e^{-\lambda\tau} \\ -\alpha_{21}e^{-\lambda\tau} & \lambda + c_2 - \alpha_{22}e^{-\lambda\tau} \end{pmatrix}$$

$$= \lambda^2 - \lambda\left(\tilde{\mathrm{c}} + \mathrm{d}\mathrm{e}^{-\lambda\mathrm{t}}\right) + \mathrm{c}_{\mathrm{d}}\mathrm{e}^{-\lambda\mathrm{t}} + \hat{c} + det(G)\;\mathrm{e}^{-2\lambda\mathrm{t}} = 0, \tag{12}$$

Where

$$\widetilde{c}=c_1+c_2,\ \ d_1=\alpha_{11}+\alpha_{22},\ \ \hat{c}=c_1c_2,\ \ c_d=c_2\alpha_{11}+c_1\alpha_{22}.$$

The stability of the trivial equilibrium point will change when the system under consideration has zero or a pair of imaginary eigenvalues. Under the conditions of Theorem 2.2, the former occurs if $\lambda=0$ in equation (12) or $c_d+\hat{c}+det\,(G)\ =0$, which can lead to the static bifurcation of the equilibrium points such that the number of equilibrium points changes when the bifurcation parameters vary. The latter deals with the Hopf bifurcation such that the dynamical behavior of the system changes from a static stable state to a periodic motion or vice versa. The dynamics becomes quite complicated when the system has two pairs of pure imaginary eigenvalues at a critical value of time delay. We will concentrate on such cases. For this, we let $c_d+\hat{c}+det\,(G)\ \neq 0$ . Thus, $\lambda=0$ is not a root of the characteristic equation (12) in the present paper. Such assumption can be realized in engineering as long as one chooses a suitable feedback controller.

Let $\det\,(D)=0$ but $c_d+c_2\neq 0$, substituting $\lambda=a+i\omega$ into (12), and equating the real and imaginary parts to zero yields

$$a^2-\omega^2-a\widetilde{c}+\hat{c}-e^{-at}\omega\sin\,(\omega\tau)\ d_1+e^{-at}\cos\,(\omega\tau)\ \ (c_d-ad_1)=0\ ,$$

$$2a\omega-\omega\widetilde{c}-e^{-at}\omega\cos\,(\omega\tau)\ d_1+e^{-at}\sin\,(\omega\tau)\ \ (ad_1-c_d)=0. \tag{13}$$

One can derive the explicit expressions for the critical stability boundaries by setting $a=0$ in equation (13) and obtain

$$-\omega^2+\hat{c}+c_d\cos\,(\omega\tau)\ -\omega d_1\sin\,(\omega\tau)\ =0,$$

$$-\omega c_1-c_d\sin\,(\omega\tau)\ -\omega d_1\cos\,(\omega\tau)\ =0. \tag{14}$$

Eliminating $t$ form equation (14), we have

$$\omega_{\pm}=\frac{\sqrt{{d_1}^2-\widetilde{c}^2+2\hat{c}\pm\sqrt{({d_1}^2-\widetilde{c}^2+2\hat{c})^2-4(\hat{c}^2-{c_d}^2)}}}{\sqrt{2}} \tag{15}$$

When the following conditions hold :

$$c_2^2-c_d^2>0,$$

$$({d_1}^2-\widetilde{c}^2+2\hat{c})^2>4(\hat{c}^2-{c_d}^2). \tag{16}$$

Then , two families of surfaces , denoted by $t_-$ and $t_+$ in terms of $c_d$ and $d_1$ corresponding to $\omega_-$ and $\omega_+$ respectively, can be derived from equation (13) and be given by

$$\cos(\omega_-\tau_-) = \frac{{\omega_-}^2 c_d - \hat{c}c_d - {\omega_-}^2 \tilde{c} d_1}{{c_d}^2 + \omega^2 - {d_1}^2}$$

$$\cos(\omega_+\tau_+) = \frac{{\omega_+}^2 c_d - \hat{c}c_d - {\omega_+}^2 \tilde{c} d_1}{{c_d}^2 + \omega^2 + {d_1}^2} \tag{17}$$

It should be noted that $\omega_- < \omega_+$. Thus, a possible double Hopf bifurcation point occurs when two such families of surfaces intersect each other where

$$\tau_- = \tau_+ \tag{18}$$

Equation (18) not only determines the linearized system around the trivial equilibrium which has two pairs of pure imaginary eigenvalues $\pm\omega_-$ and $\pm\omega_+$, but also gives a relation between $\omega_-$ and $\omega_+$. If

$$\omega_- : \omega_+ = k_1 : k_2, \tag{19}$$

then a possible double Hopf bifurcation point appears with frequencies in the ratio $\mathrm{k}_1 : \mathrm{k}_2$. If $k_1, k_2 \in Z^+$ , $k_1 < k_2$, $k_1 \neq 0$ and $k_2 \neq 0$, then such point is called the $k1 : k2$ weak or no low-oder resonant double Hopf bifurcation point. Equations (18) and (19) form the necessary conditions for the occurrence of a resonant double Hopf bifurcation point. Equation (19) yields

$${d_1}^2 = {c_1}^2 - 2c_2 + \frac{{k_1}^2 + {k_2}^2}{k_1 k_2}\sqrt{{c_2}^2 - {c_d}^2} \tag{20}$$

if conditions (14) are satisfied. Substituting (21) into (15), one can obtain the frequencies in the simple expressions given by

$$\omega_- = \sqrt{\frac{k_2}{2}\sqrt{{c_2}^2 - {\mathrm{d}}^2}}, \qquad \omega_+ = \sqrt{\frac{k_2}{k_1}\sqrt{{c_2}^2 - {c_\mathrm{d}}^2}} \tag{21}$$

The other parameters can be determined by equation (18) or the following equation

$$\arccos\left(\frac{-(c_2 c_d k_2) + \sqrt{{c_2}^2 - {c_d}^2}(c_d - c_1 d_1)k_1}{{c_d}^2 k_2 + \sqrt{{c_2}^2 - {c_d}^2}\,{d_1}^2 k_1}\right)$$

$$= \frac{k_1}{k_2}\arccos\left(\frac{-(c_2 c_\mathrm{d} k_1) + \sqrt{{c_2}^2 - {c_d}^2}(\mathrm{c}_d - c_1 d_1)k_2}{{c_d}^2 \mathrm{k}_1 + \sqrt{{c_2}^2 - {c_d}^2}\,{d_1}^2 k_2}\right). \tag{22}$$

Here $d_1$ is given in equation (21). the corresponding value of the time delay at the resonant double Hopf bifurcation point is given by

$$\tau_c = \tau_- = \tau_+$$

$$= \sqrt{\frac{k_1}{k_2\sqrt{{c_2}^2 - {c_d}^2}}} \arccos\left(\frac{-(c_2 c_d k_1) + \sqrt{{c_2}^2 - {c_d}^2}(c_d - c_1 d_1)k_2}{{c_d}^2 k_1 + \sqrt{{c_2}^2 - {c_d}^2}\ {\mathrm{d}_1}^2 k_2}\right).$$

These expressions will be used in the next section.

## 3. Numerical Simulation

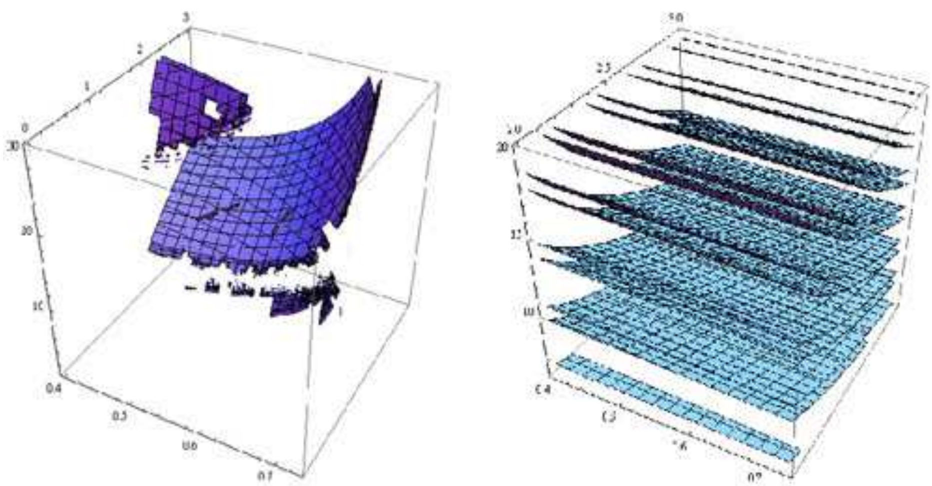

Figure 1: Two families of surfaces, $\tau_-$(left) and $\tau_+$(right)in terms of $c_d$ and $d_1$ given by equation (17).

We present some numerical simulation to verify the system solutions. We consider system in Section 2 with $n = 2$ and $f(x) = tanh(x)$ . Note that $\alpha_{ij} = a_{ij} f'(0) = a_{ij}$ and the parameters were chosen as: $c_1 = 0.8000$ and $c_2 = 0.9000$. This gives $\widetilde{c} = 1.7000$ and $\widehat{c} = 0.72$. The parameters $d_1$ and $c_d$ are considered as the variable parameters. Figure 1 shows tow families of surfaces, denoted by $\tau_-$ and $\tau_+$ in terms of $d_1$ and $c_d$ corresponding to $\omega_-$ and $\omega_+$ respectively, given in equations (17). Figure 1 shows that a possible double Hopf bifurcation point occurs because two families of surfaces intersect each other where $\tau_- = \tau_+$.

It should be noted that the parameters $d_1$ and $c_d$ cannot be solved in a closed form from equation (17) due to the trigonometric function. However, as an illustrative example, we consider a specific system with the fixed parameters $c_1 = 0.8000$, $c_2 = 0.9000$, $\alpha_{12} = 4$, $\alpha_{21} = -10.2013$ and $\alpha_{22} = -6.3520$. Time histories of the typical behaviors near the double Hopf bifurcation point are illustrated in Figure 2. Firstly, two parameters $\tau$ and $\alpha_{11}$ are fixed as $\tau = 6$ and $\alpha_{11} = 6.2850$. The numerical solution of system (1) is shown in Figure 2(B1). It shows the origin is not stable. With the increasing of parameter $\alpha_{11}$, the numerical simulation suggests that the system behavior moves to a two-frequency quasi-periodic state, as shown in Figure 2(B2). The time histories clearly shows the modulation of the peak intensities, which is also called as 2-torus. With the increasing of parameter $\alpha_{11}$, the trajectory asymptotically converges to the origin. It implies the point is locally asymptotically stable. The numerical solution is shown in Figure 2 (B3). Figure 2 (B4,B5) respectively show the stable state and unstable state of non periodical solutions.

As mentioned above, we kept on choosing the parameters as $c_1 = 0.8000$, $c_2 = 0.9000$, $\alpha_{12} = 4$, $\alpha_{21} = -10.2013$ and $\alpha_{22} = -6.3520$, respectively. Firstly, two parameters $\tau$ and $\alpha_{11}$ are fixed as $\tau = 1.1500$ and $\alpha_{11} = 6.2040$. With the increasing of delay $\tau$, the equilibrium point loses its stability in term of the supercritical Hopf bifurcation. The system periodically oscillates with the second frequency, as shown in Figure 3.

## 4. Conclusion

In neural system, action potential plays a crucial role in many information communications. To understand the information representation, many mathematical models are proposed and the mechanism of information processing is investigated by using the analytical method of nonlinear dynamics. The quiescent state, periodic spiking, quasiperiodic behavior and bursting activity are all the important biological behavior with the different neuro-computational properties. It is well known that time delay is an inevitable factor in the signal transmission between biological neurons or electronic-model-neurons. Neural systems with time delays have very rich dynamical behaviors. In this paper, we have studied the stability and numerical solutions of a Hopfield delayed neural network system which is more general than the models applied by earlier researchers. In fact, our focus here is on a Hopfield neural network with arbitrary neurons in which each neuron is bidirectionally connected to all the others. The system exhibits the double Hopf bifurcation points due to the two pairs of

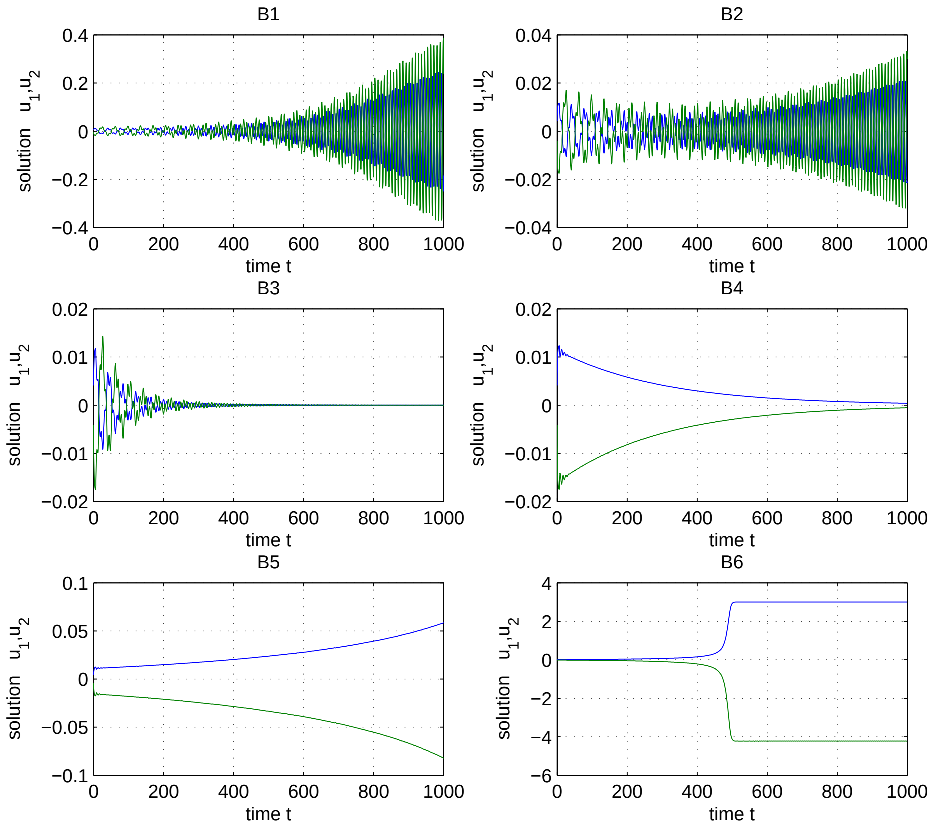


Figure 2: Time histories of the numerical simulation for the dynamical behavior in system (1) near the point of the double Hopf bifurcation, where $\alpha_{11}$ is fixed as B1(6.2850), B2(6.2920), B3(6.3100), B4(6.4240), B5(6.4280), B6(6.4320) respectively. The other parameters are chosen as $c_1 = 0.8000$, $c_2 = 0.9000$, $\alpha_{12} = 4$, $\alpha_{21} = -10.2013$, $\alpha_{22} = -6.3520$ and $\tau = 6$.

imaginary eigenvalues appearing on the margin of stability regions simultaneously. The system may exhibit the equilibrium solution, periodic solutions with the different frequencies of the Hopf bifurcations, and quasi-periodic solutions. Numerical results are given to illustrate that the double Hopf bifurcation is an interaction of the supercriticalsupercritical Hopf bifurcations.

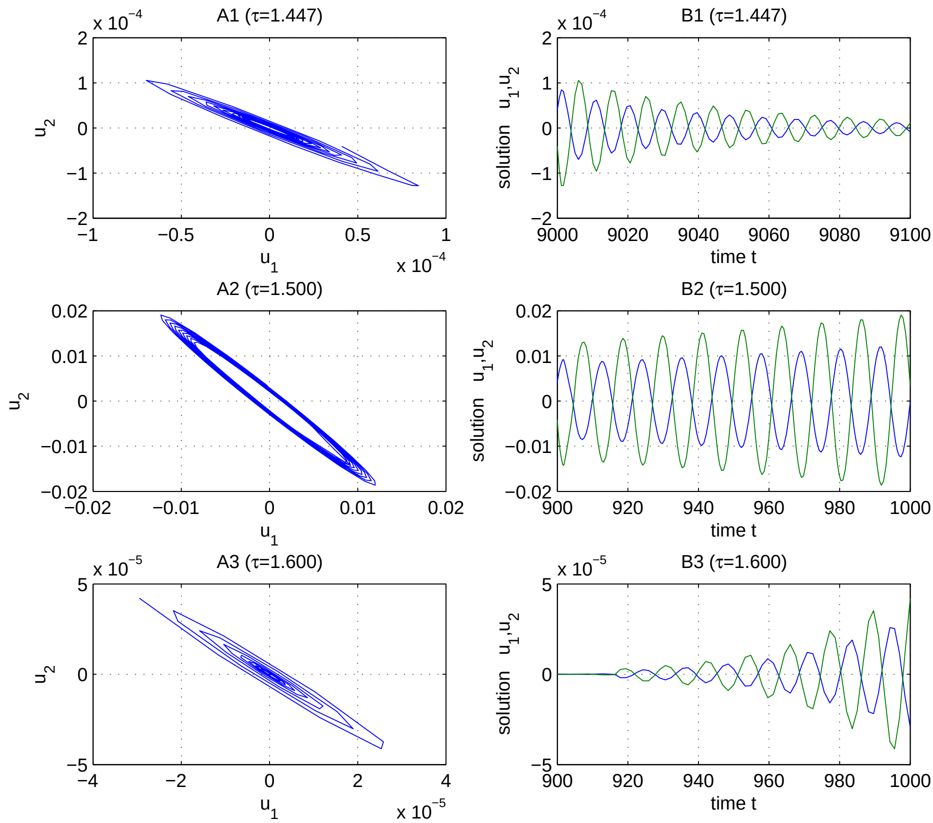


Figure 3: Time histories of the numerical simulation for the dynamical behavior in system (1), where $\alpha_{11} = 6.2040$ and $\tau$ is fixed as A1,B1(1.447), A1,B1(1.500), A1,B1(1.600), respectively. The other parameters are chosen at the same as those in Figure 2

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