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FARTHEST POINTS IN HILBERT OPERATOR SPACES WITH APPLICATIONS

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Abstract: The purpose of this paper is to Provide conditions for the existence of farthest points of closed and bounded subsets of Hilbert operator spaces. This will done by applying the concept of numerical range. We give, inter alia, some results to characterize farthest points of a subset of a C^*-algebra $\mathbb{A}$ from a fixed element $x \in \mathbb{A}$. Meanwhile, we point out the main theorems of R. Saravanan and R. Vijayaragavan[11] are incorrect, by given two counterexamples.

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1. Introduction

The problem of farthest points in normed linear spaces has been studied by Singer, Franchetti and T.D. Narang, in [4, 9]. They give some results on characterization and existence of farthest points in normed linear spaces in terms of bounded linear functionals. Also several results related with farthest points in the context of normed linear space and metric space can be obtained in[1, 3, 5, 6, 10, 11].Saravanan and R. Vijayaragavan [11] characterize farthest points from bounded sets with respect to uniform norm. We first (see Section 2) give some preliminary concepts and definitions on C^*-algebras. In Section 3 the existence of farthest points will be discussed. We also present the results

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concerning strongly farthest points. Then we characterize farthest points of a C^*-algebra $\mathbb{A}$ and state its applications in C^*-algebra $C(X)$. Finally we give two counterexamples for some results of [11].

2. Preliminaries

We recall that a complex algebra is a vector space $\mathbb{A}$ over the complex field $\mathbb{C}$ in which a multiplication is defined with $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$ which satisfies

$$x(yz) = (xy)z, \tag{2.1}$$

$$(x + y)z = xz + yz, \ x(y + z) = xy + xz \tag{2.2}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{2.3}$$

for all x, y and z in $\mathbb{A}$ and all scalars α. If in addition, $\mathbb{A}$ is a Banach space with respect to a norm which satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\| \ (x, y \in \mathbb{A}), \tag{2.4}$$

in this case the pair $(\mathbb{A}, \|.\|)$ is called a normed algebra. If $\mathbb{A}$ contains an element e such that $\|e\| = 1$ and

$$xe = ex = x \ (x \in \mathbb{A}), \tag{2.5}$$

then $\mathbb{A}$ is called a unital normed algebra. A complete unital normed algebra is called unital Banach algebra.
$a \in \mathbb{A}$ is said to be invertible if there is an element b in $\mathbb{A}$ such that $ab = ba = e$. The set of all invertible elements of $\mathbb{A}$, is denoted by $Inv(\mathbb{A})$. In fact

$$Inv(\mathbb{A}) := \{a \in \mathbb{A} : \ a \ is \ invertible\}. \tag{2.6}$$

An involution $*$ on an algebra $\mathbb{A}$ is a mapping $x \longrightarrow x^*$ from $\mathbb{A}$ onto $\mathbb{A}$ such that $(\lambda x + y)^* = \overline{\lambda} x^* + y^*, (xy)^* = y^* x^*$ and $(x^*)^* = x$, for all $x, y \in \mathbb{A}, \lambda \in \mathbb{C}$. An involutive Banach algebra is called a Banach $*$-algebra. A Banach $*$-algebra $\mathbb{A}$ is said to be a C^*-algebra if $\|xx^*\| = \|x\|^2$, for each $x \in \mathbb{A}$.
In this paper, H will denote a Hilbert space and $B(H)$ will denote the bounded linear maps on H. It is routine to show that $B(H)$ is a unital Banach-algebra with the pointwise-defined operations for addition and scalar multiplication, multiplication given by $(u, v) \to uov$, and norm is the well known operator norm,

$$\|u\| = \sup_{\|x\|=1} \|u(x)\|.$$

We recall that $B(H)$ is a C^*-algebra relative to involution $T \to T^*$ defined by:

$$\langle x, Ty\rangle = \langle T^*x, y\rangle \; for \; x, y \in H.$$

Let X be a compact space, then $C(X)$, the space of all continuous functions on X, is a C^*-algebra. For $f \in C(X)$, the uniform norm is defined by

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\|.$$

This norm is also called L_∞-norm or supremum norm.

3. Existence of Farthest Points in $B(H)$, C^*-Algebras and $C(X)$

Let X be a normed space. For a nonempty bounded subset B of X and $x \in X$, we define

$$\mathbf{S}_B(x) = \underset{g \in B}{sup} \|x - g\|. \tag{3.1}$$

An element $g_0 \in B$ is called a farthest-point to x from B if

$$\|x - g_0\| = \mathbf{S}_B(x). \tag{3.2}$$

The set of all farthest points to x from B is denoted by $\mathbf{F}_B(x)$. In fact

$$\mathbf{F}_B(x) := \{y \in B | \; \|x - y\| = \mathbf{S}_B(x)\}. \tag{3.3}$$

Let $f, g \in B(H)$. We denote by Z_f the following set:

$$Z_f := \{\{x_n\} \in H : \|x_n\| = 1, \; \lim_{n \to \infty} \|f(x_n)\| = \|f\|\}. \tag{3.4}$$

The numerical range of g^*f relative to f which is denoted by $W(g^*f)$ is defined as follows:

$$W(g^*f) := \{\lambda \in \mathbb{C} : \lambda = \lim_{n \to \infty} \langle g^*f(x_n), x_n\rangle, \;\; \{x_n\} \in Z_f\}. \tag{3.5}$$

It is well Known that $W(g^*f)$ is a compact convex subset of the complex plane [7].
Let A be a subset of $\mathbb{C}$. Then ReA, is defined by

$$ReA := \{Re(z) : \; z \in A\}.$$

Theorem 3.1. *Let B be a bounded subset of $B(H)$, $f \in B(H) \setminus B$ and $g_0 \in B$. The following statements are true:*

i) $g_0 \in \mathbf{F}_B(f)$, *if* $\min ReW((g_0 - h)^*(f - h)) \leq 0$. $(\forall h \in B)$

ii) If $g_0 \in \mathbf{F}_B(f)$ *then* $ReW((g_0 - h)^*(f - g_0)) \leq 0$. $(\forall h \in B)$.

Proof. i) We assume $\min ReW((g_0 - h)^*(f - h)) \leq 0$ for each $h \in B$. Thus for each $h \in B$, $lim_{n\to\infty} Re\langle (f - h)(x_n^h), (g_0 - h)(x_n^h)\rangle \leq 0$, for some $\{x_n^h\}_{n\in\mathbb{N}} \in Z_{f-h}$. Now we have

$$\begin{aligned}
\|f - g_0\|^2 &\geq \lim_{n\to\infty} \|(f - g_0)(x_n^h)\|^2 \\
&= \lim_{n\to\infty} \|(f - h)(x_n^h) - (g_0 - h)(x_n^h)\|^2 \\
&= \lim_{n\to\infty} [\|(f - h)(x_n^h)\|^2 + \|(g_0 - h)(x_n^h)\|^2 \\
&- 2Re\langle (f - h)(x_n^h), (g_0 - h)(x_n^h)\rangle] \\
&\geq \lim_{n\to\infty} \|(f - h)(x_n^h)\|^2 = \|f - h\|^2.
\end{aligned}$$

Hence $\|f - h\| \leq \|f - g_0\|$ for each $h \in B$. This shows $g_0 \in \mathbf{F}_B(f)$.
ii)Let $g_0 \in \mathbf{F}_B(f)$ but for some $h \in B$ there exists $\{x_n^{g_0}\}_{n\in\mathbb{N}} \in Z_{f-g_0}$ such that $\lim_{n\to\infty} Re\langle (f - g_0)(x_n^{g_0}), (g_0 - h)(x_n^{g_0})\rangle > 0$. Therefore

$$\lim_{n\to\infty} Re\langle (f - g_0)(x_n^{g_0}), (g_0 - f)(x_n^{g_0})\rangle > -\lim_{n\to\infty} Re\langle (f - g_0)(x_n^{g_0}), (f - h)(x_n^{g_0})\rangle,$$

and

$$\begin{aligned}
\|(f - g_0)(x_n^{g_0})\|^2 &< \lim_{n\to\infty} Re\langle (f - g_0)(x_n^{g_0}), (f - h)(x_n^{g_0})\rangle \\
&\leq \lim_{n\to\infty} |\langle (f - g_0)(x_n^{g_0}), (f - h)(x_n^{g_0})\rangle| \\
&\leq \|(f - g_0)(x_n^{g_0})\|\|(f - h)(x_n^{g_0})\|.
\end{aligned}$$

This shows $\|f - g_0\| < \|f - h\|$, which is inconsistent with $g_0 \in \mathbf{F}_B(f)$. □

Let U be a bounded subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. For each $h \in H$ put $v_h := (g_0 - h)^*(f - h)$. The following Corollary follows immediately from the Theorem 3.1.

Corollary 3.2. *Let U be a bounded subset of $B(H)$, $f \in B(H) \setminus U$ and $g_0 \in U$. If $v_h - t\ id_H$ be an invertible element of $B(H)$ such that $\|(v_h - t\ id_H)^{-1}\| \leq t^{-1}$ for each $t > 0$ and $h \in U$, then $g_0 \in \mathbf{F}_U(f)$.*

Proof. Let $\|(v_h - t\, id_H)^{-1}\| \leq t^{-1}$, then $t\|x\| \leq \|(v_h - t\, id_H)(x)\|$, for each $x \in H$. Thus we have $t^2\|x_n\|^2 \leq \|(v_h - t\, id_H)(x_n)\|^2$, for $\{x_n\} \in Z_{f-h}$. Hence

$$t^2\|x_n\|^2 \leq \|v_h(x_n)\|^2 + t^2\|x_n\|^2 - 2tRe\langle v_h(x_n), x_n\rangle,$$

or

$$Re\langle v_h(x_n), x_n\rangle \leq (2t)^{-1}\|v_h(x_n)\|^2.$$

Letting $t \to \infty$, we obtain $Re\langle v_h(x_n), x_n\rangle \leq 0$. The later holds for all $\{x_n\} \in Z_{f-h}$. We have $ReW((g_0 - h)^*(f - h)) \leq 0$. By part (i) of Theorem 3.1, $g_0 \in \mathbf{F}_U(f)$. This completes the proof. □

Definition 3.3. Let B be a bounded subset of a normed linear space X and $x \in X$. An element $g_0 \in B$ is called a strongly uniqe farthest point to x from B of order α, if there exists a constant $K_x > 0$ such that for every $g \in B$,

$$\|x - g_0\|^\alpha \geq \|x - g\|^\alpha + K_x\|g - g_0\|^\alpha.$$

In case $\alpha = 1$, g_0 is called only a strongly uniqe farthest point to x from B.

In the following we introduce existence and uniqueness of farthest point in $B(H)$.

Theorem 3.4. *Let B be a bounded subset of $B(H)$, $g_0 \in B$ and $f \in B(H)$. If there exists a constant $K_f > 0$ such that*

$$\min ReW((g_0 - h)^*(f - h)) < -K_f\|h - g_0\|^2, \quad (\forall h \in B) \tag{3.6}$$

then g_0 is a strongly uniqe farthest point to f from B of order 2.

Proof. Suppose that the inequality (3.6) holds for each $h \in B$. There exist $\{x_n^h\}_{n\in\mathbb{N}} \in Z_{f-h}$ such that:

$$\lim_{n\to\infty} Re\langle(f - h)(x_n^h), (g_0 - h)(x_n^h)\rangle < -K_f\|h - g_0\|^2,$$

Then we get

$$\begin{aligned}
\|f - g_0\|^2 &\geq \lim_{n\to\infty} \|(f - g_0)(x_n^h)\|^2 \\
&= \lim_{n\to\infty} \|(f - h)(x_n^h) - (g_0 - h)(x_n^h)\|^2 \\
&= \lim_{n\to\infty} [\|(f - h)(x_n^h)\|^2 + \|(g_0 - h)(x_n^h)\|^2
\end{aligned}$$

$$\begin{aligned} -2 \quad & Re\langle (f-h)(x_n^h), (h-g_0)(x_n^h)\rangle] \\ > \quad & \lim_{n\to\infty}[\|(f-h)(x_n^h)\|^2 + \|(g_0-h)(x_n^h)\|^2] + 2K_f\|h-g_0\|^2 \\ \geq \quad & \|f-h\|^2 + K_f\|h-g_0\|^2, \end{aligned}$$

Hence $\|f-g_0\|^2 > \|f-h\|^2 + K_f\|h-g_0\|^2$. Then g_0 is a strongly uniqe farthest point to f from B of order 2. □

The followng Corollary give a condition for uniqueness of farthest point of bounded subsets of $B(H)$.

Corollary 3.5. *Let B be a bounded subset of $B(H)$, $g_0 \in B$ and $f \in B(H)$. If there exists a constant $K_f > 0$ such that*

$$\min ReW((g_0-h)^*(f-h)) < -K_f\|h-g_0\|^2 \quad (\forall h \in B), \tag{3.7}$$

then g_0 is the uniqe farthest point to f from B i.e $\mathbf{F}_B(f) = \{g_0\}$.

Proof. By Theorem 3.4, g_0 is a strongly uniqe farthest point to f from B of order 2. By Theorem 3.1, g_0 is also a farthest point to f from B. Now if possible, assume g_1 be another farthest point to f from B. Then $||f-g_1|| = ||f-g_0||$. As g_0 is a strongly uniqe farthest point, there exist a constant $K_f > 0$ such that

$$\begin{aligned} ||f-g_1||^2 = ||f-g_0||^2 \quad & \geq \quad ||f-g_1||^2 + K_f||g_1-g_0||^2 \\ & \Rightarrow \quad K_f||g_1-g_0||^2 = 0 \\ & \Rightarrow \quad g_1 = g_0. \end{aligned}$$

Hence g_0 is the unique farthest point to f from B i.e $\mathbf{F}_B(f) = \{g_0\}$. □

Let $\mathbb{A}$ be a C^*-algebra, then $\mathbb{A}$ has a faithful representation, i.e. $\mathbb{A}$ is isometrically isomorphic to a concrete C^*-algebra of operators on a Hilbert space H. This result is called the "Gelfand-Naimark Theorem". (For details about C^*-algebra we refer the reader to [8]).
Let (π, H) be a faithful representation for $\mathbb{A}$. Suppose that $a, b \in \mathbb{A}$. The numerical range of a^*b relative to a which denoted by $W_{\mathbb{A}}(a^*b)$ is defined by

$$W_{\mathbb{A}}(a^*b) := \{\lambda \in \mathbb{C} : \lambda \in W(\pi(a)^*\pi(b))\}. \tag{3.8}$$

Corollary 3.6. *Let $\mathbb{B}$ be a bounded subset of a C^*-algebra $\mathbb{A}$, $a \in \mathbb{A} \setminus \mathbb{B}$ and $b_0 \in \mathbb{B}$. Then the following statements are true:*

i) $b_0 \in \mathbf{F}_{\mathbb{B}}(a)$ *if* $\min ReW_{\mathbb{A}}((b_0 - b)^*(a - b)) \leq 0, \quad (b \in \mathbb{B})$.

ii) If $b_0 \in \mathbf{F}_{\mathbb{B}}(a)$ *then* $ReW_{\mathbb{A}}((b_0 - b)^*(a - b)) \leq 0, \quad (b \in \mathbb{B})$.

Proof. i) Let for each $b \in \mathbb{B}$, $\min ReW_{\mathbb{A}}((b_0 - b)^*(a - b)) \leq 0$, by (3.8),

$$\min ReW(\pi((b_0 - b)^* \pi((a - b))) \leq 0.$$

By Theorem 3.1, $\pi(b_0) \in \mathbf{F}_{\pi(\mathbb{B})}(\pi(a))$. Thus $b_0 \in \mathbf{F}_{\mathbb{B}}(a)$, since π is isometrically isomorphism.
ii) The proof is similar to the first part. □

Now we want to discuse the existence of farthest points in $C(X)$. First we recall that the set of extreme points of a function $f \in C[a, b]$ is defined by

$$E(f) := \{x \in [a, b] : |f(x)| = \|f\|\}.$$

In case $C[a, b]$, R. Saravanan and R. Vijayaragavan in paper [11] stated the following two main Theorems.

Theorem 3.7. *[Theorem 3.2 [11]] Let* B *be a compact subset of* $C[a, b]$, *containing* 0, $f \in C[a, b] \setminus B$, *and* $g_0 \in B$. *Then the following statements are equivalent.*

i) $g_0 \in \mathbf{F}_B(f)$.

ii) for each $g \in B$, $\quad \min_{t \in E(f-g)}(f(t) - g(t))(g_0(t) - g(t)) \leq 0$.

Theorem 3.8. *[Theorem 3.3 [11]] Let* B *be a compact subset of* $C[a, b]$, *containing 0,* $f \in C[a, b] \setminus B$, *and* $g_0 \in B$. *Then the following statements are equivalent.*

i) g_0 *is a strongly unique farthest point to* f *from* B.

ii) for each $g \in B$, $\quad \min_{t \in E(f-g)}(f(t) - g(t))(g_0(t) - g(t)) < 0$.

In the following, by two counterexamples 3.1 and 3.2, we will show that Theorems 3.7 and 3.8 are incorrect.

Example 3.1. Let $[a, b] = [0, 1], f = \frac{1}{2}x$, and $B = \{0, I\}$, where $I(x) = x$ we have $F_B(f) = \{0, I\}$, we may assume without loss of generality that $g_0 = I$, but

$$(f - g)(x)(g_0 - g)(x) = \frac{1}{2}x^2 > 0, \forall x > 0.$$

in particular as $E(f-g)=\{1\}$, for $g=0$, we get $(f-g)(1)(g_0-g)(1)=\frac{1}{2}>0$.

Example 3.2. Let $[a,b]=[0,1], f(x)=x^2$, and $B=\{0,\frac{3}{2}I\}$ where $I(x)=x$. $g_0=0$ is strongly uniqe farthest point to f from B, because if we choose $K_f=\frac{1}{8}$, we get

$$\|f-g_0\|=1>\|f-g\|+K_f\|g-g_0\|=\frac{9}{16}+\frac{1}{8}\times\frac{3}{2}=\frac{12}{16},$$

when $g=\frac{3}{2}I$ and

$$\|f-g_0\|=1=\|f-g\|+K_f\|g-g_0\|,$$

when $g=0$. On the other hand $E(f-g)=\{\frac{3}{4}\}$, when $g=\frac{3}{2}I$, and in this case we have

$$(f-g)(\frac{3}{4})(g_0-g)(\frac{3}{4})=\frac{81}{128}>0.$$

Examples 3.1 and 3.2 shows that Theorems 3.7 and 3.8 are not true in general. We end this section with some results on $C(X)$ where X is a compact space.

Proposition 3.1. *Let B be a bounded subset of $C(X)$, $f\in C(X)\setminus B$, and $g_0\in \mathbf{F}_B(f)$, then for each $g\in B$ and each $x\in E(f-g_0)$ we have*

$$(f-g_0)(x)(g_0-g)(x)\leq 0. \tag{3.9}$$

Proof. Suppose (3.9) fails. Then there exist $g\in B$ and $x\in E(f-g_0)$, such that

$$(f-g_0)(x)(g_0-g)(x)>0.$$

We may assume without loss of generality, that $(f-g_0)(x)>0$ and $(g_0-g)(x)>0$. Therefore $f(x)>g_0(x)>g(x)$, so we get $|(f-g)(x)|>|(f-g_0)(x)|$ and thus $\|f-g\|\geq|(f-g)(x)|>|(f-g_0)(x)|=\|f-g_0\|$. But this is in contradiction with $g_0\in\mathbf{F}_B(f)$. □

Let X is a compact space and μ is a positive Borel measure on X, then the map $\pi: C(X)\to B(L_2(\mu))$ defined by $\pi(f)=M_f$ is a representation (see e.g [2]). where

$$M_f(h)=foh, for\ h\in L_2(\mu).$$

Corollary 3.9. *Let B be a bounded subset of $C(X)$, and $f \in C(X)\backslash B$. Then the following statements are true:*

i) $g_0 \in \mathbf{F}_B(f)$, *if for each* $g \in B$

$$\min_{h_i \in Z_{M_{f-g}}} Re \int \overline{((g_0 - g)oh_i)(x)}((f - g)oh_i)(x)d\mu \leq 0.$$

ii) If $g_0 \in \mathbf{F}_B(f)$ *then for each* $h_i \in Z_{M_{f-g_0}}$ and $g \in B$,

$$Re \int \overline{((g_0 - g)oh_i)(x)}((f - g_0)oh_i)(x)d\mu \leq 0.$$

Proof. It is a consequence of Theorem 3.1. □

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