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CYCLICITY OF WEIGHTED COMPOSITION OPERATORS ON SOME BK SPACE

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Abstract: We will investigate the cyclicity for the adjoint of a weighted composition operator acting on $(\hat{l_p}(\alpha))^*$.

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1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let ϕ, l_∞ and c_0 denote the set of all finite, bounded and null sequences. We write

$$l_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$$

for $1 \leq p < \infty$. By $e^{(n)}$ $(n \in N_0)$, we denote the sequence with $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ whenever $k \neq n$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section. Given any subset F of ω, we write $\hat{F}$ for the set of all formal power series $\hat{f}$ with $\hat{f}(z) = \sum_{k=0}^{\infty} f_k z^k$ where $f = (f_k)_{k=0}^{\infty} \in F$, regardless of whether or not the series converges for any value of z. Let $\hat{M}_z : \hat{F} \to \hat{\omega}$ be defined by $(\hat{M}_z \hat{f}) = \sum_{k=0}^{\infty} f_k z^{k+1}$.

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A BK space is a Banach sequence space with the property that convergence implies coordinatewise convergence. A BK space F containing ϕ is said to have AK if every sequence $f = (f_k)_{k=0}^{\infty} \in F$ has a unique representation $f = \sum_{k=0}^{\infty} f_k e^{(k)}$, that is $f = \lim_{n\to\infty} f^{[n]}$; it is said to have AD, if ϕ is dense in F. Given any subset F of ω, the set

$$F^{\beta} = \{a \in \omega : \sum_{k=0}^{\infty} a_k f_k \quad \text{converges for all} \quad f \in F\}$$

is called the β-dual of F.

Let F be a normed sequence space and $\hat{F}$ be the space of formal power series with coefficients in F endowed with the norm of F. Then F and $\hat{F}$ are norm isomorphic.

We say that a vector x in a Banach space X is a cyclic vector of a bounded operator A on X if $X = \text{span}\{A^n x : n = 0, 1, 2, \ldots\}$.

Consider $f = \{f_k\}_{k=0}^{\infty}$ and $g = \{g_k\}_{k=0}^{\infty}$ in ω and let $E \subset \omega$. Define $fg = \{f_k g_k\}_{k=0}^{\infty}$ and

$$g^{-1} \star E = \{f \in \omega : fg \in E\}.$$

If $\alpha = \{\alpha_k\}_{k=0}^{\infty} \in \omega$ is a given sequence with $\alpha_k \neq 0$ for all k, then by $1/\alpha$ we mean $1/\alpha = \{1/\alpha_k\}_{k=0}^{\infty}$. Write $\hat{F}(\alpha) = (\alpha^{-1} \star F)$ for any subset F of ω. From now on we suppose that $\alpha = \{\alpha_k\}_{k=0}^{\infty} \in \omega$ satisfying $\alpha_0 = 1$ and $\alpha_k \neq 0$ for all $k \geq 1$. Note that the space $\hat{l}_p(\alpha)$ is a reflexive Banach space and the dual of $\hat{l}_p(\alpha)$ is $\hat{l}_q(\alpha^{-1})$.

If λ is a complex number, then $e(\lambda)$ denotes the functional of evaluation at λ, defined on $\hat{l}_p(\alpha)$ by $e(\lambda)(\hat{f}) = \hat{f}(\lambda)$.

A complex valued function φ on Ω for which $\varphi\hat{f} \in \hat{F}$ for every $\hat{f} \in \hat{F}$ is called a multiplier of $\hat{F}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\hat{F})$. For some sources on sequence spaces, see [1–6].

2. Main Results

In this section we will investigate the cyclicity of the adjoint of weighted composition operators acting on $\hat{l}_p(\alpha)^*$. By U we mean the open unit disc.

Lemma 1. *A complex number λ is a bounded point evaluation on $\hat{l}_p(\alpha)$ if and only if $\{\lambda^n\}_{n=0}^{\infty} \in l_q(\alpha^{-1})$.*

Proof. Note that λ is a bounded point evaluation on $\hat{l}_p(\alpha)$ if and only if the functional $e(\lambda)$ is bounded on $\hat{l}_p(\alpha)$. But the dual of $\hat{l}_p(\alpha)$ is $\hat{l}_q(\alpha^{-1})$ and we can see that $e(\lambda)((e^{(k)}\hat{)}) = (e^{(k)}\hat{)}(\lambda) = \lambda^k$ for all integers $k \geq 0$. This completes the proof. $\square$

Theorem 2. *Let each point of U is a bounded point evaluation on $\hat{l}_p(\alpha)$. Then a polynomial $\hat{p}$ is cyclic for $\hat{M}_z$ if and only if $\hat{p}$ vanishes at no point in U.*

Proof. Let $\hat{p}(z) = (z-\lambda_1)...(z-\lambda_m)$ be such that $\lambda_i \notin U$ for $i = 1, ..., m$. Fix $k \in \{1, ..., m\}$ and consider $M_k \in \hat{l}_q(\alpha^{-1})$ satisfying $M_k(\hat{M}_z)^n(z-\lambda_k)) = 0$ for all integers $n \geq 0$. So there exists $h \in l_q(\alpha^{-1})$ such that $M_k\hat{f} =< \hat{f}, \hat{h} >$ for all $f \in l_p(\alpha)$. Note that

$$M_k(\hat{M}_z)^n(z-\lambda_k)) = M_k(z^{n+1} - \lambda_k z^n) = h_{n+1} - \lambda_k h_n$$

for all integers $n \geq 0$. Since $M_k(\hat{M}_z)^n(z-\lambda_k)) = 0$, we get $h_{n+1} = \lambda_k h_n$ and so $h_{n+1} = \lambda_k^{n+1} h_0^{(k)}$ for all $n \geq 0$. But $\{\lambda_k^n\}_n \notin l_q(\alpha^{-1})$ and $h \in l_q(\alpha^{-1})$, hence $h_n = 0$ for all n and so $M_k = 0$. Thus $z - \lambda_k$ is cyclic for $k = 1, ..., m$ and so $\hat{p}(z)$ is a cyclic vector for $\hat{M}_z$. The converse case is clear.

Theorem 3. *Suppose that $1 < p < \infty$, $w \in \mathcal{M}(l_p(\alpha))$, $\mathcal{M}(\hat{l}_p(\alpha)) = H^\infty$ and φ is an analytic self-map of the open unit disc U satisfying $||\varphi||_U < 1$. Also, let $\sum_{n=0}^{\infty} \alpha_n^{-q} < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$. If there exists $z_0 \in U$ satisfying $\hat{w}(\varphi_k(z_0)) \neq 0$ for all $k \geq 0$ and if the set $\{\varphi_k(z_0) : k \geq 0\}$ has limit point in U, then $e(z_0)$ is a cyclic vector for the operator $(M_w C_\varphi)^*$ acting on $\hat{l}_q(\alpha^{-1})$.*

Proof. By the property $\sum_{n=0}^{\infty} \alpha_n^{-q} < \infty$, each point of U is a bounded point evaluation and the space $\hat{l}_p(\alpha)$ consists of functions analytic in the open unit disc U. Let the map $L : \mathcal{M}(\hat{l}_p(\alpha)) \to B(\hat{l}_p(\alpha))$ be given by $L(\hat{\psi}) = \hat{M}_{\hat{\psi}}$. We prove that L is continuous. For this we use the closed graph theorem. Suppose $\hat{\psi}_n$ converges to $\hat{\psi}$ in $\mathcal{M}(\hat{l}_p(\alpha))$ and $L(\hat{\psi}_n) = \hat{M}_{\hat{\psi}_n}$ converges to A in $B(\hat{l}_p(\alpha))$. Then for each f in $l_p(\alpha)$,

$$A\hat{f} = \lim_n \hat{M}_{\hat{\psi}_n}\hat{f} = \lim_n \hat{\psi}_n\hat{f}.$$

Thus $\{\hat{\psi}_n\hat{f}\}_n$ is convergent in $\hat{l}_p(\alpha)$. Now by the continuity of point evaluations $\hat{\psi}_n\hat{f}$ converges pointwise to $\hat{\psi}\hat{f}$ on U. So $A\hat{f}$ is analytic and agree with $\hat{\psi}\hat{f}$ on U. Hence $A\hat{f} = \hat{\psi}\hat{f}$ and $A = \hat{M}_{\hat{\psi}}$. Therefore L is continuouse and there is a constant c such that $||\hat{M}_{\hat{\psi}}|| \leq c||\hat{\psi}||_U$ for all $\hat{\psi}$ in $\mathcal{M}(\hat{F})$. But $||\hat{\psi}|| \leq ||\hat{M}_{\hat{\psi}}||$ for all $\hat{\psi}$ in $\mathcal{M}(\hat{F})$. Thus $||\hat{\psi}|| \leq c||\hat{\psi}||_U$ for all $\psi \in \mathcal{M}(l_p(\alpha))$. Since $\varphi \in H^\infty$ and

$\mathcal{M}(\hat{l}_p(\alpha)) = H^\infty$, we will use $\hat{\varphi}$ instead of φ. Let $f \in l_p(\alpha)$, then $C_{\hat{\phi}}\hat{f} = \hat{f}o\hat{\varphi} \in H^\infty$ since $||\hat{\varphi}||_U < 1$. So

$$||\hat{f}o\hat{\varphi}|| \leq c||\hat{f}o\hat{\varphi}||_U \leq c||\hat{f}||_U,$$

because $\hat{\varphi}(U) \subseteq U$. On the otherhand, note that for all f in $l_p(\alpha)$, $||\hat{f}||_U \leq \gamma||\hat{f}||$ where $\gamma = \sum_{n=0}^{\infty} \alpha_n^{-q}$. Now we get $||C_{\hat{\varphi}}\hat{f}|| \leq c\gamma||\hat{f}||$ which implies that $C_{\hat{\varphi}}$ and so $\hat{M}_{\hat{w}}C_{\hat{\varphi}}$ is bounded. Now, put $A = \hat{M}_{\hat{w}}C_{\hat{\varphi}}$. To complete the proof we show that if for all $k \geq 0$, $< \hat{g}, (A^*)^k e(z_0) >= 0$, then $\hat{g}$ should be the zero constant function. For this note that

$$< \hat{g}, (A^*)^k e(z_0) >= \left(\prod_{i=0}^{k-1} \overline{\hat{w}(\hat{\varphi}_i(z_0))} \right) \hat{g} \circ \hat{\varphi}_k(z_0).$$

By the assumptions, clearly we get $\hat{g} \circ \hat{\varphi}_k(z_0) = 0$ for all $k \geq 0$. Since $\{\hat{\varphi}_k(z_0) : k \geq 0\}$ has limit point in U, it should be $\hat{g} = 0$. Thus, $e(z_0)$ is a cyclic vector for the operator $(\hat{M}_{\hat{w}}C_{\hat{\varphi}})^*$ acting on $\hat{l}_q(\alpha^{-1})$. This completes the proof. □

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