FIXED POINT THEOREM IN STRUCTURE FUZZY METRIC SPACE

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Abstract: In this short paper we prove a fixed point theorem on structure fuzzy metric space which doesn't have unique fixed point.

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1. Introduction

After Zadeh [5] introduced fuzzy sets in 1965, many researchers from various areas have developed the theory of fuzzy sets and its applications. Deng [8], Erceg [9], George and Veeramani [7] etc gave initial foundations of different forms of fuzzy metric spaces. Grebiec [3] extended Banach's [11] and Edelstein's [13] fixed point theorem in fuzzy metric space. Kramosil and Michalek [12] investigated common fixed point theorems for compatible maps.

In this paper we will study a fixed point theorem from view point of a new class of fuzzy metric defined on a set. Fixed point theorems in any areas are most useful. Mathematical economists, physicists, computer scientists etc are using fixed point theorems in their respective research. At present various types of fuzzy fixed point theorems are also playing crucial role in mathematical economics, social choices, auction theory.

2. Preliminary Definitions

In this section we discuss some existing definitions.

Definition 2.1. (see [6]) A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a *continuous t-norm* if $*$ satisfies the following conditions:

- (a) $*$ is commutative and associative;
- (b) $*$ is continuous;
- (c) $a * 1 = a \forall a \in [0, 1]$;
- (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2. (see [12]) Let X be a non-empty set, $*$ be a *continuous t-norm* and $M : X^2 \times [0, \infty) \rightarrow [0, 1]$ be a fuzzy set. Consider the following conditions for all $x, y, z \in X$ and $t, s \in [0, \infty)$:

- (M1) $M(x, y, 0) = 0$
- (M2) $M(x, x, t) = 1$
- (M3) $M(x, y, t) = 1 \Rightarrow x = y$
- (M4) $M(x, y, t) = M(y, x, t)$
- (M5) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$
- (M6) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left continuous

Then $(X, M, *)$ is said to be a fuzzy metric space.

3. Main Result

In this section we will prove a fixed point theorem which doesn't have unique fixed point by removing certain conditions from definitions of fuzzy metric space.

Definition 3.1. $(X, M, *)$ is said to be a *structure fuzzy metric space (SFMS)* if it satisfies conditions (M1), (M3), (M4), (M5) and (M6) of Definition 2.2.

Definition 3.2. A sequence $\langle x_n \rangle$ in a SFMS is said to be *structure convergent* if $\exists x \in X$ such that $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1 \forall t > 0$. Then x is said to be *structure limit* of $\langle x_n \rangle$ and denoted by $\lim_n x_n = x$.

Definition 3.3. A sequence $\langle x_n \rangle$ in a SFMS $(X, M, *)$ is said to be *structure cauchy sequence* if for each $t > 0$ and $r \in N$ such that $\lim_{n \rightarrow \infty} M(x_{n+r}, x_n, t) = 1$.

$(X, M, *)$ is said to be *structure complete* if every *structure cauchy sequence* in it is *structure convergent*.

Definition 3.4. Let $(X, M, *)$ be a SFMS and f and h are self maps on X . Then f and h are said to be *normalized* at x if and only if $M(fhx, hfx, t) = 1 \forall t \in [0, \infty)$.

f and h are said to be *normalized* on X if f and h are normalized at all point x of X .

Definition 3.5. f and h are said to be *common domain normalized* (CDN) if they are normalized at the coincidence point of f and h .

Theorem 3.1. Let $(X, M, *)$ be a SFMS, $f, h : X \rightarrow X$ be two mappings and D be the set of all coincidence points of f and h with the following conditions:

- (a) $f(X) \subset h(X)$
- (b) Either $f(X)$ or $h(X)$ is *structure complete*
- (c) $M(fx, fy, t) \geq g(M(hx, hy, t)) \forall x, y \in X, t \in (0, \infty)$ and $M(hx, hy, t) > 0$
- (d) $g : (0, 1] \rightarrow (0, 1]$ is *monotonic increasing* and $\lim_{n \rightarrow \infty} g^n(\beta) = 1, g(\beta) \geq \beta \forall \beta \in (0, 1]$
- (e) $M(hx, fx, t) > 0 \forall t > 0$ and for some fixed $x \in X$

Then f and h have a coincidence point; moreover if f and h are CDN and $M(h^2a, ha, t) > 0$ for some $a \in D$ and $t \in (0, \infty)$ then f and h have a fixed point.

Proof. Let x_o be fixed and $x_o \in X$ and $M(hx_o, fx_o, t) > 0 \forall t \in (0, \infty)$. As $f(X) \subset g(X)$ then $\exists x_1 \in X$ such that $y_1 = fx_o = hx_1$. By mathematical induction we have, $y_{n+1} = fx_n = hx_{n+1} \forall n \in N$ and $y_o = hx_o$.

Now $M(y_{n+1}, y_{n+2}, t) \geq g^n(M(hx_o, fx_o, t)) \forall n \in N, t \in (0, \infty)$. As $n \rightarrow \infty$, so by condition (d) we have $M(y_{n+1}, y_{n+2}, t) \rightarrow 1$.

Thus

$$M(y_n, y_{n+2}, t) \geq M(y_n, y_{n+1}, \frac{t}{2}) * M(y_{n+1}, y_{n+2}, \frac{t}{2}),$$

$$M(y_n, y_{n+3}, t) \geq M(y_n, y_{n+1}, \frac{t}{3}) * M(y_{n+1}, y_{n+2}, \frac{t}{3}) * M(y_{n+2}, y_{n+3}, \frac{t}{3}).$$

Proceeding by using mathematical induction we have the following step

$$M(y_n, y_{n+r}, t) \geq M(y_n, y_{n+1}, \frac{t}{r}) * M(y_{n+1}, y_{n+2}, \frac{t}{r}) * M(y_{n+2}, y_{n+3}, \frac{t}{r}) * \dots \\ * M(y_{n+r-1}, y_{n+r}, \frac{t}{r}),$$

where $r \in N$. If $n \rightarrow \infty$ then $M(y_n, y_{n+r}, t) \rightarrow 1$, i.e. $\lim_{n \rightarrow \infty} M(y_n, y_{n+r}, t) = 1$. Thus implies $\langle y_n \rangle$ is a structure Cauchy sequence.

Let us assume that $h(X)$ is *structure complete*, then $\exists u \in h(X)$ such that $\lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} hx_{n+1} = u = \lim_{n \rightarrow \infty} fx_n$. Let $hv = u$ for some $v \in X$. If $n \rightarrow \infty$ then by definition of *SFMS*, we have $M(hv, fx_n, t) \rightarrow 1 \Rightarrow M(hv, hx_{n+1}, t) \rightarrow 1$.

By condition (c) and (d), we have $M(fv, fx_{n+1}, t) \geq g(M(hv, hx_{n+1}, t)) \geq M(hv, hx_{n+1}, t)$. If $n \rightarrow \infty$ then we have $fv = hv$ i.e. f and h have a coincidence point.

Let us consider that f and h are *CDN* and $M(h^2a, ha, t) > 0$ for some $a \in D$ and $t \in (0, \infty)$. Let $fa = ha = \omega$. So, by condition (c), $M(fha, fa, t) \geq g(M(h^2a, ha, t)) > 0$. It implies $M(hfa, ha, t) > 0$, after proceeding with mathematical induction we have $M(f^2a, fa, t) \geq g^n(M(hfa, ha, t))$. As $n \rightarrow \infty$ then by condition (d), $M(f^2a, fa, t) \rightarrow 1$. It implies that $f^2a = fa$ or $f\omega = \omega$.

Now easy to verify that $hfa = fa$ or $h\omega = \omega$. Hence ω is the fixed point of f and h . In similar way we can prove for *structure completeness* of $f(X)$.

Remark 3.1. It can be easily checked that the case of uniqueness of fixed point is not possible in the above fixed point theorem.

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