COMMON FIXED POINT THEOREMS IN MODULAR METRIC SPACES

Hossein Rahimpour^{1 §}, Ali Ebadian², Madjid Eshaghi Gordji³, Ali Zohri⁴

^{1,2,4}Department of Mathematics

Payame Noor University

P.O. BOX 19395-3697, Tehran, IRAN

³Department of Mathematics

Semnan University

P.O. BOX 35195-363, Semnan, IRAN

³Center of Excellence in

online Analysis and Applications (CENAA)

Semnan University

Semnan, IRAN

alizohri@gmail.com

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1. Introduction

The study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers, see for example ([13], [14], [6]). The concept of the commutativity has generalized in several ways. For this Sessa [15] has introduced the concept of weakly commuting. Then Jungck

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[§]Correspondence author

generalized this idea, first to compatible mappings [7] and then to weakly compatible mappings [8]. It can be easily verified that when the two mappings are commuting then they are compatible but not conversely. In 1998, Jungck and Rhoades [9] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. In 2010, V. V. Chistyakov introduced the notation of modular metric spaces [2], he present some interesting applications for superposition operators by applying the theory of modular metric spaces [3]. Later, researchers have studied and extend fixed point problems in modular metric spaces, see for example ([1], [4], [5], [10], [11], [12]). In this paper we prove some coincidence and common fixed point theorems for a contractive mapping in modular metric spaces. Our results generalize and extend results of Mongkolkeha et al. [10].

Definition 1. A metric modular on a non-empty set X is a function $\omega : (0, \infty) \times X \times X \rightarrow [0, \infty]$ that will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $x, y, z \in X$ and for all $\lambda > 0$, satisfies the following three conditions:

- (i) given $x, y \in X$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ iff $x = y$;
- (ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$ and all $x, y \in X$;
- (iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the condition

(i₁) $\omega_\lambda(x, x) = 0$, then ω is said to be a (metric) pseudomodular on X and if ω satisfies (i₁) and

(i₂) given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on x and y such that $\omega_\lambda(x, y) = 0$ then $x = y$,

with this condition ω is called a strict modular on X .

If instead of (iii) we replace the following condition:

(i₃) $\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$;

then ω is called a convex modular on X .

Remark 2. A modular ω on a set X , by $0 < \lambda \rightarrow \omega_\lambda(x, y) \in [0, \infty]$ for given $x, y \in X$, is nonincreasing on $(0, \infty)$. In fact, if $0 < \lambda < \mu$, then by above definition

$$\omega_\mu(x, y) \leq \omega_{\mu-\lambda}(x, x) + \omega_\lambda(x, y) = \omega_\lambda(x, y) \quad (1)$$

for all $x, y \in X$.

If a metric modular ω on X possesses a finite valued and ω is independent of λ i.e; $\omega_\lambda(x, y) = \omega_\mu(x, y)$ for all $\lambda, \mu > 0$, then $\omega_\lambda(x, y)$ is a metric on X .

The binary relation $\overset{\omega}{\sim}$ on X defined for each $x, y \in X$ by $x \overset{\omega}{\sim} y$ if and only if $\lim_{\lambda \rightarrow \infty} \omega_\lambda(x, y) = 0$. That is an equivalence relation and denote by $X_\omega(x_0)$. The set X_ω is called a modular space. The modular space X_ω can be equipped with a metric d_ω given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq \lambda\}, \quad x, y \in X_\omega$$

and the modular set X_ω is a metric space, see Theorem 2.6 of [2].

We also put

$$X_\omega^* \equiv X_\omega^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}.$$

Theorem 3. *Given a convex modular ω on a set X and put*

$$d_\omega^*(x, y) := \inf\{\lambda > 0 : \omega_\lambda(x, y) \leq 1\}, \quad x, y \in X_\omega^*.$$

Then (X_ω^, d_ω^*) is a metric space.*

Proof. See Theorem 3.6 of [2]. □

Remark 4. $X_\omega \subset X_\omega^*$ and this inclusion is proper in general, and if ω is a convex modular on X , then two modular spaces defined as above are same, $X_\omega = X_\omega^*$. See [4].

Definition 5. [4] Given a modular ω on X . A sequence $\{x_n\} \equiv \{x_n\}_{n=1}^\infty$ in X_ω (X_ω^*) is said to be modular convergent (ω -convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x , such that $\lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0$. This will be written briefly as $x_n \overset{\omega}{\rightarrow} x$ as $n \rightarrow \infty$.

Definition 6. [4] Given a modular ω on X . A sequence $\{x_n\} \subset X_\omega$ in X_ω (X_ω^*) is said to be modular Cauchy (ω -Cauchy) if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m, n \rightarrow \infty} \omega_\lambda(x_n, x_m) = 0$. i.e., for all $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq n_0(\varepsilon) : \omega_\lambda(x_n, x_m) \leq \varepsilon$.

Definition 7. [4] Given a modular ω on X , the modular space X_ω is said to be modular complete (ω -complete) if each modular Cauchy sequence from X_ω is modular convergent. More precisely; if $\{x_n\} \subset X_\omega$ and there exists $\lambda = \lambda\{x_n\} > 0$ such that

$$\lim_{m, n \rightarrow \infty} \omega_\lambda(x_n, x_m) = 0,$$

then there exists $x \in X_\omega$ such that $\lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0$.

Remark 8. [2] Let X_ω be a modular metric space, then the metric convergent (with respect d_ω) implies the modular convergent. i.e;

$$\lim_{n \rightarrow \infty} d_\omega(x_n, x) = 0 \text{ if and only if } \lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0 \text{ for all } \lambda > 0.$$

Definition 9. [4] A modular ω on X is said to satisfy the Δ_2 -condition if for a sequence $\{x_n\} \subset X_\omega$ and $x \in X_\omega$, there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and x , such that $\lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0$, then $\lim_{n \rightarrow \infty} \omega_{\frac{\lambda}{2}}(x_n, x) = 0$. This implies that $\lim_{n \rightarrow \infty} \omega_\lambda(x_n, x) = 0$ for all $\lambda > 0$.

Note that in the all of this paper we suppose that ω is a modular on X and satisfy in the Δ_2 -condition on X .

2. Main result

Let X_ω be a modular metric space and $S, T : X_\omega \longrightarrow X_\omega$. A point $x \in X_\omega$ is called a coincidence point of S and T if $Tx = Sx$. The mapping S and T are said to be weakly compatible if they commute at their coincidence point (i.e., $TSx = STx$ whenever $Tx = Sx$). Suppose $T(X_\omega) \subset S(X_\omega)$. Let x_0 be an arbitrary point in X_ω . Since $T(X_\omega) \subseteq S(X_\omega)$ there exists a point $x_1 \in X_\omega$ such that $Tx_0 = Sx_1$. By repeating this process we construct the sequence $\{x_n\}$ in X_ω such that $Sx_n = Tx_{n-1}$ for all $n \geq 1$, we say that $\{Tx_n\}$ is a $T-S$ -sequence with initial point x_0 .

Theorem 10. Let X_ω be a modular metric space and $S, T : X_\omega \longrightarrow X_\omega$ be two mapping such that $T(X_\omega) \subseteq S(X_\omega)$ and $S(X_\omega)$ be a ω -complete subspace of X_ω . Suppose there exists numbers $\alpha, \beta, \gamma \in [0, 1)$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:

1. $(\alpha + 2\beta + 2\gamma) < 1$ for all $0 \leq \alpha, \beta, \gamma < 1$;
- 2.

$$\begin{aligned} \omega_\lambda(Tx, Ty) &\leq \alpha\omega_\lambda(Sx, Sy) + \beta[\omega_\lambda(Sx, Tx) + \omega_\lambda(Sy, Ty)] \\ &\quad + \gamma[\omega_{2\lambda}(Sx, Ty) + \omega_\lambda(Sy, Tx)]; \end{aligned}$$

3. $\omega_\lambda(Sx, Ty) < \infty$.

Then T and S have a coincidence point.

Proof. Let x_0 be an arbitrary point in X_ω . Since $T(X_\omega) \subseteq S(X_\omega)$ there exists a $T-S$ -sequence $\{Tx_n\}$ in X_ω such that

$$Sx_n = Tx_{n-1}$$

for all $n \geq 1$. Now we take $x = x_n$ and $y = x_{n+1}$ in (2), we get

$$\begin{aligned} \omega_\lambda(Tx_n, Tx_{n+1}) &\leq \alpha\omega_\lambda(Sx_n, Sx_{n+1}) \\ &\quad + \beta[\omega_\lambda(Sx_n, Sx_{n+1}) + \omega_\lambda(Tx_n, Tx_{n+1})] \\ &\quad + \gamma[\omega_{2\lambda}(Tx_{n-1}, Tx_{n+1}) + \omega_\lambda(Sx_{n+1}, Sx_{n+1})] \end{aligned}$$

for all $\lambda > 0$. On the other hand

$$\begin{aligned} \omega_{2\lambda}(Tx_{n-1}, Tx_{n+1}) &\leq \omega_\lambda(Tx_{n-1}, Tx_n) + \omega_\lambda(Tx_n, Tx_{n+1}) \\ &= \omega_\lambda(Sx_n, Sx_{n+1}) + \omega_\lambda(Tx_n, Tx_{n+1}) \end{aligned}$$

so we obtain,

$$\begin{aligned} \omega_\lambda(Tx_n, Tx_{n+1}) &\leq \alpha\omega_\lambda(Sx_n, Sx_{n+1}) \\ &\quad + \beta[\omega_\lambda(Sx_n, Sx_{n+1}) + \omega_\lambda(Tx_n, Tx_{n+1})] \\ &\quad + \gamma[\omega_\lambda(Tx_n, Tx_{n+1}) + \omega_\lambda(Sx_n, Sx_{n+1})]. \end{aligned}$$

This implies that

$$\omega_\lambda(Tx_n, Tx_{n+1}) \leq k\omega_\lambda(Sx_n, Sx_{n+1}) \quad \text{for all } n \in \mathbb{N},$$

where $k = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} < 1$. So by induction we get

$$\omega_\lambda(Tx_n, Tx_{n+1}) \leq k^n \omega_\lambda(Tx_0, Tx_1) \quad \text{for all } n \in \mathbb{N}. \quad (2)$$

By 2, in a straightforward way, we imply that $\{Tx_n\}$ is a ω -Cauchy sequence. Since $S(X_\omega)$ is ω -complete, there exists $u, v \in X_\omega$ such that $u = S(v)$ and $Tx_n \xrightarrow{\omega} u$ as $n \rightarrow \infty$. Since ω satisfy in the Δ_2 -condition on X we get $\lim_{n \rightarrow \infty} \omega_\lambda(Tx_n, u) = 0$ for all $\lambda > 0$, therefore

$$\lim_{n \rightarrow \infty} \omega_\lambda(Tx_n, u) = \lim_{n \rightarrow \infty} \omega_\lambda(Sx_n, u) = 0 \quad \text{for all } \lambda > 0. \quad (3)$$

Now by taking $x = x_n$ and $y = v$ in (2), we obtain that

$$\begin{aligned} \omega_\lambda(Tx_n, Tv) &\leq \alpha\omega_\lambda(Sx_n, Sv) + \beta[\omega_\lambda(Sx_n, Tx_n) + \omega_\lambda(Sv, Tv)] \\ &\quad + \gamma[\omega_{2\lambda}(Sx_n, Tv) + \omega_\lambda(Sv, Tx_n)] \end{aligned}$$

by Remark 2 the function $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing, so we have

$$\begin{aligned}\omega_\lambda(Tx_n, Tv) &\leq \alpha\omega_\lambda(Sx_n, Sv) + \beta[\omega_\lambda(Sx_n, Tx_n) + \omega_\lambda(Sv, Tv)] \\ &\quad + \gamma[\omega_\lambda(Sx_n, Tx_n) + \omega_\lambda(Tx_n, Tv) + \omega_\lambda(Sv, Tx_n)]\end{aligned}$$

By using 3 and letting $n \rightarrow \infty$ in the above inequality, we get

$$\begin{aligned}\omega_\lambda(Sv, Tv) &\leq \alpha\omega_\lambda(Sv, Sv) + \beta[\omega_\lambda(Sv, Tv) + \omega_\lambda(Sv, Tv)] \\ &\quad + \gamma[\omega_\lambda(Sv, Tv) + \omega_\lambda(Sv, Tv) + \omega_\lambda(Sv, Sv)].\end{aligned}$$

So $(1 - 2\beta - 2\gamma)\omega_\lambda(Sv, Tv) \leq 0$, for all $\lambda > 0$, and hence

$$Sv = Tv = u.$$

Thus we have proved that S and T have a coincidence point. $\square$

Theorem 11. *In addition to the hypotheses of Theorem 10, suppose that T and S are weakly compatible, then T and S have a unique common fixed point. Further, for any $x_0 \in X_\omega$, the $T-S$ -sequence $\{Tx_n\}$ with initial point x_0 modular converges to the common fixed point.*

Proof. Assume that S, T are weakly compatible, then

$$Su = STv = T Sv = Tu,$$

we will show that $Tu = u = Tv$.

Suppose $\omega_\lambda(Tu, Tv) > 0$ for all $\lambda > 0$, by taking $x = u$ and $y = v$ in (2) we get

$$\begin{aligned}\omega_\lambda(Tu, Tv) &\leq \alpha\omega_\lambda(Su, Sv) + \beta[\omega_\lambda(Su, Tu) + \omega_\lambda(Sv, Tv)] \\ &\quad + \gamma[\omega_{2\lambda}(Su, Tv) + \omega_\lambda(Sv, Tu)]\end{aligned}$$

for all $u, v \in X_\omega$ and $\lambda > 0$, i.e.,

$$\begin{aligned}\omega_\lambda(Tu, Tv) &\leq \alpha\omega_\lambda(Tu, Tv) + \beta[\omega_\lambda(Tu, Tu) + \omega_\lambda(Tv, Tv)] \\ &\quad + \gamma[\omega_{2\lambda}(Tu, Tv) + \omega_\lambda(Tv, Tu)]\end{aligned}$$

by Remark 2 we have

$$\omega_\lambda(Tu, Tv) \leq \alpha\omega_\lambda(Tu, Tv) + \gamma[\omega_\lambda(Tu, Tv) + \omega_\lambda(Tv, Tu)]$$

for all $\lambda > 0$. This implies that $\omega_\lambda(Tu, Tv)(1 - \alpha - 2\gamma) \leq 0$, which is a contradiction by assumption.

Therefore $Su = Tu = Tv = u$, and hence S, T have common fixed point. For the uniqueness of the common fixed point, suppose that u and z be two common fixed point, i.e.,

$$Tu = Su = u \quad \text{and} \quad Tz = Sz = z.$$

By taking $x = u$ and $y = z$ in (2), we obtain that

$$\begin{aligned} \omega_\lambda(Tu, Tz) &\leq \alpha\omega_\lambda(Su, Sz) + \beta[\omega_\lambda(Su, Tu) + \omega_\lambda(Sz, Tz)] \\ &\quad + \gamma[\omega_{2\lambda}(Su, Tz) + \omega_\lambda(Sz, Tu)] \end{aligned}$$

for all $\lambda > 0$. So, $\omega_\lambda(Tu, Tz)(1 - \alpha - 2\gamma) \leq 0$ for all $\lambda > 0$, which is contradiction. Therefore $\omega_\lambda(u, z) = 0$ for all $\lambda > 0$ and so $u = z$. Clearly, for any $x_0 \in X_\omega$, the $T - S$ -sequence $\{Tx_n\}$ with initial point x_0 converges to the unique common fixed point. $\square$

By setting $S = I_{X_\omega}$, we deduce the following result of fixed point for one self-mapping from Theorem 10.

Corollary 12. *Let X_ω be a ω -complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and*

$$\begin{aligned} \omega_\lambda(Tx, Ty) &\leq \alpha\omega_\lambda(x, y) + \beta[\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)] \\ &\quad + \gamma[\omega_{2\lambda}(x, Ty) + \omega_\lambda(y, Tx)], \end{aligned}$$

where $(\alpha + 2\beta + 2\gamma) < 1$ and $0 \leq \alpha, \beta, \gamma < 1$. Then T has a unique fixed point. Further, for any $x_0 \in X_\omega$, the Picard sequence $\{Tx_n\}$ with initial point x_0 modular converges to the fixed point.

Corollary 13. *Let X_ω be a ω -complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and*

$$\omega_\lambda(Tx, Ty) \leq \alpha\omega_\lambda(x, y)$$

where $0 \leq \alpha < 1$. Then T has a unique fixed point.

Corollary 14. *Let X_ω be a ω -complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and*

$$\omega_\lambda(Tx, Ty) \leq \beta[\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)]$$

where $\beta \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Recently, Mongkolkeha et al. [10] has introduced some notions and established some fixed point results in modular metric spaces. The Corollaries of 13 and 14 are results of [10].

Theorem 15. *Let X_ω be a modular metric space and $S, T : X_\omega \longrightarrow X_\omega$ be two mapping such that $T(X_\omega) \subseteq S(X_\omega)$ and $S(X_\omega)$ be a ω -complete subspace of X_ω . Suppose there exists mappings $\alpha, \beta, \gamma, \mu : X_\omega \longrightarrow [0, 1)$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

1. $\alpha(Tx) \leq \alpha(Sx), \beta(Tx) \leq \beta(Sx), \gamma(Tx) \leq \gamma(Sx), \mu(Tx) \leq \mu(Sx);$
2. $(\alpha + 2\beta + \gamma + \mu) < 1$ for all $0 \leq \alpha, \beta, \gamma, \mu < 1;$
3. $\omega_\lambda(Tx, Ty) \leq \alpha(Sx)\omega_\lambda(Sx, Sy) + \beta(Sx)\omega_{2\lambda}(Sx, Ty) + \gamma(Sx)\omega_\lambda(Sx, Tx) + \mu(Sx)\omega_\lambda(Sy, Ty);$
4. $\omega_\lambda(Sx, Ty) < \infty.$

Then T and S have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X_ω . Since $T(X_\omega) \subseteq S(X_\omega)$ there exists a $T - S$ - sequence $\{Tx_n\}$ in X_ω such that

$$Sx_n = Tx_{n-1} \tag{4}$$

for all $n \geq 1$. From (1),(3) and 4, we have

$$\begin{aligned} \omega_\lambda(Sx_n, Sx_{n+1}) &= \omega_\lambda(Tx_{n-1}, Tx_n) \\ &\leq \alpha(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Sx_{n-1})\omega_{2\lambda}(Sx_{n-1}, Tx_n) \\ &\quad + \gamma(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Tx_{n-1}) + \mu(Sx_{n-1})\omega_\lambda(Sx_n, Tx_n) \\ &= \alpha(Tx_{n-2})\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Tx_{n-2})\omega_{2\lambda}(Sx_{n-1}, Sx_{n+1}) \\ &\quad + \gamma(Tx_{n-2})\omega_\lambda(Sx_{n-1}, Sx_n) + \mu(Tx_{n-2})\omega_\lambda(Sx_n, Sx_{n+1}) \\ &\leq \alpha(Sx_{n-2})\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Sx_{n-2})\omega_{2\lambda}(Sx_{n-1}, Sx_{n+1}) \\ &\quad + \gamma(Sx_{n-2})\omega_\lambda(Sx_{n-1}, Sx_n) + \mu(Sx_{n-2})\omega_\lambda(Sx_n, Sx_{n+1}) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\leq \alpha(Sx_0)\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Sx_0)\omega_{2\lambda}(Sx_{n-1}, Sx_{n+1}) \\ &\quad + \gamma(Sx_0)\omega_\lambda(Sx_{n-1}, Sx_n) + \mu(Sx_0)\omega_\lambda(Sx_n, Sx_{n+1}) \\ &\leq \alpha(Sx_0)\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Sx_0)[\omega_\lambda(Sx_{n-1}, Sx_n) \\ &\quad + \omega_\lambda(Sx_n, Sx_{n+1})] \\ &\quad + \gamma(Sx_0)\omega_\lambda(Sx_{n-1}, Sx_n) + \mu(Sx_0)\omega_\lambda(Sx_n, Sx_{n+1}). \end{aligned}$$

This implies that

$$\omega_\lambda(Sx_n, Sx_{n+1}) \leq \left(\frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} \right) \omega_\lambda(Sx_{n-1}, Sx_n) \quad (5)$$

for all $n \geq 1$. Now, we let

$$k = \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} < 1.$$

By repeating 5, we get

$$\omega_\lambda(Sx_n, Sx_{n+1}) \leq k^n \omega_\lambda(Sx_1, Sx_0). \quad (6)$$

Now for $m > n \geq 1$, it follows from 6 that

$$\begin{aligned} \omega_\lambda(Sx_n, Sx_m) &\leq \omega_{\frac{\lambda}{m-n}}(Sx_n, Sx_{n+1}) + \omega_{\frac{\lambda}{m-n}}(Sx_{n+1}, Sx_{n+2}) \\ &\quad + \dots + \omega_{\frac{\lambda}{m-n}}(Sx_{m-1}, Sx_m) \\ &\leq (k^n + k^{n+1} + \dots + k^{m-1}) \omega_{\frac{\lambda}{m-n}}(Sx_0, Sx_1) \\ &= \frac{k^n - k^m}{1 - k} \omega_{\frac{\lambda}{m-n}}(Sx_0, Sx_1). \end{aligned}$$

Since $0 \leq k < 1$, we conclude that $\{Sx_n\}$ is a ω -Cauchy sequence in $S(X_\omega)$, on the other hand by hypothesis $S(X_\omega)$ is a ω -complete subspace of X_ω , therefore there exists a point $u \in S(X_\omega)$ such that $Sx_n \xrightarrow{\omega} u$ as $n \rightarrow \infty$. By hypothesis ω satisfies in Δ_2 -condition on X_ω , so $\lim_{n \rightarrow \infty} \omega_\lambda(Sx_n, u) = 0$ for all $\lambda > 0$.

Now we claim that u is common fixed point. Suppose that $Tu \neq u$ or $Su \neq u$. Then we have

$$\begin{aligned} 0 &< \inf\{\omega_\lambda(Tx, u) + \omega_\lambda(Sx, u) + \omega_\lambda(Tx, Sx) : x \in X_\omega\} \\ &\leq \inf\{\omega_\lambda(Tx_n, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(Tx_n, Sx_n) : n \geq 1\} \\ &= \inf\{\omega_\lambda(Sx_{n+1}, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(Sx_n, Sx_{n+1})\} \\ &\leq \inf\{\omega_\lambda(Sx_{n+1}, u) + \omega_\lambda(Sx_n, u) + \omega_{\frac{\lambda}{2}}(Sx_n, u) + \omega_{\frac{\lambda}{2}}(u, Sx_{n+1})\} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ for all $\lambda > 0$, which is contradiction. Therefore, this implies that $u = Su = Tu$.

For the uniqueness of the common fixed point, suppose $Tu = Su = u$ and $Tz = Sz = z$ be two common fixed points, then by taking $x = u$ and $y = z$ in (3), we obtain that

$$\omega_\lambda(u, z) \leq \alpha(u) \omega_\lambda(u, z) + \beta(u) \omega_{2\lambda}(u, z) + \gamma(u) \omega_\lambda(u, u) + \mu(u) \omega_\lambda(z, z)$$

$$\leq \alpha(u)\omega_\lambda(u, z) + \beta(u)\omega_\lambda(u, z).$$

This implies that $(1 - \alpha - \beta)(u)(\omega_\lambda(u, z)) \leq 0$, which is a contradiction by assumption. $\square$

By taking the mapping S in Theorem 15 as I_{X_ω} where I_{X_ω} is an identity mapping on X_ω , we have the following corollary.

Corollary 16. *Let X_ω be a ω -complete modular metric space and $T : X_\omega \rightarrow X_\omega$. Suppose there exists mappings $\alpha, \beta, \gamma, \mu : X_\omega \rightarrow [0, 1)$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:*

1. $\alpha(Tx) \leq \alpha(x), \beta(Tx) \leq \beta(x), \gamma(Tx) \leq \gamma(x), \mu(Tx) \leq \mu(x)$;
2. $(\alpha + 2\beta + \gamma + \mu) < 1$ for all $0 \leq \alpha, \beta, \gamma, \mu < 1$;
3. $\omega_\lambda(Tx, Ty) \leq \alpha(x)\omega_\lambda(x, y) + \beta(x)\omega_{2\lambda}(x, Ty) + \gamma(x)\omega_\lambda(x, Tx) + \mu(x)\omega_\lambda(y, Ty)$;
4. $\omega_\lambda(x, Ty) < \infty$.

Then T has a unique fixed point.

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