COMMON FIXED POINT THEOREMS IN MODULAR METRIC SPACES

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1. Introduction

The study of fixed and common fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers, see for example ([13], [14], [6]). The concept of the commutativity has generalized in several ways. For this Sessa [15] has introduced the concept of weakly commuting. Then Jungck

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generalized this idea, first to compatible mappings [7] and then to weakly compatible mappings [8]. It can be easily verified that when the two mappings are commuting then they are compatible but not conversely. In 1998, Jungck and Rhoades [9] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. In 2010, V. V. Chistyakov introduced the notion of modular metric spaces [2], he present some interesting applications for superposition operators by applying the theory of modular metric spaces [3]. Later, researchers have studied and extend fixed point problems in modular metric spaces, see for example ([1], [4], [5], [10], [11], [12]). In this paper we prove some coincidence and common fixed point theorems for a contractive mapping in modular metric spaces. Our results generalize and extend results of Mongkolkeha et al. [10].

Definition 1. A metric modular on a non-empty set $X$ is a function $\omega: (0, \infty) \times X \times X \rightarrow [0, \infty]$ that will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $x, y, z \in X$ and for all $\lambda > 0$, satisfies the following three conditions:

(i) given $x, y \in X$, $\omega_\lambda(x, y) = 0$ for all $\lambda > 0$ iff $x = y$;

(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$ and all $x, y \in X$;

(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the condition

(i$_1$) $\omega_\lambda(x, x) = 0$, then $\omega$ is said to be a (metric) pseudomodular on $X$ and if $\omega$ satisfies (i$_1$) and

(i$_2$) given $x, y \in X$, if there exists $\lambda > 0$, possibly depending on $x$ and $y$ such that $\omega_\lambda(x, y) = 0$ then $x = y$,

with this condition $\omega$ is called a strict modular on $X$.

If instead of (iii) we replace the following condition:

(i$_3$) $\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu} \omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu} \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$;

then $\omega$ is called a convex modular on $X$.

Remark 2. A modular $\omega$ on a set $X$, by $0 < \lambda \rightarrow \omega_\lambda(x, y) \in [0, \infty]$ for given $x, y \in X$, is nonincreasing on $(0, \infty)$. In fact, if $0 < \lambda < \mu$, then by above definition

$$\omega_\mu(x, y) \leq \omega_{\mu-\lambda}(x, x) + \omega_\lambda(x, y) = \omega_\lambda(x, y)$$

(1)

for all $x, y \in X$.

If a metric modular $\omega$ on $X$ possesses a finite valued and $\omega$ is independent of $\lambda$ i.e; $\omega_\lambda(x, y) = \omega_\mu(x, y)$ for all $\lambda, \mu > 0$, then $\omega_\lambda(x, y)$ is a metric on $X$. 
The binary relation $\sim$ on $X$ defined for each $x, y \in X$ by $x \sim y$ if and only if $\lim_{\lambda \to \infty} \omega_{\lambda}(x, y) = 0$. That is an equivalence relation and denote by $X_\omega(x_0)$. The set $X_\omega$ is called a modular space. The modular space $X_\omega$ can be equipped with a metric $d_\omega$ given by

$$d_\omega(x, y) = \inf\{\lambda > 0 : \omega_{\lambda}(x, y) \leq \lambda\}, \quad x, y \in X_\omega$$

and the modular set $X_\omega$ is a metric space, see Theorem 2.6 of [2].

We also put

$$X_\ast \equiv X_\ast(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_{\lambda}(x, x_0) < \infty\}.$$ 

**Theorem 3.** Given a convex modular $\omega$ on a set $X$ and put

$$d_{\ast\omega}(x, y) := \inf\{\lambda > 0 : \omega_{\lambda}(x, y) \leq 1\}, \quad x, y \in X_\ast.$$ 

Then $(X_\ast, d_{\ast\omega})$ is a metric space.

**Proof.** See Theorem 3.6 of [2]. \qed

**Remark 4.** $X_\omega \subset X_\ast$ and this inclusion is proper in general, and if $\omega$ is a convex modular on $X$, then two modular spaces defined as above are same, $X_\omega = X_\ast$. See [4].

**Definition 5.** [4] Given a modular $\omega$ on $X$. A sequence $\{x_n\} \equiv \{x_n\}_{n=1}^\infty$ in $X_\omega (X_\ast)$ is said to be modular convergent ($\omega$-convergent) to an element $x \in X$ if there exists a number $\lambda > 0$, possibly depending on $\{x_n\}$ and $x$, such that $\lim_{n \to \infty} \omega_{\lambda}(x_n, x) = 0$. This will be written briefly as $x_n \omega^{\sim} x$ as $n \to \infty$.

**Definition 6.** [4] Given a modular $\omega$ on $X$. A sequence $\{x_n\} \subset X_\omega$ in $X_\omega (X_\ast)$ is said to be modular Cauchy ($\omega$-Cauchy) if there exists a number $\lambda = \lambda(\{x_n\}) > 0$ such that $\lim_{m, n \to \infty} \omega_{\lambda}(x_n, x_m) = 0$. i.e., for all $\varepsilon > 0$ there exists $n_0(\varepsilon) \in \mathbb{N}$ such that for all $n, m \geq n_0(\varepsilon)$ : $\omega_{\lambda}(x_n, x_m) \leq \varepsilon$.

**Definition 7.** [4] Given a modular $\omega$ on $X$, the modular space $X_\omega$ is said to be modular complete ($\omega$-complete) if each modular Cauchy sequence from $X_\omega$ is modular convergent. More precisely; if $\{x_n\} \subset X_\omega$ and there exists $\lambda = \lambda\{x_n\} > 0$ such that

$$\lim_{m, n \to \infty} \omega_{\lambda}(x_n, x_m) = 0,$$
then there exists \( x \in X_\omega \) such that \( \lim_{n \to \infty} \omega_\lambda(x_n, x) = 0 \).

**Remark 8.** [2] Let \( X_\omega \) be a modular metric space, then the metric convergent (with respect \( d_\omega \)) implies the modular convergent. i.e;

\[
\lim_{n \to \infty} d_\omega(x_n, x) = 0 \text{ if and only if } \lim_{n \to \infty} \omega_\lambda(x_n, x) = 0 \text{ for all } \lambda > 0.
\]

**Definition 9.** [4] A modular \( \omega \) on \( X \) is said to satisfy the \( \Delta_2 \)-condition if for a sequence \( \{x_n\} \subset X_\omega \) and \( x \in X_\omega \), there exists a number \( \lambda > 0 \), possibly depending on \( \{x_n\} \) and \( x \), such that \( \lim_{n \to \infty} \omega_\lambda(x_n, x) = 0 \), then \( \lim_{n \to \infty} \omega_\lambda^2(x_n, x) = 0 \). This implies that \( \lim_{n \to \infty} \omega_\lambda(x_n, x) = 0 \) for all \( \lambda > 0 \).

Note that in the all of this paper we suppose that \( \omega \) is a modular on \( X \) and satisfy in the \( \Delta_2 \)-condition on \( X \).

2. Main result

Let \( X_\omega \) be a modular metric space and \( S, T : X_\omega \to X_\omega \). A point \( x \in X_\omega \) is called a coincidence point of \( S \) and \( T \) if \( Tx = Sx \). The mapping \( S \) and \( T \) are said to be weakly compatible if they commute at their coincidence point (i.e., \( T Sx = STx \) whenever \( Tx = Sx \)). Suppose \( T(X_\omega) \subset S(X_\omega) \). Let \( x_0 \) be an arbitrary point in \( X_\omega \). Since \( T(X_\omega) \subset S(X_\omega) \) there exists a point \( x_1 \in X_\omega \) such that \( Tx_0 = Sx_1 \). By repeating this process we construct the sequence \( \{x_n\} \) in \( X_\omega \) such that \( Sx_n = Tx_{n-1} \) for all \( n \geq 1 \), we say that \( \{Tx_n\} \) is a \( T - S \)-sequence with initial point \( x_0 \).

**Theorem 10.** Let \( X_\omega \) be a modular metric space and \( S, T : X_\omega \to X_\omega \) be two mapping such that \( T(X_\omega) \subseteq S(X_\omega) \) and \( S(X_\omega) \) be a \( \omega \)-complete subspace of \( X_\omega \). Suppose there exists numbers \( \alpha, \beta, \gamma \in [0, 1) \) such that the following assertion for all \( x, y \in X_\omega \) and \( \lambda > 0 \) hold:

1. \((\alpha + 2\beta + 2\gamma) < 1 \text{ for all } 0 \leq \alpha, \beta, \gamma < 1; \)
2. \[
\omega_\lambda(Tx, Ty) \leq \alpha \omega_\lambda(Sx, Sy) + \beta[\omega_\lambda(Sx, Tx) + \omega_\lambda(Sy, Ty)] + \gamma[\omega_\lambda^2(Sx, Ty) + \omega_\lambda(Sy, Tx)];
\]
3. \( \omega_\lambda(Sx, Ty) < \infty. \)

Then \( T \) and \( S \) have a coincidence point.
Proof. Let $x_0$ be an arbitrary point in $X_\omega$. Since $T(X_\omega) \subseteq S(X_\omega)$ there exists a $T - S-$ sequence $\{T x_n\}$ in $X_\omega$ such that

$$S x_n = T x_{n-1}$$

for all $n \geq 1$. Now we take $x = x_n$ and $y = x_{n+1}$ in (2), we get

$$\omega_\lambda(T x_n, T x_{n+1}) \leq \alpha \omega_\lambda(S x_n, S x_{n+1})$$

$$+ \beta[\omega_\lambda(S x_n, S x_{n+1}) + \omega_\lambda(T x_n, T x_{n+1})]$$

$$+ \gamma[\omega_2(T x_{n-1}, T x_{n+1}) + \omega_\lambda(S x_{n+1}, S x_{n+1})]$$

for all $\lambda > 0$. On the other hand

$$\omega_2(T x_{n-1}, T_{n+1}) \leq \omega_\lambda(T x_{n-1}, T x_n) + \omega_\lambda(T x_n, T x_{n+1})$$

$$= \omega_\lambda(S x_n, S x_{n+1}) + \omega_\lambda(T x_n, T x_{n+1})$$

so we obtain,

$$\omega_\lambda(T x_n, T x_{n+1}) \leq \alpha \omega_\lambda(S x_n, S x_{n+1})$$

$$+ \beta[\omega_\lambda(S x_n, S x_{n+1}) + \omega_\lambda(T x_n, T x_{n+1})]$$

$$+ \gamma[\omega_\lambda(T x_n, T x_{n+1}) + \omega_\lambda(S x_n, S x_{n+1})].$$

This implies that

$$\omega_\lambda(T x_n, T x_{n+1}) \leq k \omega_\lambda(S x_n, S x_{n+1})$$

for all $n \in \mathbb{N}$,

where $k = \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} < 1$. So by induction we get

$$\omega_\lambda(T x_n, T x_{n+1}) \leq k^n \omega_\lambda(T x_0, T x_1)$$

for all $n \in \mathbb{N}$. (2)

By 2, in a straightforward way, we imply that $\{T x_n\}$ is a $\omega$-Cauchy sequence. Since $S(X_\omega)$ is $\omega$-complete, there exists $u, v \in X_\omega$ such that $u = S(v)$ and $T x_n \xrightarrow{\omega} u$ as $n \to \infty$. Since $\omega$ satisfy in the $\Delta_2$-condition on $X$ we get

$$\lim_{n \to \infty} \omega_\lambda(T x_n, u) = 0$$

for all $\lambda > 0$, therefore

$$\lim_{n \to \infty} \omega_\lambda(T x_n, u) = \lim_{n \to \infty} \omega_\lambda(S x_n, u) = 0$$

for all $\lambda > 0$. (3)

Now by taking $x = x_n$ and $y = v$ in (2), we obtain that

$$\omega_\lambda(T x_n, T v) \leq \alpha \omega_\lambda(S x_n, S v)$$

$$+ \beta[\omega_\lambda(S x_n, T x_n) + \omega_\lambda(S v, T v)]$$

$$+ \gamma[\omega_2(S x_n, T v) + \omega_\lambda(S v, T x_n)]$$
by Remark 2 the function $\lambda \mapsto \omega_\lambda(x, y)$ is non-increasing, so we have

$$\omega_\lambda(Tx_n, Tv) \leq \alpha \omega_\lambda(Sx_n, Sv) + \beta [\omega_\lambda(Sx_n, Tx_n) + \omega_\lambda(Sv, Tv)] + \gamma [\omega_\lambda(Sx_n, Tx_n) + \omega_\lambda(Tx_n, Tv) + \omega_\lambda(Sv, Tx_n)]$$

By using $3$ and letting $n \to \infty$ in the above inequality, we get

$$\omega_\lambda(Sv, Tv) \leq \alpha \omega_\lambda(Sv, Sv) + \beta [\omega_\lambda(Sv, Tv) + \omega_\lambda(Sv, Tv)] + \gamma [\omega_\lambda(Tv, Tv) + \omega_\lambda(Sv, Sv)].$$

So $(1 - 2\beta - 2\gamma) \omega_\lambda(Sv, Tv) \leq 0$, for all $\lambda > 0$, and hence

$$Sv = Tv = u.$$

Thus we have proved that $S$ and $T$ have a coincidence point. \hfill \square

**Theorem 11.** In addition to the hypotheses of Theorem 10, suppose that $T$ and $S$ are weakly compatible, then $T$ and $S$ have a unique common fixed point. Further, for any $x_0 \in X_\omega$, the $T - S -$ sequence $\{Tx_n\}$ with initial point $x_0$ modular converges to the common fixed point.

**Proof.** Assume that $S, T$ are weakly compatible, then

$$Su = STv = TSv = Tu,$$

we will show that $Tu = u = Tv$.

Suppose $\omega_\lambda(Tu, Tv) > 0$ for all $\lambda > 0$, by taking $x = u$ and $y = v$ in (2) we get

$$\omega_\lambda(Tu, Tv) \leq \alpha \omega_\lambda(Su, Sv) + \beta [\omega_\lambda(Su, Tu) + \omega_\lambda(Sv, Tv)] + \gamma [\omega_\lambda(Tu, Tv) + \omega_\lambda(Sv, Tu)]$$

for all $u, v \in X_\omega$ and $\lambda > 0$, i.e.,

$$\omega_\lambda(Tu, Tv) \leq \alpha \omega_\lambda(Tu, Tv) + \beta [\omega_\lambda(Tu, Tu) + \omega_\lambda(Tv, Tv)] + \gamma [\omega_\lambda(Tu, Tv) + \omega_\lambda(Tv, Tu)]$$

by Remark 2 we have

$$\omega_\lambda(Tu, Tv) \leq \alpha \omega_\lambda(Tu, Tv) + \gamma [\omega_\lambda(Tu, Tv) + \omega_\lambda(Tv, Tu)]$$

for all $\lambda > 0$. This implies that $\omega_\lambda(Tu, Tv)(1 - \alpha - 2\gamma) \leq 0$, which is a contradiction by assumption.
Therefore $Su = Tu = Tv = u$, and hence $S, T$ have common fixed point. For the uniqueness of the common fixed point, suppose that $u$ and $z$ be two common fixed point, i.e.,

$$Tu = Su = u \quad \text{and} \quad Tz = Sz = z.$$  

By taking $x = u$ and $y = z$ in (2), we obtain that

$$\omega_\lambda(Tu, Tz) \leq \alpha \omega_\lambda(Su, Sz) + \beta [\omega_\lambda(Su, Tu) + \omega_\lambda(Sz, Tz)]$$

plus

$$\gamma [\omega_2\lambda(Su, Tz) + \omega_\lambda(Sz, Tz)],$$

for all $\lambda > 0$. So, $\omega_\lambda(Tu, Tz)(1-\alpha-2\gamma) \leq 0$ for all $\lambda > 0$, which is contradiction. Therefore $\omega_\lambda(u, z) = 0$ for all $\lambda > 0$ and so $u = z$. Clearly, for any $x_0 \in X$, the $T - S$ sequence $\{T x_n\}$ with initial point $x_0$ converges to the unique common fixed point. □

By setting $S = I_{X_\omega}$, we deduce the following result of fixed point for one self-mapping from Theorem 10.

**Corollary 12.** Let $X_\omega$ be a $\omega$-complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and

$$\omega_\lambda(Tx, Ty) \leq \alpha \omega_\lambda(x, y) + \beta [\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)] + \gamma [\omega_2\lambda(x, Ty) + \omega_\lambda(y, Tx)],$$

where $(\alpha + 2\beta + 2\gamma) < 1$ and $0 \leq \alpha, \beta, \gamma < 1$. Then $T$ has a unique fixed point. Further, for any $x_0 \in X_\omega$, the Picard sequence $\{T x_n\}$ with initial point $x_0$ modular converges to the fixed point.

**Corollary 13.** Let $X_\omega$ be a $\omega$-complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and

$$\omega_\lambda(Tx, Ty) \leq \alpha \omega_\lambda(x, y)$$

where $0 \leq \alpha < 1$. Then $T$ has a unique fixed point.

**Corollary 14.** Let $X_\omega$ be a $\omega$-complete modular metric space and $T : X_\omega \rightarrow X_\omega$, such that for all $\lambda > 0$ and $x, y \in X_\omega$, $\omega_\lambda(x, Ty) < \infty$ and

$$\omega_\lambda(Tx, Ty) \leq \beta [\omega_\lambda(x, Tx) + \omega_\lambda(y, Ty)]$$

where $\beta \in \left[0, \frac{1}{2}\right)$. Then $T$ has a unique fixed point.
Recently, Mongkolkeha et al. [10] has introduced some notions and established some fixed point results in modular metric spaces. The Corollaries of 13 and 14 are results of [10].

**Theorem 15.** Let $X_\omega$ be a modular metric space and $S, T : X_\omega \to X_\omega$ be two mapping such that $T(X_\omega) \subseteq S(X_\omega)$ and $S(X_\omega)$ be a $\omega$-complete subspace of $X_\omega$. Suppose there exists mappings $\alpha, \beta, \gamma, \mu : X_\omega \to [0, 1)$ such that the following assertion for all $x, y \in X_\omega$ and $\lambda > 0$ hold:

1. $\alpha(Tx) \leq \alpha(Sx), \beta(Tx) \leq \beta(Sx), \gamma(Tx) \leq \gamma(Sx), \mu(Tx) \leq \mu(Sx)$;
2. $(\alpha + 2\beta + \gamma + \mu) < 1$ for all $0 \leq \alpha, \beta, \gamma, \mu < 1$;
3. $\omega_\lambda(Tx, Ty) \leq \alpha(Sx)\omega_\lambda(Sx, Sy) + \beta(Sx)\omega_\lambda(Sx, Ty) + \gamma(Sx)\omega_\lambda(Sx, Tx) + \mu(Sx)\omega_\lambda(Sy, Ty)$;
4. $\omega_\lambda(Sx, Ty) < \infty$.

Then $T$ and $S$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X_\omega$. Since $T(X_\omega) \subseteq S(X_\omega)$ there exists a $T - S$- sequence $\{Tx_n\}$ in $X_\omega$ such that

$$Sx_n = Tx_{n-1}$$

for all $n \geq 1$. From (1),(3) and 4, we have

$$\omega_\lambda(Sx_n, Sx_{n+1}) = \omega_\lambda(Tx_{n-1}, Tx_n)$$

$$\leq \alpha(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Sx_n) + \beta(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Tx_n) + \gamma(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Tx_{n-1}) + \mu(Sx_{n-1})\omega_\lambda(Sx_{n-1}, Tx_{n-1})$$

$$= \alpha(Tx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_n) + \beta(Tx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1}) + \gamma(Tx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1}) + \mu(Tx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1})$$

$$\leq \alpha(Sx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_n) + \beta(Sx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1}) + \gamma(Sx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1}) + \mu(Sx_{n-2})\omega_\lambda(Sx_{n-2}, Sx_{n+1})$$

$$\leq \alpha(Sx_0)\omega_\lambda(Sx_0, Sx_n) + \beta(Sx_0)\omega_\lambda(Sx_0, Sx_{n+1}) + \gamma(Sx_0)\omega_\lambda(Sx_0, Sx_{n+1}) + \mu(Sx_0)\omega_\lambda(Sx_0, Sx_{n+1})$$

$$\leq \alpha(Sx_0)\omega_\lambda(Sx_0, Sx_n) + \beta(Sx_0)[\omega_\lambda(Sx_{n-1}, Sx_n) + \omega_\lambda(Sx_{n+1}, Sx_n)] + \gamma(Sx_0)[\omega_\lambda(Sx_{n-1}, Sx_n) + \omega_\lambda(Sx_{n+1}, Sx_n)] + \mu(Sx_0)[\omega_\lambda(Sx_{n-1}, Sx_n) + \omega_\lambda(Sx_{n+1}, Sx_n)].$$
This implies that
\[ \omega_\lambda(Sx_n, Sx_{n+1}) \leq \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} \omega_\lambda(Sx_{n-1}, Sx_n) \] (5)
for all \( n \geq 1 \). Now, we let
\[ k = \frac{\alpha(Sx_0) + \beta(Sx_0) + \gamma(Sx_0)}{1 - \beta(Sx_0) - \mu(Sx_0)} < 1. \]
By repeating 5, we get
\[ \omega_\lambda(Sx_n, Sx_{n+1}) \leq k^n \omega_\lambda(Sx_1, Sx_0). \] (6)
Now for \( m > n \geq 1 \), it follows from 6 that
\[ \omega_\lambda(Sx_n, Sx_m) \leq \omega_{\frac{k}{m-n}}(Sx_n, Sx_{n+1}) + \omega_{\frac{k}{m-n}}(Sx_{n+1}, Sx_{n+2}) \]
\[ + \cdots + \omega_{\frac{k}{m-n}}(Sx_{m-1}, Sx_m) \]
\[ \leq (k^n + k^{n+1} + \cdots + k^{m-1}) \omega_{\frac{k}{m-n}}(Sx_0, Sx_1) \]
\[ = \frac{k^n - k^m}{1 - k} \omega_{\frac{k}{m-n}}(Sx_0, Sx_1). \]
Since \( 0 \leq k < 1 \), we conclude that \( \{Sx_n\} \) is a \( \omega \)-Cauchy sequence in \( S(X_\omega) \), on the other hand by hypothesis \( S(X_\omega) \) is a \( \omega \)-complete subspace of \( X_\omega \), therefore there exists a point \( u \in S(X_\omega) \) such that \( Sx_n \xrightarrow{\omega} u \) as \( n \to \infty \). By hypothesis \( \omega \) satisfies in \( \Delta_2 \)-condition on \( X_\omega \), so \( \lim_{n \to \infty} \omega_\lambda(Sx_n, u) = 0 \) for all \( \lambda > 0 \).
Now we claim that \( u \) is common fixed point. Suppose that \( Tu \neq u \) or \( Su \neq u \). Then we have
\[ 0 < \inf \{\omega_\lambda(Tx, u) + \omega_\lambda(Sx, u) + \omega_\lambda(Tx, Sx) : x \in X_\omega\} \]
\[ \leq \inf \{\omega_\lambda(Tx_n, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(Tx_n, Sx_n) : n \geq 1\} \]
\[ = \inf \{\omega_\lambda(Sx_{n+1}, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(Sx_n, Sx_{n+1})\} \]
\[ \leq \inf \{\omega_\lambda(Sx_{n+1}, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(Sx_n, u) + \omega_\lambda(u, Sx_{n+1})\} \to 0 \]
as \( n \to \infty \) for all \( \lambda > 0 \), which is contradiction. Therefore, this implies that \( u = Su = Tu \).
For the uniqueness of the common fixed point, suppose \( Tu = Su = u \) and \( Tz = Sz = z \) be two common fixed points, then by taking \( x = u \) and \( y = z \) in (3), we obtain that
\[ \omega_\lambda(u, z) \leq \alpha(u) \omega_\lambda(u, z) + \beta(u) \omega_2 \lambda(u, z) + \gamma(u) \omega_\lambda(u, u) + \mu(u) \omega_\lambda(z, z) \]
\[
\leq \alpha(u)\omega_{\lambda}(u, z) + \beta(u)\omega_{\lambda}(u, z).
\]

This implies that \((1 - \alpha - \beta)(u)(\omega_{\lambda}(u, z)) \leq 0\), which is a contradiction by assumption. \(\square\)

By taking the mapping \(S\) in Theorem 15 as \(I_{X_{\omega}}\) where \(I_{X_{\omega}}\) is an identity mapping on \(X_{\omega}\), we have the following corollary.

**Corollary 16.** Let \(X_{\omega}\) be a \(\omega\)-complete modular metric space and \(T : X_{\omega} \rightarrow X_{\omega}\). Suppose there exists mappings \(\alpha, \beta, \gamma, \mu : X_{\omega} \rightarrow [0, 1)\) such that the following assertion for all \(x, y \in X_{\omega}\) and \(\lambda > 0\) hold:

1. \(\alpha(Tx) \leq \alpha(x), \beta(Tx) \leq \beta(x), \gamma(Tx) \leq \gamma(x), \mu(Tx) \leq \mu(x)\);
2. \((\alpha + 2\beta + \gamma + \mu) < 1\) for all \(0 \leq \alpha, \beta, \gamma, \mu < 1\);
3. \(\omega_{\lambda}(Tx, Ty) \leq \alpha(x)\omega_{\lambda}(x, y) + \beta(x)\omega_{2\lambda}(x, Ty) + \gamma(x)\omega_{\lambda}(x, Tx) + \mu(x)\omega_{\lambda}(y, Ty)\);
4. \(\omega_{\lambda}(x, Ty) < \infty\).

Then \(T\) has a unique fixed point.

**References**


