TRUST REGION WITH NONLINEAR CONJUGATE GRADIENT METHOD

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Abstract: Descent direction methods and trust region methods are usually used to solve the unconstrained optimization problem (p)

$$(P) \left\{ \min_{x \in \mathbb{R}^n} f(x) . \right.$$

In this work, we are interested in convergence results that use trust region methods which employ the conjugate gradient method Day-Yuan version as a subprogram for each iteration. Further, we penalize the quadratic problems with constraints and convert them into series of unconstrained problems.

AMS Subject Classification: 65K05, 90C30

Key Words: unconstrained optimization, conjugate gradient method Day-Yuan version, Wolf's rule, trust region methods, penalty method

1. Introduction

The aim of this paper is to solve the following unconstrained optimization problem

$$(P) : \left\{ \min_{x \in \mathbb{R}^n} f(x) , \right. \tag{1.1}$$

where the function f is assumed to be nonlinear and differentiable.

At the current point $x_k \in \mathbb{R}^n$, a model of the variation of f , for an increment d of x_k , is supposed to be given. In differentiable optimization, it is reasonable to consider a quadratic model of the form

$$f(x_k + d) \simeq \Psi_k(d) \simeq f(x_k) + g_k^T d + \frac{1}{2} d^T H_k d, \quad (1.2)$$

where $g_k = \nabla f(x_k)$ is the gradient of f at x_k and $H_k = \nabla^2 f(x_k)$ is the Hessian of f at x_k . g_k and H_k are assumed to be computed using the Euclidean scalar product $\langle u, v \rangle = \sum_i u_i v_i$.

In trust region methods[1, 2, 3, 7], we consider that Ψ_k is a model of the variation of f which is acceptable in a neighborhood of the form

$$\overline{B}(0, \triangle_k) = \{x \in \mathbb{R}^n : \|x\| \leq \triangle_k\},$$

where $\triangle_k > 0$ and $\|\cdot\|$ is the Euclidean norm. The domain $\overline{B}(0, \triangle_k)$ is called the **Trust region** of the model Ψ_k and the positive number $\triangle_k$ is called **the radius** of trust.

To find the increase d_k to be given to x_k , we minimize the quadratic model Ψ_k on the trust region [7]. Therefore, we have to solve the quadratic subproblem

$$(p1) \begin{cases} \min \Psi_k(d) \\ \|d\| \leq \triangle_k \end{cases} . \quad (1.3)$$

A possible approach for solving constraint problems is the **penalty method** [7]. Clearly, we introduce the constraints in the objective under penalties form by starting from a non-feasible solution and try to impose on the unconstrained optimum to arrive to qualifying set.

By solving a problem of the form

$$(p2) \begin{cases} \min_{d \in \mathbb{R}^n} \Phi(d, \mu_k) = \min_{d \in \mathbb{R}^n} \{\Psi(d) + \mu_k p(d)\} , \end{cases} \quad (1.4)$$

the sequence of values $\{\mu_k\}$ has the following properties

- i) $\mu_k > 0, \forall k$,
- ii) the sequence $\{\mu_k\}$ is increasing. That is $\mu_{k+1} > \mu_k$ for every k ,
- iii) the sequence $\{\mu_k\}$ diverges as $k \rightarrow \infty$,

and the $p(d)$ is a continuous penalty function such that

$$p(d) = \begin{cases} = 0 & \text{if } \|d\| \leq \Delta \\ > 0 & \text{otherwise.} \end{cases} \quad (1.5)$$

We choose

$$p(d) = (\max\{0, \|d\| - \Delta\})^2$$

2. Algorithm 1.

Initialization step. Set constants $\eta_1, \eta_2, \gamma_1, \gamma_2$ and γ_3 such as

$$0 < \eta_1 < \eta_2 < 1 \text{ et } 0 < \gamma_1 < \gamma_2 < 1 \leq \gamma_3. \quad (2.1)$$

Set the trust radius Δ_0 and an iterate x_0 . Compute of $f(x_0)$. Put $k = 0$ and go to step 1.

Step 1. Construct the model function $\Phi(d, \mu_k)$ according to the relation (1.4) and determine an approximate solution d_k of the problem (p_2)

$$(p_2) \left\{ \min_{d \in \mathbb{R}^n} \Phi(d, \mu_k) \right\}.$$

Step 1.0. Set a penalized problem $\phi_k(d, \mu_k)$ and $\mu_0 > 0$, $\beta > 1$, the start point s_0 , $M_0 = \nabla \phi(d_0, \mu_0)$, put $s_0 = -M_0$. Set $k = 0$ and go to step 1.1.

Step 1.1. If $M_k = 0$: stop ($d^* = d_k$). “Stop test”.
Otherwise go to step 1.2.

Step 1.2. Put $\mu_{k+1} = \beta \mu_k$.

Define $d_{k+1} = d_k + \alpha_k s_k$ with α_k satisfies the conditions of low Wolfe

$$\begin{cases} \phi(d_k + \alpha_k s_k, \mu_k) \leq \phi(d_k, \mu_k) + \rho \alpha s_k^T M_k \\ s_k^T M_{k+1} \geq \sigma s_k^T M_k \end{cases} \quad \text{such that } 0 < \rho < \sigma < 1.$$

$$s_k = -M_{k+1} + B_{k+1} s_k$$

$$B_{k+1}^{DY} = \frac{\|M_{k+1}\|^2}{s_k^T [M_{k+1} - M_k]} = \frac{\|M_{k+1}\|^2}{s_k^T y_k}.$$

Set $k = k + 1$ and go to step 1.1.

Step 2. Compute of $f(x_k + d_k)$ and evaluate the performance criteria

$$\rho_k = \frac{f(x_k) - f(x_k + d_k)}{\Psi(x_k) - \Psi(x_k + d_k)}.$$

Step 3. Update the radius of trust

- (a) If $\rho_k \geq \eta_2$: The step is a success and d_k is accepted. $x_{k+1} = x_k + d_k$ and choose $\Delta_k \in [\Delta_k, \gamma_3 \Delta_k]$.
- (b) If $\eta_1 \leq \rho_k \leq \eta_2$: The step is a success and d_k is accepted. $x_{k+1} = x_k + d_k$ and choose $\Delta_k \in [\gamma_2 \Delta_k, \Delta_k]$.
- (c) If $\rho_k < \eta_1$: The step is a failure and d_k is rejected. $x_{k+1} = x_k$ and choose $\Delta_k \in [\gamma_1 \Delta_k, \gamma_2 \Delta_k]$.

Step 4. If $\|\nabla f(x_k)\| \leq \varepsilon$, then the algorithm is stopped. Otherwise, we set $k = k + 1$ and go back to step 1.

We usually choose $\varepsilon = 10^{-5}$, $\eta_1 = 0,25$, $\eta_2 = 0,75$, $\gamma_1 = 0,25$, $\gamma_2 = 0,75$ and $\gamma_3 = 2$.

3. Convergence Results

By a similar manner to the linear search methods, the determination of optimal steps d_k is not a necessary condition for global convergence. Under certain conditions, a good approximation of these steps may be acceptable. It suffices to determine an approximate solution d_k , in the interior of the trust region, which produces a sufficient reduction of the model function. This reduction can be achieved by the method of Cauchy point d_k^c .

Definition 1. [3] We call Cauchy point of quadratic subproblem (RC_k) , the point noted d_k^c solution of the problem

$$(p_3) \begin{cases} \min \Psi_k(d) \\ \|d\| \leq \Delta_k . \\ d = -\alpha g_k , \alpha \in \mathbb{R}. \end{cases}$$

Therefore, such a point is the point minimizing Ψ_k in the trust region. That is, along the right side of the strongest slope of Ψ_k .

Proposition 2. [3] *The Cauchy point d_k^c is unique and it is given by*

$$d_k^c = \begin{cases} 0 & \text{if } g_k = 0 \\ -\frac{\Delta_k}{\|g_k\|} g_k & \text{if } g_k \neq 0 \text{ et } g_k^T H_k g_k \leq 0 \\ -\frac{\Delta_k}{\|g_k\|} g_k \min \left(\frac{\|g_k\|^3}{\Delta_k g_k^T H_k g_k}, 1 \right) & \text{otherwise.} \end{cases} \quad (3.1)$$

Proof. Indeed, the case where $g_k = 0$ is obvious. Suppose now that $g_k \neq 0$. If $g_k^T H_k g_k \leq 0$, then $\alpha \mapsto \Psi_k(-\alpha g_k)$ is concave and $\nabla \Psi_k(0)^T(-g_k) = -\|g_k\|^2 < 0$, therefore, α must be taken as large as possible while keeping $\|-\alpha g_k\| \leq \Delta_k$. This gives $\alpha = \frac{\Delta_k}{\|g_k\|}$.

It remains to examine the case $g_k^T H_k g_k > 0$. The result follows from the fact that, in this case, the minimum of the function $\alpha \mapsto \Psi_k(-\alpha g_k)$ is attained for $\alpha = \frac{\|g_k\|^2}{g_k^T H_k g_k}$.

As we have seen, the point of Cauchy d_k^c allows whether an approximate solution of the problem in trust region to be validated or not. So, the point d_k must verify the decrease of the model function

$$\Psi_k(0) - \Psi_k(d_k) \geq c_1 (\Psi_k(0) - \Psi_k(d_k^c)). \quad c_1 > 0 \quad (3.2)$$

□

The following proposition is known as Powell Condition.

Proposition 3. [1] *The Cauchy point d_k^c verifies the condition of Powell with $C = \frac{1}{2}$, such as*

$$\Psi_k(0) - \Psi_k(d_k^c) \geq C \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \Delta_k \right\}. \quad (3.3)$$

Proof. We consider two cases for the Hessian matrix H_k . That is,

$$\text{a) } g_k^T H_k g_k \leq 0$$

$$\Psi_k(d_k^c) - \Psi_k(0) = \Psi_k \left(-\frac{\Delta_k}{\|g_k\|} g_k \right) - f_k$$

$$\begin{aligned}
&= -\frac{\Delta_k}{\|g_k\|} \|g_k\|^2 + \frac{1}{2} \frac{\Delta_k^2}{\|g_k\|^2} g_k^T H_k g_k \\
&\leq -\frac{\Delta_k}{\|g_k\|} \|g_k\|^2 \\
&\leq -\Delta_k \|g_k\| \\
&\leq -\|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \Delta_k \right\} \\
&\leq -\frac{1}{2} \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \Delta_k \right\}
\end{aligned}$$

so, we have (3.3).

b) $\mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k > 0$ and $\frac{\|g_k\|^3}{\Delta_k \mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k} \leq 1$:

$$\begin{aligned}
\Psi_k(d_k^c) - \Psi_k(0) &= -\frac{\|g_k\|^4}{\Delta_k \mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k} + \frac{1}{2} g_k^T H_k g_k \frac{\|g_k\|^4}{(\mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k)^2} \\
&= -\frac{1}{2} \frac{\|g_k\|^4}{\mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k} \\
&\leq -\frac{1}{2} \frac{\|g_k\|^4}{\|\mathbf{H}_k\| \|\mathbf{g}_k\|^2} \\
&\leq -\frac{1}{2} \frac{\|g_k\|^2}{\|\mathbf{H}_k\|} \\
&\leq -\frac{1}{2} \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \Delta_k \right\}
\end{aligned}$$

so, we have (3.3).

c) $\mathbf{g}_k^T \mathbf{H}_k \mathbf{g}_k < \frac{\|g_k\|^3}{\Delta_k}$, $d_k^c = -\frac{\Delta_k}{\|g_k\|} g_k$

we have (3.3).

□

Theorem 4. [1] Let d_k be an arbitrary vector such that

$$\|d_k\| \leq \Delta_k$$

and

$$\Psi_k(0) - \Psi_k(d_k) \geq c_2 (\Psi_k(0) - \Psi_k(d_k^c)). \quad (3.4)$$

Then, d_k satisfies (3.3) with $c_2 = \frac{c_1}{2}$.

Proof. Since $\|d_k\| \leq \Delta_k$, from (3.2), we have,

$$\Psi_k(0) - \Psi_k(d_k) \geq c_1 (\Psi_k(0) - \Psi_k(d_k^c)) \geq \frac{1}{2} c_1 \|g_k\| \min \left\{ \frac{\|g_k\|}{\|H_k\|}, \Delta_k \right\}$$

In particular, if d_k is the exact solution of (RC_k) , then it satisfies (3.3) with $c_2 = \frac{1}{2}$. $\square$

3.1. Convergence of penalty methods

The following two lemmas are used to prove the convergence of the method.

By d_k we denote the solution of (p_2) .

Lemma 5. [7] *For any value of k , we have*

- i) $\Phi(d_k, \mu_k) \leq \Phi(d_{k+1}, \mu_{k+1})$
- ii) $p(d_k) \geq p(d_{k+1})$
- iii) $\Psi(d_k) \leq \Psi(d_{k+1})$

Proof. •

- i) Since the sequence $\{\mu_k\}$ is increasing and $p(d_{k+1}) \geq 0$, we have

$$\begin{aligned} \Phi(d_{k+1}, \mu_{k+1}) &= \Psi(d_{k+1}) + \mu_{k+1}p(d_{k+1}) \\ &\geq \Psi(d_{k+1}) + \mu_k p(d_{k+1}) \\ &\geq \Psi(d_k) + \mu_k p(d_k) \end{aligned}$$

and d_k minimizes $\Phi(d, \mu_k)$, we obtain

$$\Phi(d_{k+1}, \mu_{k+1}) \geq \Phi(d_k, \mu_k).$$

- ii) Since d_k minimizes $\Phi(d, \mu_k)$

$$\Phi(d, \mu_k) \Psi(d_k) + \mu_k p(d_k) \leq \Psi(d_{k+1}) + \mu_k p(d_{k+1})$$

and d_{k+1} minimizes $\Phi(d, \mu_{k+1})$

$$\Psi(d_{k+1}) + \mu_{k+1}p(d_{k+1}) \leq \Psi(d_k) + \mu_{k+1}p(d_k),$$

adding the two inequalities we obtain,

$$(\mu_{k+1} - \mu_k)p(d_{k+1}) \leq (\mu_{k+1} - \mu_k)p(d_k).$$

Since $\mu_{k+1} > \mu_k$, we have

$$p(d_{k+1}) \leq p(d_k).$$

iii) $\Psi(d_{k+1}) + \mu_k p(d_{k+1}) \geq \Psi(d_k) + \mu_k p(d_k)$ using ii) and the fact that $\mu_k \geq 0$, we have

$$\Psi(d_{k+1}) + \mu_k p(d_{k+1}) \geq \Psi(d_k) + \mu_k p(d_{k+1}).$$

Consequently, $\Psi(d_{k+1}) \geq \Psi(d_k)$.

□

Lemma 6. [7] d^* be an optimal solution of (p_1) . Then, for each k we have,

$$\Psi(d^*) \geq \Phi(d_k, \mu_k) \geq \Psi(d_k).$$

Proof.

$$\begin{aligned} \Psi(d^*) &= \Psi(d^*) + \mu_k p(d^*) \quad \forall k, \text{ since } p(d^*) = 0 \\ &\geq \Psi(d_k) + \mu_k p(d_k) = \Phi(d_k, \mu_k) \quad \forall k \text{ (since } d_k \text{ minimizes } \Phi(d, \mu_k)) \\ &\geq \Psi(d_k) \quad \forall k \text{ (since } \mu_k \geq 0 \text{ and } p(d_k) \geq 0). \end{aligned}$$

So, $\Psi(d^*) \geq \Phi(d_k, \mu_k) \geq \Psi(d_k) \quad \forall k$.

□

Theorem 7. Suppose that d_k is an exact global minimum of (p_2) and assume that $\{\mu_k\}$ is a divergent sequence. Then, every limit point d^* of the sequence $\{d_k\}$ is a solution of (p_1) .

Proof. Let $\bar{d}$ be the solution of (p_1) such that

$$\Psi(\bar{d}) \leq \Psi(d) \quad \forall \|d\| \leq \Delta.$$

Since d_k minimizes $\Phi(d, \mu_k)$ for all k , we have,

$$\Phi(d_k, \mu_k) \leq \Phi(\bar{d}, \mu_k).$$

So,

$$\Psi(d_k) + \mu_k p(d_k) \leq \Psi(\bar{d}) + \mu_k p(\bar{d}) = \Psi(\bar{d}) \quad (\text{because } p(\bar{d}) = 0).$$

Then,

$$p(d_k) \leq \frac{1}{\mu_k} [\Psi(\bar{d}) - \Psi(d_k)] \quad (3.5)$$

It is assumed that d^* is the limit point $\{d_k\}$. Then, there exists an infinite subsequence of K such that

$$\lim_{k \in K} d_k = d^*.$$

By taking the limit of inequality (3.5), we find

$$p(d^*) = \lim_{k \in K} p(d_k) \leq \lim_{k \in K} \frac{1}{\mu_k} [\Psi(\bar{d}) - \Psi(d_k)] = 0$$

(because $\{\mu_k\}$ is a divergent sequence).

So, we have $p(d^*) = 0$ and consequently $\|d^*\| \leq \Delta$.

The optimality of d^* follows directly from lemma 6. Indeed, the relation $\Psi(d_k) \leq \Psi(d^*)$ being satisfied for any k , it follows that

$$\Psi(d^*) = \lim_{k \in K} \Psi(d_k) \leq \Psi(\bar{d}).$$

□

3.2. Global Convergence

Theorem 8. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function on the set*

$$X = \{x \in \mathbb{R}^n : f(x) < f(x_0)\},$$

and let $\{x_k\}$ be a sequence generated by the algorithm 1, with H_k uniformly bounded. If d_k satisfies the condition (3.2) then we have

$$\liminf_{k \rightarrow \infty} \|\nabla f_k\| = 0. \quad (3.6)$$

Proof. Assume that there exists an indice k_1 and a constant $\gamma > 0$ such that

$$\|\nabla f_k\| \geq \gamma, \forall k \geq k_1. \quad (3.7)$$

We claim that

$$\Delta_k \rightarrow 0 \quad \text{and} \quad \sum_{k \geq 1} \|d_k\| < \infty. \quad (3.8)$$

According to the condition of Powell (3.3) and the fact that $\{H_k\}$ is bounded [3], we have

$$\begin{aligned}\Psi_k(d_k) &\leq -C \|\nabla f_k\| \min \left\{ \frac{\|\nabla f_k\|}{\|H_k\|}, \Delta_k \right\} \\ &\leq -C \min \{1, \Delta_k\},\end{aligned}$$

where C is a positive constant which is independent of k . Using the step 3 of the algorithm 1 ($\rho_k \geq \eta_2$), we have

$$f(x_k + d_k) \leq f(x_k) + \eta_2 \Psi_k(d_k).$$

The last inequality can be written as

$$f(x_{k+1}) \leq f(x_k) - C \min \{1, \Delta_k\},$$

$$C \min \{1, \Delta_k\} \leq f(x_k) - f(x_{k+1}).$$

Since f is bounded below, the sequence $\{f(x_k) - f(x_{k+1})\}$ is convergent and we deduce from the above inequality that $\sum_{k \geq 1} \Delta_k < \infty$, and since $\|d_k\| \leq \Delta_k$, the assertions of (3.8) are deduced.

We now show that the ratio $\rho_k \rightarrow 1$. Since f is bounded and $\|d_k\| \leq \Delta_k$ then,

$$\frac{\left| f(x_k + d_k) - f(x_k) - \nabla f(x_k)_k^T d_k \right|}{\|d_k\|} \rightarrow 0. \quad (3.9)$$

On the other hand, since $\{H_k\}$ is bounded, we have:

$$\left| \Psi_k(d_k) - \nabla f(x_k)_k^T d_k \right| \leq c \|d_k\|^2. \quad (3.10)$$

Finally, we can successively write

$$\begin{aligned}\rho_k = \frac{f(x_{k+1}) - f(x_k)}{\Psi_k(d_k)} &= \frac{f(x_k + d_k) - f(x_k) - \nabla f(x_k)_k^T d_k}{\Psi_k(d_k)} \\ &\quad + \frac{\nabla f(x_k)_k^T d_k - \Psi_k(d_k)}{\Psi_k(d_k)} + 1\end{aligned}$$

$$|\rho_k - 1| \leq \frac{\left| f(x_k + d_k) - f(x_k) - \nabla f(x_k)_k^T d_k \right|}{|\Psi_k(d_k)|}$$

$$+ \frac{\left| \nabla f(x_k)_k^T d_k - \Psi_k(d_k) \right|}{\left| \Psi_k(d_k) \right|}.$$

From (3.9), $\Delta \rightarrow 0$ and $\|s_k\| \leq \Delta_k$, and so,

$$\left| \Psi_k(d_k) \right| \geq c \|d_k\|.$$

Then, combining (3.9) and (3.10), we get

$$|\rho_k - 1| \leq \frac{c \|d_k\|^2}{c \|d_k\|} \leq \|d_k\|.$$

This shows that $\rho_k \rightarrow 1$.

Accordingly, $\rho_k > \eta_1$ for k large enough. Due to the updating rule of Δ_k (see the step 3 of the algorithm 1), this implies that $\Delta_k > \Delta > 0$. But this contradicts the first assertion of (3.8). This contradiction completes the proof. $\square$

Theorem 9. *Suppose, in addition to the assumptions and conditions of Theorem 8, that ∇f_k is continuous in the sense of Lipchitz and bounded below on the set*

$$X = \{x \in \mathbb{R}^n : f(x) < f(x_0)\},$$

then we have

$$\lim_{k \rightarrow \infty} \|\nabla f_k\| = 0. \quad (3.11)$$

Proof. Suppose that there exists a constant $\varepsilon > 0$ and a subsequence $\{x_k\}_{k \geq m} \subset B(x_m, R)$ where $B(x_m, R)$ is the closed ball of center x_m and radius R , for which we have

$$\|\nabla f_k\| \geq \varepsilon, \quad \text{for all } k \geq m. \quad (3.12)$$

Let $l \geq m$ such that x_{k+1} is the first iterate after x_m outside $B(x_m, R)$. Since $\|\nabla f_k\| \geq \varepsilon$ for all $k = m, m+1, \dots, l$, we can write

$$f(x_m) - f(x_{l+1}) = \sum_{k=m}^l f(x_k) - f(x_{k+1}).$$

Using the condition of Powell (3.3) and the fact that $\{H_k\}$ is bounded [3], we have as in the proof of Theorem 8,

$$f(x_{k+1}) - f(x_k) \leq -C \min \{\|\nabla f_k\|, \Delta_k\}.$$

So, for all $k \geq m$, we have

$$f(x_{l+1}) - f(x_m) \leq -C \min \{1, \Delta_k\}.$$

Since $f(x_{l+1}) - f(x_m) \rightarrow 0$, $\min \{1, \Delta_k\} = \Delta_k$, for k large enough [3]. Finally, using the fact that $\|d_k\| \leq \Delta_k$, we have

$$\|d_k\| \leq C(f(x_k) - f(x_{l+1})), \quad k = m, m+1, \dots, l.$$

$$\|x_{l+1} - x_m\| \leq \sum_{k=m}^l \|d_k\| \leq C(f(x_m) - f(x_{l+1})).$$

So $\|x_{l+1} - x_m\| \rightarrow 0$. Then by the uniform continuity of ∇f_k , we have

$$\|f(x_{l+1}) - f(x_m)\| \rightarrow 0.$$

This is in contradiction with (3.12). Hence the theorem is proved. $\square$

4. Numerical tests

We have performed numerical tests using the trust region algorithm penalized by the following functions

Problem 1: The generalized function of Rosenbrock

$$f(x) = \sum_{i=1}^{n-1} 100(x_i^2 - x_{i+1})^2 + (x_i - 1)^2$$

Problem 2: The generalized function of Dixon

$$f(x) = (x_1 - 1)^2 + \sum_{i=2}^n i(2x_i^2 - x_{i-1})^2$$

Problem 3: The function of Oren

$$f(x) = \left[\sum_{i=1}^n i x_i^2 \right]^2$$

Dimension n	iteration k	$f(x_k)$	$\ g(x_k)\ $	Time of convergence
2	26	$2,59.10^{-12}$	$9,56.10^{-6}$	$4,68.10^{-2}$ s
10	29	$4,82.10^{-11}$	$8,71.10^{-6}$	0,26 s
30	29	$8,16.10^{-13}$	$1,27.10^{-6}$	2,51 s
50	28	$2,72.10^{-11}$	$7,91.10^{-6}$	5,47 s
80	30	3.10^{-11}	$6,46.10^{-6}$	20,57 s
100	28	$2,71.10^{-11}$	$8,59.10^{-6}$	22 s
150	27	$5,67.10^{-12}$	$6,53.10^{-6}$	50,66 s
200	25	$2,11.10^{-13}$	$5,46.10^{-6}$	129,62 s

Table 1: Rosenbrock function

Dimension n	iteration k	$f(x_k)$	$\ g(x_k)\ $	Time of convergence
2	8	$5,62.10^{-12}$	$8,72.10^{-6}$	$1,56.10^{-2}$ s
6	25	$1,90.10^{-12}$	$9,97.10^{-6}$	1,06 s
10	41	0,66	$8,43.10^{-6}$	1,29 s
30	68	0,66	$9,79.10^{-6}$	2,72 s
50	111	0,66	$9,45.10^{-6}$	6,83 s
100	36	0,66	$8,84.10^{-6}$	39,18 s

Table 2: Dixon function

Problem 4: The function of Powell

$$f(x) = \sum_{i=1}^{n/4} [(x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4]$$

In all tests the stopping criterion of the algorithm is $\varepsilon = 10^{-5}$. The trust regions algorithm is applied with $\triangle_0 = 0.5$, $\eta_1 = 0,25$, $\eta_2 = 0,75$, $\gamma_1 = 0,25$, $\gamma_2 = 0,75$ and $\gamma_3 = 2$. The conjugate gradient algorithm is applied to the linear search of wolfe with the parameters: $m_1 = 0.3$, $m_2 = 0.7$. The penalty method is applied with $\mu_0 = 2$.

The results showing the number of iterations and time of convergence are illustrated in the following tables.

Dimension n	iteration k	$f(x_k)$	$\ g(x_k)\ $	Time of convergence
2	7	$2,68.10^{-8}$	$9,84.10^{-6}$	$1,56.10^{-2}$ s
10	19	$1,10.10^{-8}$	$9,66.10^{-6}$	0,60 s
50	23	$4,20.10^{-9}$	$9,74.10^{-6}$	103,13 s
80	25	$3,13.10^{-8}$	$9,94.10^{-6}$	470 s
100	25	$2,52.10^{-9}$	$9,32.10^{-6}$	967 s

Table 3: Oren Function

Dimension n	iteration k	$f(x_k)$	$\ g(x_k)\ $	Time of convergence
4	20	$1,15.10^{-8}$	$9,90.10^{-6}$	0,15 s
8	53	$1,76.10^{-8}$	$8,89.10^{-6}$	0,24 s
16	79	$9,38.10^{-9}$	$9,28.10^{-6}$	1,04 s
32	18	$1,25.10^{-8}$	$4,40.10^{-6}$	2,72 s
100	19	$2,02.10^{-8}$	$9,30.10^{-6}$	31,90 s
200	20	$3,86.10^{-8}$	$5,92.10^{-6}$	138,37 s

Table 4: Powell function

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