CHARACTERIZATIONS OF OPERATOR ORDER FOR THREE POSITIVE DEFINITE OPERATORS VIA OPERATOR MEAN

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Abstract: Motivated by Lin and Cho’s characterizations of $A \geq B \geq C$ via extended grand Furuta inequality, we present two characterizations of $A \geq B \geq C$ via operator mean.

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1. Introduction

A capital letter (such as $T$) stands for a bounded linear operator on a Hilbert space. $T > 0$ and $T \geq 0$ mean $T$ is a positive definite operator and $T$ is a positive semidefinite operator, respectively.

As an important and historic extension of Löwner-Heinz inequality ($A \geq B \geq 0 \Rightarrow A^\alpha \geq B^\alpha$, $\alpha \in [0, 1]$), T. Furuta proved the following theorem in 1987.

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Theorem 1.1. (see Furuta Inequality, [2, 7]) If $A \geq B \geq 0$, then $(A^{\frac{r}{p}}B^pA^\frac{r}{q})^{\frac{1}{q}} \geq (A^{\frac{r}{p}}B^pA^\frac{r}{q})^{\frac{1}{q}}$ holds for $p \leq 0$, $r \geq 0$, $q \geq 1$ with $(1+r)q \geq p+r$.

In 1995, T. Furuta obtained the following grand form of Theorem 1.1.

Theorem 1.2. (see Grand Furuta Inequality, [3, 8]) If $A \geq B \geq 0$ with $A > 0$, then $A^{1-t+r} \geq [A^{\frac{r}{p}}(B^{-\frac{t}{p}}C^pB^{-\frac{t}{q}})^sA^{\frac{r}{q}}]^{\frac{1-t+r}{p-t}}$ holds for $p, s \geq 1$, $t \in [0, 1]$ and $r \geq t$.

In 2003, M. Uchiyama showed the following extended form of Theorem 1.2.

Theorem 1.3. (Extended Grand Furuta Inequality, [9]) If $A \geq B \geq C \geq 0$ with $B > 0$, then $A^{1-t+r} \geq [A^{\frac{r}{p}}(B^{-\frac{t}{p}}C^pB^{-\frac{t}{q}})^sA^{\frac{r}{q}}]^{\frac{1-t+r}{p-t}}$ holds for $p, s \geq 1$, $t \in [0, 1]$ and $r \geq t$.

For $S, T > 0$, operator mean of $S$ and $T$ is defined by F. Kubo and T. Ando in [4] as $S^\alpha T = S^{\frac{\alpha}{2}}(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^\alpha S^{\frac{\alpha}{2}}$, where $\alpha \in [0, 1]$. Generally, if $\alpha \in \mathbb{R}$, $S^{\frac{\alpha}{2}}(S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^\alpha S^{\frac{\alpha}{2}}$ is denoted by $S^\alpha T$.

Recently, C.-S. Lin and Y. J. Cho in [5] showed characterizations of $A \geq B \geq C$ via extended grand Furuta inequality. Motivated by [5], we present two characterizations of $A \geq B \geq C$ via operator mean.

2. Main Results

C.-S. Lin in 2010 showed the following results on operator mean.

Lemma 2.1. (see [6]) For $p, s \geq 1$, $t \in [0, 1]$ and $r \geq 1$, if $A \geq B \geq C \geq 0$, then

$$C^\frac{r}{p-t} r \geq s \geq t \geq \frac{r}{p(t-s+t)}(B^\frac{r}{p}A^{-p}B^\frac{r}{q})^s \leq C \leq B \leq A \leq A^\frac{r}{p-t} r \geq s \geq t \geq \frac{r}{p(t-s+t)}(B^\frac{r}{p}C^{-p}B^\frac{r}{q})^s.$$

Lemma 2.2. (see [6]) For $p, s \geq 1$, $t \in [0, 1]$ and $r \geq 1+t$, if $A \geq B \geq C > 0$, then

$$C^{r-t} r \geq s \geq t \geq \frac{r}{p(t-s+t)}C^{-\frac{r}{p}} B^\frac{r}{p} (B^{-t}A^{-p}) B^\frac{r}{p} C^{-\frac{r}{p}} \leq C \leq B \leq A \leq A^{r-t} r \geq s \geq t \geq \frac{r}{p(t-s+t)}A^{-\frac{r}{p}} B^\frac{r}{p} (B^{-t}C^{-p}) B^\frac{r}{p} A^{-\frac{r}{p}}.$$
Next we will show two characterizations of operator order for three positive definite operators via Lemma 2.1 and Lemma 2.2.

**Theorem 2.1.** For $A, B, C > 0$. $A \geq B \geq C$ if and only if the following two inequalities
\[
C^r \frac{(r-1-t)}{(p-t)s+r} (B^\frac{t}{2} A^{-p} B^\frac{t}{2})^s \leq C,
\]
\[
A \leq A^r \frac{(r-1-t)}{(p-t)s+r} (B^\frac{t}{2} C^{-p} B^\frac{t}{2})^s
\]
hold for $p, s \geq 1, t \in [0, 1]$ and $r \geq 1$.

**Theorem 2.2.** For $A, B, C > 0$. $A \geq B \geq C$ if and only if the following two inequalities
\[
C^r - t \frac{(r-1-t)}{(p-t)s+r} C^{-\frac{t}{2}} B^\frac{t}{2} (B^{-t} A^{-p})(B^\frac{s}{2} C^{-p}) B^\frac{s}{2} A^{-\frac{t}{2}} \leq C,
\]
\[
A \leq A^{r} - t \frac{(r-1-t)}{(p-t)s+r} A^{-\frac{t}{2}} B^\frac{t}{2} (B^{-t} C^{-p})(B^\frac{s}{2} C^{-p}) B^\frac{s}{2} A^{-\frac{t}{2}}
\]
hold for $p, s \geq 1, t \in [0, 1]$ and $r \geq 1 + t$.

**Proof of Theorem 2.1.** The necessity is obviously by Lemma 2.1. We only need to prove the sufficiency. We adopt the same method as in [5].

Putting $p = t = 1, r = 2$ in (2.1), we have $C^2 \frac{(r-1-t)}{(p-t)s+r} (B^\frac{t}{2} A^{-p} B^\frac{t}{2})^s \leq C$. By the definition of $\sharp$, the following inequality holds.
\[
(C^{-1} (B^\frac{t}{2} A^{-1} B^\frac{t}{2})^s C^{-1})^\frac{1}{2} \leq C^{-1}.
\]

Because $C > 0$ and $C$ is bounded, there exist two positive numbers $m_C$ and $n_C$ such that $m_C I \geq C \geq n_C I > 0$. According to Theorem 6 in [1] ($X \geq Y \geq 0$ with $mI \geq X \geq nI > 0 \Rightarrow \frac{(m+n)^2}{4mn} X^2 \geq Y^2$), we have
\[
C^{-1} (B^\frac{t}{2} A^{-1} B^\frac{t}{2})^s C^{-1} \leq \frac{(m_c^{-1} + n_c^{-1})^2}{4m_c^{-1} n_c^{-1}} C^{-2}.
\]

Deleting $C^{-1}$ in the both side of the inequality above, and applying Löwner-Heinz inequality, the following inequality holds.
\[
B^\frac{t}{2} A^{-1} B^\frac{t}{2} \leq \left(\frac{(m_c^{-1} + n_c^{-1})^2}{4m_c^{-1} n_c^{-1}}\right)^\frac{1}{2} I.
\]

Letting $s \to +\infty$ above, then $A^{-1} \leq B^{-1}$, which ensures $A \geq B$. 


By the same way, we can obtain $B \geq C$ from (2.2).

**Proof of Theorem 2.2.** The necessity is obviously by Lemma 2.2. We only need to prove the sufficiency.

Putting $p = t = 1, r = 4$ in (2.3), we have $C^{3/2}B^{1/2}(B^{-1/2}A^{-1})B^{1/2}C^{-1/2} \leq C$. By the definitions of $\sharp$ and $\natural$, the following inequality holds.

$$
(C^{-2}(B^{1/2}A^{-1}B^{1/2})sC^{-2})^{1/2} \leq C^{-1/2}.
$$

(2.8)

According to Theorem 6 in [1], we have

$$
C^{-2}(B^{1/2}A^{-1}B^{1/2})sC^{-2} \leq \left(\frac{(m_c^{-2} + n_c^{-2})^2}{4m_c^{-2}n_c^{-2}}\right)^{1/2} C^{-4}.
$$

(2.9)

Deleting $C^{-2}$ in the both side of the inequality above, and applying Löwner-Heinz inequality, the following inequality holds.

$$
B^{1/2}A^{-1}B^{1/2} \leq \left(\frac{(m_c^{-2} + n_c^{-2})^2}{4m_c^{-2}n_c^{-2}}\right)^{1/2} I.
$$

(2.10)

Letting $s \to +\infty$ above, then $A^{-1} \leq B^{-1}$, which ensures $A \geq B$.

By the same way, we can obtain $B \geq C$ from (2.4).

**Remark 2.1.** We can also obtain $A \geq B$ from (2.5) and (2.8) by Theorem 3.1 in [10](For $C, D > 0, r > 0, \delta > -r$ and $0 < w \leq 1$, if $C^{\delta+r} \geq (C^{\natural}D^{s}C^{\sharp})^{w}$ holds for any $s > 1$, then $D \leq I$). We leave the details to readers.

**References**


[2] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.*, 101 (1987), 85-88.


