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SUFFICIENT CONDITIONS FOR SUBSPACE-HYPERCYCLICITY

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Abstract: In this paper we give some new sufficient conditions for an operator to be subspace-hypercyclic and subspace-mixing. We construct various examples with interesting properties, by using these conditions. Also we show subspace-hypercyclicity of a class of weighted shifts.

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1. Introduction and Preliminaries

An operator T on a Banach space X is called hypercyclic, if there is a vector whose orbit under T is dense in X. A good reference about hypercyclicity is the book of Grosse-Erdmann and Peris [3]. For more information one can see [1-2] and [9]. Recently Madore and Martinez-Avendano in [6] introduced subspace-hypercyclic operators and subspace-transitive operators as follows.

Definition 1.1. Let $T \in B(X)$ and let M be a closed subspace of X. We say T is M-hypercyclic, if there exists a vector $x \in X$ such that $orb(T, x) \cap M$ is dense in M. Such a vector x is called an M-hypercyclic vector for T.

Definition 1.2. Let $T \in B(X)$ and let M be a closed subspace of X. We

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say T is M-transitive, if for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ contains a relatively open nonempty subset of M.

The following lemma states two equivalent conditions for subspace-transitivity.

Lemma 1.3. *(see [6]) Let $T \in B(X)$. The following conditions are equivalent:*

(i) T is subspace-transitive with respect to M.

(ii) for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is a relatively open non-empty subset of M.

(iii) for any non-empty sets $U \subseteq M$ and $V \subseteq M$ both relatively open, there exists $n \in \mathbb{N}_0$ such that $T^{-n}(U) \cap V$ is non-empty and $T^n(M) \subseteq M$.

Theorem 1.4. *(see [6]) Let $T \in B(X)$ and M be a nonzero closed subspace of X. If T is M-transitive, then T is M-hypercyclic.*

It is proved in [6] by Madore and Martinez-Avendano that the converse of Theorem1.4 is not always true. So there are subspace-hypercyclic operators that are not subspace-transitive.

Some properties of subspace-hypercyclic operators are similar to properties of hypercyclic operators. For example hypercyclicity is an infinite-dimensional concept. This is true for subspace-hypercyclic operators as it proved in [6].

Theorem 1.5. *(see [6]) Let X be finite-dimensional. If $T \in B(X)$, then T is not subspace-hypercyclic for any nonzero closed subspace M.*

Theorem 1.6. *(see [6]) Let $T \in B(X)$. If T is subspace-hypercyclic for a nonzero closed subspace M, then M is not finite dimensional.*

But some properties of hypercyclic and subspace-hypercyclic operators are different. For example if an operator T be hypercyclic, then the point spectrum of its adjoint, $\sigma_p(T^*)$, must be empty. But if an operator T be subspace-hypercyclic, then $\sigma_p(T^*)$ may be empty or not (see [7]).

Theorem 1.7. *(see [7]) For every scalar $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, there exists a subspace-hypercyclic operator T such that $||T|| = |\lambda|$ and $\sigma_p(T^*) \neq \phi$.*

Lemma 1.8. *(see [7]) There is a subspace-hypercyclic operator T such that, $\sigma_p(T^*)$ is empty.*

For more information about subspace-hypercyclic operators, one can also see [4] and [8-9] and the references there in. In this paper we state some new sufficient conditions for an operator to be subspace-hypercyclic. Also we construct various examples with interesting properties of subspace-hypercyclic operators by using these conditions. Also we show subspace-hypercyclicity of a class of weighted shifts.

2. Main Results and Examples

In what follows X always is an F-space, a complex and complete metrizable topological vector space. $B(X)$ is the space of bounded linear operators on X. M always is a closed nonempty subspace of X. We also assume that M is separable, since subspace-hypercyclicity can only occur with respect to separable and infinite dimensional subspaces.

Lemma 2.1. *Let $T \in B(X)$ and let M be a closed subspace of X. Suppose that for any nonempty open subsets $U \subseteq M$ and $V \subseteq M$, there is $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \phi$. Then $\cup_{n \geq n_0} T^n(W)$ is dense in M for every open subset $W \subseteq M$ and $n_0 \in \mathbb{N}$.*

Proof. Let W be a nonempty open subset of M. By hypothesis for any nonempty open subset U of M, There is $n \in \mathbb{N}_0$ such that $T^n(W) \cap U \neq \phi$. So $\cup_{n=0}^{\infty} T^k(W)$ is dense in M.

Now let $n_0 \in \mathbb{N}$, $\varepsilon > 0$ and let $x \neq 0$ be an element of M. We can assume that $\varepsilon < ||x||$ and choose:

$$s > \varepsilon^{-1}||x|| + max\{||T^i|| : 0 \leq i \leq n_0\}.$$

By first part of the proof we can find $w \in W$ and $n \in \mathbb{N}_0$ such that

$$||T^n w - \frac{sx}{||x||}|| < 1.$$

If $n \leq n_0$, then:

$$||T^n w - \frac{sx}{||x||}|| \geq ||s - T^n w|| > \varepsilon^{-1}||x|| > 1.$$

Hence n must be greater or equal to n_0. This completes the proof. □

Subspace-hypercyclic vectors seem to be very strange and exceptional. But in fact they are common. By using Lemma2.1, we state our first sufficient condition for subspace-hypercyclicity.

Theorem 2.2. *Let $T \in B(X)$ and let M be a closed subspace of X. Suppose that T satisfies the following conditions:*

(i) For any nonempty open subsets $U \subseteq M$ and $V \subseteq M$ there is $n \in \mathbb{N}_0$ such that $T^n(U) \cap V \neq \phi$.

(ii) M has a dense subset X_0 such that $T^n x \to 0$ as $n \to \infty$, for every $x \in X_0$.

Then T is M-hypercyclic.

Proof. By Lemma2.1, for every $c > 0$, $\cup_{n=0}^{\infty} T^n(cB_M)$ is dense in M, where B_M is the open unit ball of M. Let $(x_i)_{i=1}^{\infty}$ be a dense subset of M. We can suppose that each x_i appears infinitely in this sequence. By induction we find an increasing sequence s_n and also find u_n's belong to X_0 such that:

$$||u_n|| < \frac{1}{2^n max\{1, ||T^{s_1}||, ..., ||T^{s_{n-1}}||} \quad (1)$$

and

$$||T^{s_n}(u_n) - x_n|| < \frac{1}{2^n} \quad (2)$$

for each $n \in \mathbb{N}_0$.

For this, set formally $s_0 = 0$. Let $u_1, u_2, ..., u_{n-1}$ and $0 < s_1 < s_2 < ... < s_{n-1}$ are found. By condition (ii) of hypothesis, we can find $m \in \mathbb{N}$ such that $||T^j(u_i)|| < \frac{1}{2^{i+n}} (j \geq m, i = 1, 2, ..., n-1)$. Consider $s_n > max\{s_{n-1}, m\}$. We can find $u_n \in X_0$ such that satisfies conditions (1) and (2).

Let $u = \sum_{i=1}^{\infty} u_n$. It is not hard to see that the series is converges and $u \in M$. Also:

$$||T^{s_n}(u) - x_n|| \leq \sum_{i=1}^{n-1} ||T^{s_i}(u_i)|| + ||T^{s_n}(u_n) - x_n|| + \sum_{i=n+1}^{\infty} ||T^{s_i}(u_i)||$$

$$\leq \sum_{i=1}^{n-1} \frac{1}{2^{i+n}} + \frac{1}{2^n} + \sum_{i=n+1}^{\infty} \frac{1}{2^i} \leq 3.2^{-n}.$$

Each x_k can appear infinitely times. So u is an M-hypercyclic vector. □

Let $T \in B(X)$ be an operator. We say that T is M-mixing, if for any non-empty open sets $U \subseteq M$ and $V \subseteq M$, there exists a positive integer N such that $T^n(U) \cap V$ is non-empty for all $n > N$ (see [12]).

Hence if an operator be M-mixing, it is satisfies condition (i) of Theorem2.2. So we have the following corollary:

Corollary 2.3. *Let $T \in B(X)$ and let M be a closed subspace of X. If T is M-mixing and there exists a dense subset $X_0 \subseteq M$ such that for every $x \in X_0$, $T^n(x) \to 0$ as $n \to \infty$, then T is M-hypercyclic.*

You see in the next theorem a sufficient condition for an operator to be subspace-mixing that also is a sufficient condition for subspace-hypercyclicity.

Theorem 2.4. *Let $T \in B(X)$ and let M be a closed subspace of X. Suppose that, there are subsets $X_0 \subseteq M$ and $Y_0 \subseteq M$ such that X_0 and Y_0 are dense in M and there is a map $S : Y_0 \to Y_0$ such that:*

(i) $T^n x \to 0$ *for any* $x \in X_0$,

(ii) $S^n y \to 0$ *for any* $y \in Y_0$,

(iii) $TSy = y$ *for any* $y \in Y_0$.

Then T is M-mixing. Specially T is M-hypercyclic.

Proof. Let U and V be nonempty open subsets of M. There is $x \in U \cap X_0$ and $y \in V \cap Y_0$. If we consider $u_n = S^n y$, then by hypothesis $u_n \in Y_0$. Also:

$$u_n \to 0 \quad and \quad x + u_n \to x$$

as $n \to \infty$. So:

$$T^n(x + u_n) = T^n(x) + T^n(u_n) \to y \quad as \quad n \to \infty.$$

Therefore if we choose N large enough, for every $n \geq N$:

$$x + u_n \in U \quad and \quad T^n(x + u_n) \in V$$

That means $T^n(U) \cap V \neq \phi$ for every $n \geq N$. Hence T is an M-mixing operator.

Note that $T^n x \to 0$ for every $x \in X_0$. Hence by Theorem2.2, T is M-hypercyclic. □

By using Theorem2.4, we construct an operator such that for every $m \in \mathbb{N}$, T^m is subspace-mixing and subspace-hypercyclic as follows.

Example 2.5. Let B be the backward shift on l^2, that is for $(x_1, x_2, x_3, ...) \in l^2$ defined as

$$B(x_1, x_2, x_3, ...) = (x_2, x_3, ...).$$

Let λ be a scalar with $|\lambda| > 1$ and consider $T = \lambda B$. Then for every $m \in \mathbb{N}$, $T^m = (\lambda B)^m$ is subspace-mixing with respect to:

$$M = \{\{a_n\}_{n=1}^{\infty} : a_{2k+1} = 0 \quad for \quad all \quad k \in \mathbb{N}\}.$$

Specially T^m is M-hypercyclic for any $m \in \mathbb{N}$.

Proof. Consider a natural number m. Let $X_0 = Y_0$ be the subsets of M, that consist all finite sequences. Suppose that $x \in X_0$. Since x has only finitely nonzero enteries, $T^{mn}(x)$ will be zero for n large enough. Thus $(T^m)^n(x) = T^{mn}(x) \to 0$ as $n \to \infty$. If we define $S = (\frac{1}{\lambda}F)^m$, where F is the forward shift on l^2, that is defined as:

$$F(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...),$$

then the conditions (ii) and (iii) are satisfied. So T^m is M-mixing. Moreover for every $x \in X_0$, we have $(T^m)^n(x) \to 0$. So T^m is M-hypercyclic. □

In the next example you see an operator T, such that both T and T^*, the adjoint of T, are subspace-hypercyclic and subspace-mixing with respect to same subspace.

Example 2.6. Let $T = B$ be the backward shift on:

$$l^1(\mathbb{N}, v) = \{\{x_n\}_{n \in N} : ||x_n|| = \sum |x_n| v_n < \infty\},$$

where for every $n \in \mathbb{N}$, $v_n = \frac{1}{n+1}$. Let $m \in \mathbb{N}$ and

$$M = \{\{x_n\}_{n \in \mathbb{N}} \in l^1(\mathbb{N}, v) : x_n = 0 \quad for \quad n < m\}.$$

Similar to Example2.5, T is M-mixing and M-hypercyclic.

The adjoint of T is the forward shift F. If we consider $X_0 = Y_0$, the set of finite sequence and consider $S = B$. Then X_0, Y_0 and S satisfies three conditions of Theorem2.4. So $T^* = F$ is M-mixing. Also for every $x \in X_0$, we have $T^n(x) \to 0$ as $n \to \infty$. So T^* is M-hypercyclic too.

Rezaei in [10] asked this question:
Question: Does there exist a subspace-hypercyclic operator T for a nontrivial subspace M such that we have neither $T^n(M) \subseteq M$ nor $M \subseteq T^n(M)$ for any integer $n \geq 1$?

Jimenez-Munguia, Martinez-Avendano and Peris in [4], construct an M-hypercyclic operator such that for any $n \in \mathbb{N}$, neither $T^n(M) \subseteq M$ nor $M \subseteq T^n(M)$.

Now we construct another example with this property and show the answer of Rezaei question is negative.

Example 2.7. Consider $l^1(\mathbb{Z}, w) = \{\{x_n\}_{n\in\mathbb{Z}} : ||x_n|| = \sum |x_n|w_n < \infty\}$, where for every $n \in \mathbb{Z}$, $w_n = \frac{1}{|n|+1}$. Let $m \in \mathbb{N}$. Consider:

$$M = \{\{x_n\}_{n\in\mathbb{Z}} \in l^1(\mathbb{Z}, w) : x_n = 0 \quad for \quad -m < n < m\}.$$

Let $T = B$ the backward shit, where $B(x_n) = (x_{n+1})$. Similar to Example2.6, T is M-hypercyclic. Moreover for any $k \in \mathbb{N}$,

$$T^k(M) = \{\{x_n\}_{n\in\mathbb{Z}} \in l^1(\mathbb{Z}, w) : x_n = 0 \quad for \quad -m+k < n < m+k\}.$$

If $k \geq m$, then $-m+k \geq 0$. Thus $T^k(M)$ have elements $\{x_n\}$ with $x_0 \neq 0$. But for any $\{x_n\} \in M$, x_0 must be equal to zero. Hence $T^k(M)$ is not a subset of M. On the other hand M have elements $\{x_n\}$ with $x_{m+k-1} \neq 0$. But this is impossible for $T^k(M)$, since $k \geq 1$. Hence M is not a subspace of $T^k(M)$. It is not hard to see that for any natural number k with $k < m$, $T^k(M)$ is not a subset of M and M is not a subspace of $T^k(M)$.

Thus neither $T^k(M) \subset M$ nor $M \subset T^k(M)$ for any $k \in \mathbb{N}$.

Note. A weighted backward shift on l^2 is an operator of the form:

$$B_w(x_1, x_2, x_3, ...) = (w_2x_2, w_3x_3, w_4x_4, ...)$$

where $(w_n)_n$, its weight, is a sequence of nonzero scalars.

A weighted forward shift on l^2 with weight $(w_n)_n$, is defined as:

$$F_w(x_1, x_2, x_3, ...) = (0, w_1x_1, w_2x_2, w_3x_3, ...)$$

One reason of studying weighted shifts is that they provide us a good source of examples. Salas in [11] investigated and charactrized hypercyclic weighted shifts. In the next corollary, we state a sufficient condition for a weighted backward shift to be subspace-hypercyclic.

Corollary 2.8. *Let B_w be a weighted backward shift on l^2 with the weight sequence (w_n). Let (w_n) be bounded and $inf_n w_n > 0$. Then for every $m \in N$, B_w is subspace-hypercyclic with respect to subspace*

$$M = \{\{a_n\}_{n=1}^{\infty} : a_n = 0 \quad for \quad n < m\}.$$

Proof. Let $X_0 = Y_0$ be a subset of M that contains finite sequences. That is not hard to see that $T^n x \to 0$ as $n \to \infty$.

If we consider $S = F_w$ the weighted forward shift on l^2, with weight $(\frac{1}{w_n})$, then $B_wF_w = I$ on l^2. So B_w is M-hypercyclic. □

Madore and Martinez-Avendano in [6], state a subspace-hypercyclicity criterion as follows:

Theorem 2.9. *(see [6]) Let $T \in B(X)$ and let M be a non-zero subspace of X. Assume there exist Y and Z, dense subsets of M and an increasing sequence of positive integers $\{n_k\}$ such that:*

(i) $T^{n_k} y \to 0$ *for all* $y \in Y$,

(ii) for each $z \in Z$, *there exists a sequence* $\{x_k\}$ *in* M *such that*

$$x_k \to 0 \quad and \quad T^{n_k} x_k \to z.$$

(iii) M *is an invariant subspace for* T^{n_k} *for all* $k \in \mathbb{N}$.

Then T is subspace transitive with respect to M and hence T is subspace-hypercyclic for M.

Le showed in [5] that condition(iii) in the above theorem is necessary. In the following theorem you see that we can get condition(iii) away in Theorem2.9, if we consider increasing sequence $\{n\}$, instead of an arbitrary increasing sequence $\{n_k\}$.

By doing this we give another sufficient condition for subspace-hypercyclicity.

Theorem 2.10. *Let $T \in B(X)$ and let M be a closed subspace of X. Let X_0 and Y_0 be dense subsets of M such that:*

(i) $T^n x \to 0$ *for any* $x \in X_0$, *as* $n \to \infty$,

(ii) For any $y \in Y_0$, *there exists a sequence* $\{x_n\}$ *in* M *such that* $x_n \to 0$ *and* $T^n x_n \to y$ *as* $n \to \infty$.

Then T is subspace-hypercyclic with respect to M.

Proof. First we prove that for any nonempty open subsets $U \subset M$ and $V \subset M$, there is a positive integer n such that $T^n(U) \cap V \neq \phi$. By hypothesis X_0 and Y_0 are dense subsets of M, so $U \cap X_0 \neq \phi$ and $V \cap Y_0 \neq \phi$. Let $x \in U \cap X_0$ and $y \in V \cap Y_0$. Since U and V are open in M, there is some $\varepsilon > 0$ such that:

$$B(x, \varepsilon) \cap M \subset U \quad and \quad B(y, \varepsilon) \cap M \subset V.$$

By hypothesis, for this x and y, one can find large enough n such that:

$$||T^n x|| < \frac{\varepsilon}{2}, \quad ||x_n|| < \varepsilon \quad and \quad ||T^n x_n - y|| < \frac{\varepsilon}{2}.$$

Thus $x + x_n \in U$ and $T^n(x + x_n) = T^n x + T^n(x_n) \in V$. Hence $T^n(U) \cap V \neq \phi$. Therefore by Theorem2.2, T is M-hypercyclic. ∎

The following theorem shows that a large supply of eigenvectors is conductive to subspace-hypercyclicity. It is stated of the style of Godefroy-Shapiro criterion in [3].

Theorem 2.11. *Let $T \in B(X)$ and let M be a closed subspace of X. If the subspaces:*

$$X_0 = span\{x \in M : Tx = \lambda x \quad for \quad some \quad \lambda \in \mathbb{C} \quad with \quad |\lambda| < 1\}$$

$$Y_0 = span\{x \in M : Tx = \lambda x \quad for \quad some \quad \lambda \in \mathbb{C} \quad with \quad |\lambda| > 1\}$$

are dense in M, then T is M-mixing. Specially T is M-hypercyclic.

Proof. Let $U \subset M$ and $V \subset M$, be arbitrary open sets. By hypothesis X_0 and Y_0 are dense in M, so $U \cap X_0$ and $V \cap Y_0$ are nonempty. Consider $x \in U \cap X_0$ and $v \in V \cap Y_0$. So we can write:

$$x = \sum_{k=1}^{m} a_k x_k \quad and \quad y = \sum_{k=1}^{m} b_k y_k,$$

where a_k and b_k are scalars and $x_k \in X_0$ and $y_k \in Y_0$, for $1 \leq k \leq m$. So one can find scalars λ_k with $|\lambda_k| < 1$ and μ_k with $|\mu_k| > 1$ such that $Tx_k = \lambda_k x_k$ and $Ty_k = \mu_k y_k$, for $1 \leq k \leq m$.

It is not hard to see that:

$$T^n x = \sum_{k=1}^{m} a_k {\lambda_k}^n x_k \to 0 \quad and \quad u_n := \sum_{k=1}^{m} b_k \frac{1}{\mu_k^n} y_k \to 0$$

as $n \to \infty$. Also note that $T^n x \in M$, $u_n \in M$ and $T^n u_n = y$. Hence there is some natural number N such that for every $n \geq N$,

$$x + u_n \in U \quad and \quad T^n(x + u_n) = T^n x + y \in V.$$

That means $T^n(U) \cap V \neq \phi$ for every $n \geq N$. Therefore T is M-mixing.

Moreover for each $x \in X_0$ we have $T^n x = \lambda^n x$. Hence $T^n x \to 0$ as $n \to \infty$, since $|\lambda| < 1$. So by Corollary2.3, T is M-hypercyclic. □

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