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ON THE GERBER-SHIU PENALTY FUNCTION FOR THE COMPOUND BINOMIAL RISK MODEL WITH DELAYED CLAIMS

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Abstract: In this paper we consider the Gerber-Shiu penalty function in the compound binomial risk model with time-correlated claims. It is assumed that each main claim will induce a by-claim but the occurrence of the by-claim may be delayed with a certain probability. Formulas for the probability generating function of the penalty function are obtained, together with the expression for the penalty function with zero initial surplus. Then we show that the penalty function satisfies a defective renewal equation. When the claims follow geometric distributions, explicit expression for the Gerber-Shiu function is derived. Numerical example is provided to illustrate the application of the result discussed in the paper.

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1. Introduction

The classical compound binomial risk model has been studied in actuarial lit-

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erature by [1, 2, 3] among many others. Some ruin quantities such as ruin probabilities, the deficit at ruin and the surplus prior to ruin are investigated extensively, based on the assumption of stationary and independent increment in the surplus process. In practical contexts, this assumption can be restrictive. Recently, many authors consider the compound binomial risk model with time-correlated claims. Yuen and Guo [4] present a compound binomial model by assuming that every main claim will produce a by-claim but the occurrence of the by-claim may be delayed for one time period with a certain probability. They derive recursive formulas for the finite time ruin probabilities and explicit expressions for ultimate ruin probabilities. Xiao and Guo [5] further investigate the joint distribution of the surplus immediately prior to ruin and deficit at ruin. Wu and Li [6] consider a compound binomial model with time-correlated individual claims and dividend payments that are ruled by a constant dividend barrier, the expected present value of total dividends are studied. Other dependence structures in terms of main claims and by-claims are studied in Wu and Yuen [7] and the references therein.

Motivated by Yuen and Guo [4], we explore the compound binomial model with different framework of time-correlated claims. It is assumed that each main claim induces a by-claim. The number of main claim process $\{N(n),\ n \in \mathbb{N}\}$ is a binomial process with independent and identically distributed (i.i.d.) interclaim times $\{T_1, T_2, \ldots\}$ having probability function (p.f.) $k(x) = (1-q)q^{x-1}$ for $x \in \mathbb{N}^+$ and $0 < q < 1$. The main claim amounts $\{X_1, X_2, \ldots\}$ are i.i.d. positive and integer valued random variables with common cumulative distribution function (c.d.f.) $F(x) = 1 - \overline{F}(x)$, p.f. $f(x)$ and mean μ_F.

Suppose that the main claim X_i in epoch T_i will induce a by-claim Y_i. Moreover, the by-claim Y_i may occur simultaneously with main claim X_i with probability θ ($0 \le \theta \le 1$), or the occurrence of the by-claim may be delayed to T_{i+1} with probability $1-\theta$. In the later case, we assume that the occurrence of the delayed by-claim Y_i is independent of the occurrence of the next main claim X_{i+1}.

The by-claims $\{Y_1, Y_2, \ldots\}$ are also i.i.d. positive and integer valued random variables having c.d.f., p.f. and mean denoted by $G(y)$, $g(y)$ and μ_G, respectively. The independence between $\{X_1, X_2, \ldots\}$, $\{Y_1, Y_2, \ldots\}$ and $\{N(n),\ n \in \mathbb{N}\}$ are assumed. Suppose that the premium rate received per period is 1, then the surplus process $U(n)$ of an insurance company at time n is described as

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i - R(n), \quad n \in \mathbb{N}, \tag{1}$$

where $U(0) = u \in \mathbb{N}$ is the initial surplus and $R(n)$ is the total by-claim amounts up to time n. With $p = 1 - q$, it is easy to see that

$$\mathrm{E}\left(\sum_{i=1}^{N(n)} X_i + R(n)\right) = np\mu_F + np\mu_G - (1-\theta)\mu_G(1-q^n).$$

Therefore, we further assume that $p(\mu_F + \mu_G) < 1$, providing a positive loading condition.

Let $T = \inf\{n \in \mathbb{N}^+ : \; U(n) < 0\}$ be the time of ruin with $T = \infty$ if ruin does not occur. Note that if ruin occurs, $U(T-1)$ is the surplus one period prior to ruin and $|U(T)|$ is the deficit at ruin. Denote by

$$m(u) = \mathrm{E}\left[\omega(U(T-1), |U(T)|)I(T < \infty) \,|\, U(0) = u\right]$$

the Gerber-Shiu expected discounted free penalty function (e.g., Gerber and Shiu [8]). Where $\omega(u_1, u_2) : \mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ and $I(A)$ is the indicator function of an event A.

The rest of the paper is structured as follows. In Section 2, both the Gerber-Shiu penalty function with zero initial surplus and the generating function of the penalty function are obtained. The defective renewal equation satisfied by the penalty function is derived in Section 3. In Section 4 the explicit expression for the penalty function is given when the individual claim amounts from both classes follow geometric distribution. Numerical result is also provided to illustrate the application of the result discussed in this section.

2. The Probability Generating Function

Throughout the rest of the entire paper, unless otherwise stated, we adopt the convention that $\sum_m^n = 0$ when $n < m$. For any function $a(x)$, we will use $\hat{a}(z)$ to denote the corresponding probability generating function (p.g.f.).

In order to derive the p.g.f. of the expected penalty function $m(u)$, we consider a complementary surplus process as follows. Instead of having one main claim and a by-claim with probability θ in the first epoch T_1, another by-claim distributed as Y_1 occurs simultaneously at this time. We denote the corresponding expected discounted free Gerber-Shiu function for this auxiliary model by $m_1(u)$. For the risk process (1), there are two possibilities at time T_1:

1. A main claim and its by-claim occur concurrently, then the surplus process gets renewed except for the initial value;

2. A main claim occurs and its associated by-claim is delayed to T_2, then the surplus process transfers to the auxiliary model described as above.

Considering what will happen at time T_1, by the law of total probability we have

$$m(u) = p\sum_{t=1}^{\infty} q^{t-1}[\theta\gamma(u+t) + (1-\theta)\gamma_1(u+t)], \tag{2}$$

where

$$\gamma(i) = \sum_{k=2}^{i} m(i-k) f * g(k) + \omega(i), \quad \omega(i) = \sum_{k=i+1}^{\infty} \omega(i-1, k-i) f * g(k),$$

$$\gamma_1(i) = \sum_{k=1}^{i} m_1(i-k) f(k) + \omega_1(i), \quad \omega_1(i) = \sum_{k=i+1}^{\infty} \omega(i-1, k-i) f(k),$$

for $i \in \mathbb{N}^+$, and $*$ denotes convolution of the p.f.s. Equation (2) can be rewritten as

$$qm(u) = p\sum_{t=1}^{\infty} q^{t}[\theta\gamma(u+t) + (1-\theta)\gamma_1(u+t)]. \tag{3}$$

But from (2) with u replaced by $u-1$, it follows that

$$m(u-1) = p\sum_{t=0}^{\infty} q^{t}[\theta\gamma(u+t) + (1-\theta)\gamma_1(u+t)], \quad u \in \mathbb{N}^+. \tag{4}$$

Subtracting (3) from (4) yields

$$m(u-1) - qm(u) = p[\theta\gamma(u) + (1-\theta)\gamma_1(u)], \quad u \in \mathbb{N}^+. \tag{5}$$

Multiplying (5) by z^u and summing over u from 1 to ∞, it follows that

$$(z-q)\hat{m}(z) + qm(0) = p\big\{\theta\hat{m}(z)\hat{f}(z)\hat{g}(z) + (1-\theta)\hat{m}_1(z)\hat{f}(z) + \hat{\eta}(z)\big\}, \tag{6}$$

which is equivalent to

$$\big[z - q - p\theta\hat{f}(z)\hat{g}(z)\big]\hat{m}(z) - p(1-\theta)\hat{f}(z)\hat{m}_1(z) = p\hat{\eta}(z) - qm(0), \tag{7}$$

where $\hat{\eta}(z) = \theta\hat{\omega}(z) + (1-\theta)\hat{\omega}_1(z)$.

For the auxiliary process we have analogously

$$m_1(u) = p\sum_{t=1}^{\infty} q^{t-1}\{\theta\gamma_2(u+t) + (1-\theta)\gamma_3(u+t)\},$$

where

$$\gamma_2(i) = \sum_{k=3}^{i} m(i-k) f * g * g(k) + \omega_2(i),$$

$$\omega_2(i) = \sum_{k=i+1}^{\infty} \omega(i-1, k-i) f * g * g(k),$$

$$\gamma_3(i) = \sum_{k=2}^{i} m_1(i-k) f * g(k) + \omega(i),$$

for $i \in \mathbb{N}^+$. Imitating closely the procedure to derive the Equation (7), we have

$$\big[z - q - p(1-\theta)\hat{f}(z)\hat{g}(z)\big]\hat{m}_1(z) - p\theta\hat{f}(z)\hat{g}^2(z)\hat{m}(z) = p\hat{\eta}_1(z) - qm_1(0), \quad (8)$$

where $\hat{\eta}_1(z) = \theta\hat{\omega}_2(z) + (1-\theta)\hat{\omega}(z)$.

Solving the simultaneous Equations (7) and (8), we obtain the expressions for $\hat{m}(z)$ and $\hat{m}_1(z)$ as follows:

$$\hat{m}(z) = \frac{\left\{\begin{array}{c} \big(z - q - p(1-\theta)\hat{f}(z)\hat{g}(z)\big)\big(p\hat{\eta}(z) - qm(0)\big) \\ +p(1-\theta)\hat{f}(z)\big(p\hat{\eta}_1(z) - qm_1(0)\big) \end{array}\right\}}{(z-q)\big[z - q - p\hat{f}(z)\hat{g}(z)\big]}, \quad (9)$$

and

$$\hat{m}_1(z) = \frac{\left\{\begin{array}{c} \big(z - q - p\theta\hat{f}(z)\hat{g}(z)\big)\big(p\hat{\eta}_1(z) - qm_1(0)\big) \\ +p\theta\hat{f}(z)\hat{g}^2(z)\big(p\hat{\eta}(z) - qm(0)\big) \end{array}\right\}}{(z-q)\big[z - q - p\hat{f}(z)\hat{g}(z)\big]}. \quad (10)$$

To determine the constants $m(0)$ and $m_1(0)$, we consider the roots of the common denominator of (9) and (10), *i.e.*, the roots of the following equation:

$$(z-q)\big[z - q - p\hat{f}(z)\hat{g}(z)\big] = 0. \quad (11)$$

Obviously, 1 and q are two roots to the Equation (11). Now let $l(z) = z - q$ and $\kappa(z) = p\hat{f}(z)\hat{g}(z)$. It is easy to see that $\kappa'(z) \geq 0, \kappa''(z) \geq 0$. Also, $l(q) = 0 < \kappa(q)$ and $l(1) = 1 - q = p = \kappa(1)$. Furthermore, $\kappa'(z)|_{z=1} = p(\mu_F + \mu_G) < 1$ by noting the safety loading condition. Therefore, we conclude that 1 and q are unique roots to the equation (11) in the interval $(0, 1]$.

Since $\hat{m}(z)$ and $\hat{m}_1(z)$ are analytic, 1 and q must be zeros of the numerators of (9) and (10). Then we can obtain the expressions for $m(0)$ and $m_1(0)$ after some calculations

$$m(0) = \frac{p\{\theta\hat{\eta}(1) + (1-\theta)[\hat{g}(q)\hat{\eta}(q) + \hat{\eta}_1(1) - \hat{\eta}_1(q)]\}}{q[\theta + (1-\theta)\hat{g}(q)]}, \quad (12)$$

and

$$m_1(0) = \frac{p\{(1-\theta)\hat{\eta}_1(1)\hat{g}(q) + \theta[\hat{g}(q)(\hat{\eta}(1) - \hat{\eta}(q)) + \hat{\eta}_1(q)]\}}{q[\theta + (1-\theta)\hat{g}(q)]}. \tag{13}$$

Hence, $\hat{m}(z)$ and $\hat{m}_1(z)$ can be determined by substituting (12) and (13) into (9) and (10) respectively.

In the sequel, we denote the two distinct roots to (11) by $\rho_1 = 1$ and $\rho_2 = q$ for notational convenience.

3. The Defective Renewal Equation

In this section we aim to derive the defective renewal equation for the Gerber-Shiu function $m(u)$. In order to simplify calculations, we will use the discrete operator proposed by Li [9]. More precisely, define T_r to be the operator of any real valued function $\xi(x), x \in \mathbb{N}^+$, by

$$T_r\xi(y) = \sum_{x=y}^{\infty} r^{x-y}\xi(x), \quad y \in \mathbb{N}^+.$$

Some nice properties for this operator have been given in [9].

Now we denote the denominator of (9) by $\hat{h}_1(z) - \hat{h}_2(z)$, where $\hat{h}_1(z) = (z-q)^2$ and $\hat{h}_2(z) = p(z-q)\hat{f}(z)\hat{g}(z)$. Let $h_1(x)$ and $h_2(x)$ are the inverse function of $\hat{h}_1(z)$ and $\hat{h}_2(z)$, respectively. Using the Lagrange interpolating polynomial, $\hat{h}_1(z)$ can be rewritten as

$$\begin{aligned}
\hat{h}_1(z) &= \hat{h}_1(0)\frac{(z-\rho_1)(z-\rho_2)}{\rho_1\rho_2} + \hat{h}_1(\rho_1)\frac{z(z-\rho_2)}{\rho_1(\rho_1-\rho_2)} + \hat{h}_1(\rho_2)\frac{z(z-\rho_1)}{\rho_2(\rho_2-\rho_1)} \\
&= \hat{h}_1(0)\frac{(z-\rho_1)(z-\rho_2)}{\rho_1\rho_2} + \hat{h}_2(\rho_1)\frac{z-\rho_2}{\rho_1-\rho_2} + \hat{h}_2(\rho_2)\frac{z-\rho_1}{\rho_2-\rho_1} \\
&\quad + (z-\rho_1)(z-\rho_2)\left(\frac{\hat{h}_2(\rho_1)}{\rho_1(\rho_1-\rho_2)} + \frac{\hat{h}_2(\rho_2)}{\rho_2(\rho_2-\rho_1)}\right).
\end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}
\hat{h}_1(z) - \hat{h}_2(z) &= (z-\rho_1)(z-\rho_2)\left(\frac{\hat{h}_1(0)}{\rho_1\rho_2} + \frac{\hat{h}_1(\rho_1)}{\rho_1(\rho_1-\rho_2)} + \frac{\hat{h}_1(\rho_2)}{\rho_2(\rho_2-\rho_1)}\right) \\
&\quad - \left(\hat{h}_2(z) + \frac{(z-\rho_1)\hat{h}_2(\rho_2) - (z-\rho_2)\hat{h}_2(\rho_1)}{\rho_1-\rho_2}\right).
\end{aligned} \tag{15}$$

By noting that $\rho_1 = 1$, it is easy to see that

$$\frac{\hat{h}_1(0)}{\rho_1\rho_2} + \frac{\hat{h}_1(\rho_1)}{\rho_1(\rho_1-\rho_2)} + \frac{\hat{h}_1(\rho_2)}{\rho_2(\rho_2-\rho_1)} = 1. \tag{16}$$

On the other hand, by the properties of the discrete operator, we obtain

$$\begin{aligned}\hat{h}_2(z) &+ \frac{(z-\rho_1)\hat{h}_2(\rho_2)-(z-\rho_2)\hat{h}_2(\rho_1)}{\rho_1-\rho_2} \\ &= (z-\rho_1)(z-\rho_2)\left[T_zT_{\rho_2}T_{\rho_1}h_2(1) - T_zT_{\rho_2}h_2(1)\right].\end{aligned} \tag{17}$$

Equations (14)–(17) imply that

$$\hat{h}_1(z) - \hat{h}_2(z) = (z-\rho_1)(z-\rho_2)[1 + T_zT_{\rho_2}h_2(1) - T_zT_{\rho_2}T_{\rho_1}h_2(1)]. \tag{18}$$

Now we deal with the numerator of (9). Define

$$\begin{aligned}\hat{\tau}_1(z) &= qm(0)(z-q),\\ \hat{\tau}_2(z) &= p\big\{\big[z\hat{\eta}(z) - q\hat{\eta}(z)\big] + p(1-\theta)\big[\hat{Q}_2(z)-\hat{Q}_1(z)\big]\\ &\quad + q(1-\theta)\big[\hat{f}(z)\hat{g}(z)m(0) - \hat{f}(z)m_1(0)\big]\big\},\end{aligned}$$

where $\hat{Q}_1(z) = \hat{f}(z)\hat{g}(z)\hat{\eta}(z)$ and $\hat{Q}_2(z) = \hat{f}(z)\hat{\eta}_1(z)$.

Noting that $\hat{\tau}_2(z) - \hat{\tau}_1(z)$ is an alternative expression for the numerator of (9), $\hat{\tau}_2(\rho_i) = \hat{\tau}_1(\rho_i)$ holds for $i = 1, 2$. Since $\hat{\tau}_1(z)$ is a polynomial of degree 1, using the Lagrange interpolating formula once again, we have

$$\hat{\tau}_1(z) = \frac{\hat{\tau}_2(\rho_1)(z-\rho_2) - \hat{\tau}_2(\rho_2)(z-\rho_1)}{\rho_1-\rho_2}. \tag{19}$$

Consequently, by (19) we get

$$\hat{\tau}_2(z) - \hat{\tau}_1(z) = (z-\rho_1)(z-\rho_2)[T_zT_{\rho_2}T_{\rho_1}\tau_2(1) - T_zT_{\rho_2}\tau_2(1)]. \tag{20}$$

By (18) and (20) we can rewrite the expression $\hat{m}(z)$ as follows

$$\hat{m}(z) = \frac{\hat{\tau}_2(z)-\hat{\tau}_1(z)}{\hat{h}_1(z)-\hat{h}_2(z)} = \frac{T_zT_{\rho_2}T_{\rho_1}\tau_2(1) - T_zT_{\rho_2}\tau_2(1)}{1 + T_zT_{\rho_2}h_2(1) - T_zT_{\rho_2}T_{\rho_1}h_2(1)}, \tag{21}$$

and (21) implies that

$$\hat{m}(z) = [T_zT_{\rho_2}T_{\rho_1}h_2(1) - T_zT_{\rho_2}h_2(1)]\hat{m}(z) + [T_zT_{\rho_2}T_{\rho_1}\tau_2(1) - T_zT_{\rho_2}\tau_2(1)]. \tag{22}$$

Denote by $Q_1(k)$ and $Q_2(k)$ the inverse image of $\hat{Q}_1(z)$ and $\hat{Q}_2(z)$ respectively, then we can show that the Geber-Shiu function $m(u)$ satisfies a defective renewal equation.

Theorem. *For $u \in \mathbb{N}$, we have*

$$m(u) = p \sum_{k=0}^{u} m(u-k)\zeta(k) + \varphi(u), \tag{23}$$

where

$$\zeta(k) = \sum_{j=k+1}^{\infty} f * g(j),$$

$$\varphi(u) = p\delta(u) + q(1-\theta) \sum_{k=u+1}^{\infty} (1 - q^{k-u-1})[m(0)f * g(k) - m_1(0)f(k)],$$

$$\delta(u) = \sum_{k=u+1}^{\infty} \{\eta(k) + (1-\theta)(1 - q^{k-u-1})[Q_2(k) - Q_1(k)]\}, \tag{24}$$

and $m(0)$ and $m_1(0)$ are determined by (12) and (13), respectively.

Proof. For notational convenience, let $\hat{b}(z) = \hat{f}(z)\hat{g}(z)$ and $b(x)$ is the inverse function of $\hat{b}(z)$. Then $\hat{h}_2(z)$ can be expressed as $\hat{h}_2(z) = p[z\hat{b}(z) - q\hat{b}(z)]$. Thus we have

$$\begin{aligned} T_z T_{\rho_2} T_{\rho_1} h_2(1) = \frac{p}{\rho_1 - \rho_2} \Bigg\{ & \rho_1 \frac{z\hat{b}(z) - \rho_1 \hat{b}(\rho_1)}{z - \rho_1} - \rho_2 \frac{z\hat{b}(z) - \rho_2 \hat{b}(\rho_2)}{z - \rho_2} \\ & - q\left(\rho_1 T_z T_{\rho_1} b(1) - \rho_2 T_z T_{\rho_2} b(1)\right) \Bigg\}. \end{aligned} \tag{25}$$

For $i = 1, 2$, one deduces

$$\frac{z\hat{b}(z) - \rho_i \hat{b}(\rho_i)}{z - \rho_i} = \hat{b}(z) + \rho_i T_z T_{\rho_i} b(1). \tag{26}$$

Substituting (26) into (25) yields

$$T_z T_{\rho_2} T_{\rho_1} h_2(1) = p\big[\hat{b}(z) + \rho_1 T_z T_{\rho_1} b(1)\big]. \tag{27}$$

Similarly, by (26) we have

$$T_z T_{\rho_2} h_2(1) = p \left\{ \frac{z\hat{b}(z) - \rho_2 \hat{b}(\rho_2)}{z - \rho_2} - q \frac{\hat{b}(z) - \hat{b}(\rho_2)}{z - \rho_2} \right\} = p\hat{b}(z),$$

which together with (27) gives

$$T_zT_{\rho_2}T_{\rho_1}h_2(1) - T_zT_{\rho_2}h_2(1) = p\rho_1 T_zT_{\rho_1}b(1). \tag{28}$$

If we denote by $\hat{W}(z) = p(z\hat{\eta}(z) - q\hat{\eta}(z))$ and $W(x)$ the corresponding inverse function, then (28) implies that

$$T_zT_{\rho_2}T_{\rho_1}W(1) - T_zT_{\rho_2}W(1) = p\rho_1 T_zT_{\rho_1}\eta(1). \tag{29}$$

On the other hand, it is easy to see that

$$T_zT_{\rho_2}T_{\rho_1}Q_i(1) - T_zT_{\rho_2}Q_i(1) = \frac{T_zT_{\rho_2}Q_i(1) - T_zT_{\rho_1}Q_i(1)}{\rho_2 - \rho_1}, \quad i = 1, 2, \tag{30}$$

$$T_zT_{\rho_2}T_{\rho_1}f(1) - T_zT_{\rho_2}f(1) = \frac{T_zT_{\rho_2}f(1) - T_zT_{\rho_1}f(1)}{\rho_2 - \rho_1}, \tag{31}$$

$$T_zT_{\rho_2}T_{\rho_1}f * g(1) - T_zT_{\rho_2}f * g(1) = \frac{T_zT_{\rho_2}f * g(1) - T_zT_{\rho_1}f * g(1)}{\rho_2 - \rho_1}. \tag{32}$$

Recall the fact that $\rho_1 - \rho_2 = p$, combine (30)–(32) leads to

$$\begin{aligned} T_zT_{\rho_2}T_{\rho_1}\tau_2(1) - T_zT_{\rho_2}\tau_2(1) = {} & p\{\rho_1 T_zT_{\rho_1}\eta(1) + (1-\theta)[T_zT_{\rho_1}Q_2(1) \\ & - T_zT_{\rho_2}Q_2(1) - (T_zT_{\rho_1}Q_1(1) - T_zT_{\rho_2}Q_1(1))]\} \\ & + q(1-\theta)\{m(0)[T_zT_{\rho_1}f * g(1) - T_zT_{\rho_2}f * g(1)] \\ & - m_1(0)[T_zT_{\rho_1}f(1) - T_zT_{\rho_2}f(1)]\}. \end{aligned} \tag{33}$$

Remember that $\rho_1 = 1$ and $\rho_2 = q$. Substitute (28) and (33) into (22), the renewal equation then follows by inverting the p.g.f. of $\hat{m}(z)$ immediately.

Now we only need to show that the renewal Equation (23) is defective. In fact, by the positive relative safety condition we deduce that

$$p\sum_{k=0}^{\infty}\zeta(k) = p\sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty} f * g(j) = p(\mu_F + \mu_G) < 1,$$

and theorem is proved. □

We remark that explicit solution of the defective renewal equation (23) can be obtained by using results in Li [10] (Section 3) or Wu and Li [11] (Section 2). In the special case when $\theta = 1$, the risk model studied in the present paper degenerates to the classical compound binomial risk process, and the only difference is that the claim amount variable is $X_i + Y_i$. Let $\theta = 1$ in (23), the result coincides with Theorem 4.1 in [12] with some modifications.

4. Geometric Claim Amount Distribution

In this section we assume that main claims and by-claims are both geometrically distributed with $f(x) = p_1 q_1^{x-1}$, $g(x) = p_2 q_2^{x-1}$, $x \in \mathbb{N}^+$ for $0 < q_i < 1$ and $p_i = 1 - q_i$, $i = 1, 2$. It can be easily verified that

$$f * g(k) = \frac{p_1 p_2 \left(q_2^{k-1} - q_1^{k-1}\right)}{q_2 - q_1}, \quad k = 2, 3, \cdots,$$

$$f * g * g(k) = \frac{p_1 p_2^2 \{[(k-2)q_2 - (k-1)q_1] q_2^{k-2} + q_1^{k-1}\}}{(q_2 - q_1)^2}, \quad k = 3, 4, \cdots.$$

From (21) we know that

$$\hat{m}(z) = \frac{(z-\rho_1)(z-\rho_2)[T_z T_{\rho_2} T_{\rho_1} \tau_2(1) - T_z T_{\rho_2} \tau_2(1)]}{(z-q)[z - q - p\hat{f}(z)\hat{g}(z)]}. \tag{34}$$

Noting that $\hat{f}(z) = \frac{z p_1}{1 - z q_1}$, $\hat{f}(z)\hat{g}(z) = \frac{z^2 p_1 p_2}{(1-zq_1)(1-zq_2)}$, after some careful calculations we obtain

$$T_z T_1 f(1) - T_z T_q f(1) = \frac{p q_1}{(1 - z q_1)(1 - q q_1)}, \tag{35}$$

and

$$T_z T_{\rho_1} f * g(1) - T_z T_{\rho_2} f * g(1) = \frac{p\{1 - q q_1 q_2 - q_1 q_2 [1 + q(p_1 - q_2)] z\}}{(1 - z q_1)(1 - z q_2)(1 - q q_1)(1 - q q_2)}. \tag{36}$$

Denote by

$$\varrho(z) = (1 - z q_1)(1 - z q_2)(z - q)^2 - p p_1 p_2 z^2 (z - q),$$

which is a polynomial of degree 4 with leading coefficient $q_1 q_2$. Obviously, 1 and q are two roots of $\varrho(z)$. Therefore, if R_1 and R_2 are another two roots, we can rewrite $\varrho(z)$ as

$$\varrho(z) = q_1 q_2 (z - 1)(z - q)(R_1 - z)(R_2 - z).$$

In what follows, we assume that R_1 and R_2 are distinct, by partial fraction we have

$$\frac{1}{(R_1 - z)(R_2 - z)} = \sum_{i=1}^{2} \frac{a_i}{R_i - z}, \quad a_1 = \frac{1}{R_2 - R_1}, \; a_2 = \frac{1}{R_1 - R_2},$$

$$\frac{z}{(R_1 - z)(R_2 - z)} = \sum_{i=1}^{2} \frac{b_i}{R_i - z}, \quad b_1 = \frac{R_1}{R_2 - R_1}, \ b_2 = \frac{R_2}{R_1 - R_2},$$

$$\frac{1 - zq_2}{(R_1 - z)(R_2 - z)} = \sum_{i=1}^{2} \frac{c_i}{R_i - z}, \quad c_1 = \frac{1 - q_2 R_1}{R_2 - R_1}, \ c_2 = \frac{1 - q_2 R_2}{R_1 - R_2},$$

$$\frac{(1 - zq_1)(1 - zq_2)}{(R_1 - z)(R_2 - z)} = q_1 q_2 + \sum_{i=1}^{2} \frac{d_i}{R_i - z},$$

$$d_1 = \frac{1 - R_1[q_1 + q_2 - q_1 q_2 R_1]}{R_2 - R_1}, \ d_2 = \frac{1 - R_2[q_1 + q_2 - q_1 q_2 R_2]}{R_1 - R_2}.$$

Multiplying both the denominator and numerator of (34) by $(1 - zq_1)(1 - zq_2)$, (33), (35) and (36) imply that

$$\hat{m}(z) = \frac{pq(1-\theta)}{q_1 q_2 (1 - qq_1)(1 - qq_2)} \Bigg\{ m(0) \Bigg[(1 - qq_1 q_2) \sum_{i=1}^{2} \frac{a_i}{R_i - z} - q_1 q_2 (1 + q(p_1 - q_2)) \sum_{i=1}^{2} \frac{b_i}{R_i - z} \Bigg] - m_1(0) q_1 (1 - qq_2) \sum_{i=1}^{2} \frac{c_i}{R_i - z} \Bigg\} + \frac{p}{q_1 q_2} \left(q_1 q_2 + \sum_{i=1}^{2} \frac{d_i}{R_i - z} \right) \hat{\delta}(z), \tag{37}$$

where

$$\hat{\delta}(z) = T_z T_1 \eta(1) + (1-\theta)[T_z T_1 Q_2(1) - T_z T_q Q_2(1) - (T_z T_1 Q_1(1) - T_z T_q Q_1(1))],$$

and the corresponding inverse image $\delta(u)$ is determined by (24).

Inverting the p.g.f. in (37) gives an explicit expression for $m(u)$ as follows:

$$m(u) = \frac{pq(1-\theta)}{q_1 q_2 (1 - qq_1)(1 - qq_2)} \Bigg\{ m(0) \Bigg[(1 - qq_1 q_2) \sum_{i=1}^{2} a_i R_i^{-(u+1)} - q_1 q_2 (1 + q(p_1 - q_2)) \sum_{i=1}^{2} b_i R_i^{-(u+1)} \Bigg] - m_1(0) q_1 (1 - qq_2) \sum_{i=1}^{2} c_i R_i^{-(u+1)} \Bigg\} + \frac{p}{q_1 q_2} \left[q_1 q_2 \delta(u) + \sum_{i=1}^{2} d_i \sum_{j=0}^{u} \delta(u-j) R_i^{-(j+1)} \right].$$

Example. Let $\omega(x,y)=1$, then the Gerber-Shiu function $m(u)$ and $m_1(u)$ reduces to the ruin probabilities $\psi(u)$ and $\psi_1(u)$, respectively. Assume that $p=0.25$, $p_1=0.6$ and $p_2=0.75$. It is easy to check that the net profit condition holds. Now in this case, we obtain $R_1=6.1568433$ and $R_2=1.2181567$. The initial value $\psi(0)$ and $\psi_1(0)$, given in (12) and (13) respectively, are also obtained in Table 1.

Figure 1 gives the ruin probabilities $\psi(u)$ in Example 1, for fixed value $\theta=0,0.25,0.5,0.75,1$ and different initial values of $u=0,1,2\ldots,12$. It shows that these probabilities decrease generally as the initial surplus u increases. Moreover, with fixed u, ruin probabilities increase as θ increases, which is consistent with our intuitive knowledge.

Table 1: $p=0.25$, $p_1=0.6$, $p_2=0.75$.

θ	0	0.25	0.5	0.75	1
$\psi(0)$	0.5185185	0.5666667	0.6060606	0.6388889	0.6666667
$\psi_1(0)$	0.6666667	0.7000000	0.7272727	0.7500000	0.7692308

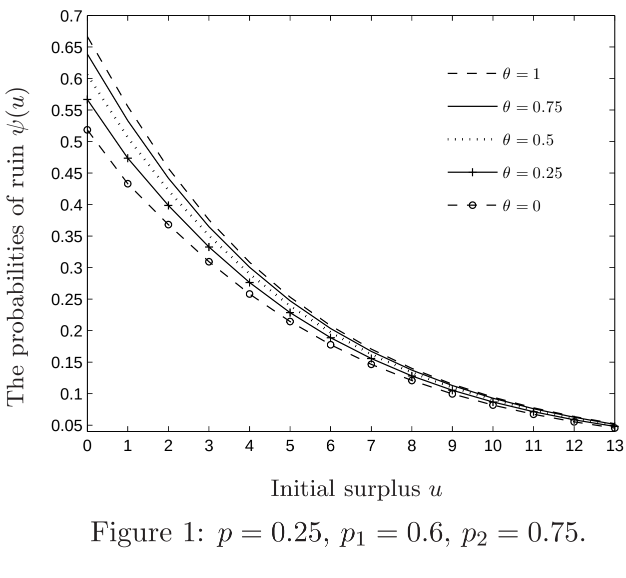

Figure 1: $p=0.25$, $p_1=0.6$, $p_2=0.75$.

5. Conclusions

In this paper a framework of time-correlated claims is proposed by introducing two kinds of dependent claims, namely main claims and by-claims, and allowing possible delays of the occurrence of the by-claims. Some analytic techniques are applied to study the Gerber-Shiu function. We show that the generating function and defective renewal equation for the Gerber-Shiu function can be obtained. When the claim amounts follow geometric distribution, the explicit expression for the Gerber-Shiu function is also derived. The work of this paper can be regarded as an extension to the prior work on time-correlated claims studied by Xiao and Guo [5] and the references therein.

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