CHARACTERIZATION OF ADJOINTABLE OPERATORS ON HILBERT C^* -modules

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Abstract: Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. In this paper, we find the general form of the mappings $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfying

$$2\langle T(x), T(y) \rangle = \langle T(x), y \rangle + \langle x, T(y) \rangle \quad (x, y \in \mathcal{H}),$$

as adjointable (bounded) $\mathcal{A}$ -linear operators. The generalized Hyers-Ulam stability of the functional equation is discussed.

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1. Introduction and Preliminaries

The notion of a Hilbert C^* -module initiated by Kaplansky [3] as a generalization of a Hilbert space in which the inner product takes its values in a C^* -algebra.

Let $\mathcal{A}$ be a C^* -algebra. A pre-Hilbert $\mathcal{A}$ -module or an inner product $\mathcal{A}$ -module is a complex linear space $\mathcal{H}$ which is a left $\mathcal{A}$ -module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x)$ ($\lambda \in \mathbb{C}, x \in \mathcal{H}, a \in \mathcal{A}$), together with an $\mathcal{A}$ -valued inner product $(x, y) \mapsto \langle x, y \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ such that for each $x, y, z \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

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- (i) $\langle x, x \rangle \geq 0$ and the equality holds if and only if $x = 0$.
- (ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- (iii) $\langle ax, y \rangle = a \langle x, y \rangle$,
- (iv) $\langle x, y \rangle^* = \langle y, x \rangle$.

The notion of a right Hilbert $\mathcal{A}$ -module can be defined similarly. Note that the condition (i) is understood as a statement in the C^* -algebra $\mathcal{A}$, where an element a is called positive if it can be represented as bb^* for some $b \in \mathcal{A}$. The conditions (ii) and (iv) imply the inner product to be conjugate-linear in its second variable. Validity of a useful version of the classical Cauchy-Schwartz inequality follows that $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ defines a norm on $\mathcal{H}$ making it into a normed left $\mathcal{A}$ -module. An inner product $\mathcal{A}$ -module $\mathcal{H}$ which is complete with respect to the norm $\|x\|$ is called a Hilbert $\mathcal{A}$ -module or a Hilbert C^* -module over the C^* -algebra $\mathcal{A}$. Every C^* -algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$ -module under the $\mathcal{A}$ -valued inner product $\langle a, b \rangle = a^*b$ ($a, b \in \mathcal{A}$). Every complex Hilbert space is a left Hilbert $\mathbb{C}$ -module.

One may define an $\mathcal{A}$ -valued norm $|x| = \langle x, x \rangle^{\frac{1}{2}}$ (where, $|a|$ denotes the unique square root of the positive element a in $\mathcal{A}$). Clearly, $\||x|\| = \|x\|$, for each $x \in \mathcal{H}$. The $\mathcal{A}$ -valued norm $|x|$ is a useful device but it needs to be handled with care. For example, it is known that $|\cdot|$ does not satisfy the triangle inequality $|x + y| \leq |x| + |y|$ for each $x, y \in \mathcal{H}$; cf. [4].

Let $\mathcal{H}$ be a Hilbert C^* -module over a C^* -algebra $\mathcal{A}$. A system $(e_i)_{i \in I}$ in $\mathcal{H}$ is called orthogonal, if $\langle e_i, e_j \rangle = 0$ whenever $i \neq j$. An orthogonal system $(e_i)_{i \in I}$ in $\mathcal{H}$ is said to be an orthonormal, provided $\mathcal{A}$ is unital and for the inner squares it happens that $\langle e_i, e_i \rangle = 1$ for all $i \in I$. Let $(e_i)_{i \in I}$ be an orthonormal system in a Hilbert module $\mathcal{H}$ over a unital C^* -algebra $\mathcal{A}$. Landi and Pavlov showed in Theorem 2.10 of [5] that the following conditions are equivalent:

- (i) For any x of $\mathcal{H}$ there are elements a_i of $\mathcal{A}$ such that

$$x = \sum_{i \in I} a_i e_i \tag{1}$$

where convergence in norm is meant and

$$\sum_{i \in I} a_i e_i = \lim_{F \in \mathcal{F}} \sum_{i \in F} a_i e_i$$

indicates the limit over the set $\mathcal{F}$ of all finite subsets of I , directed by inclusions.

- (ii) The system $(e_i)_{i \in I}$ generates $\mathcal{H}$ over $\mathcal{A}$, that is to say, the closure of its $\mathcal{A}$ -linear span coincides with $\mathcal{H}$.
- (iii) The system $(e_i)_{i \in I}$ is closed, that is to say, for any $x \in \mathcal{H}$

$$\langle x, x \rangle = \sum_{i \in I} \langle x, e_i \rangle \langle e_i, x \rangle,$$

where the series converges in norm.

An orthonormal system $(e_i)_{i \in I}$ satisfying the equivalent conditions (i)-(iii) is called a *Schauder basis* for $\mathcal{H}$ over $\mathcal{A}$. If $(e_i)_{i \in I}$ is a Schauder basis for $\mathcal{H}$, then the coefficients in the decomposition (1) are unique for any vector x of $\mathcal{H}$. In fact for any $i \in I$, $a_i = \langle x, e_i \rangle$. Thus any vector of $\mathcal{H}$ is the limit in norm of its Fourier series. Any Schauder basis $(e_i)_{i \in I}$ is complete, i.e. there is no non-zero vector x of $\mathcal{H}$ such that $\langle x, e_i \rangle = 0$ for all $i \in I$. Note that by Proposition 3.1. of [5] any two closed orthonormal systems of a Hilbert module over a unital C^* -algebra have the same cardinality.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert C^* -modules over a C^* -algebra $\mathcal{A}$. A mapping $T : \mathcal{H} \rightarrow \mathcal{K}$ is said to be $\mathcal{A}$ -linear, if

$$T(ax + \lambda y) = aT(x) + \lambda T(y)$$

for all $x, y \in \mathcal{H}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. A mapping $T : \mathcal{H} \rightarrow \mathcal{K}$ is said to be adjointable, if there exists a mapping $S : \mathcal{K} \rightarrow \mathcal{H}$ such that

$$\langle T(x), y \rangle = \langle x, S(y) \rangle$$

for all $x \in D_T \subseteq \mathcal{H}$, $y \in D_S \subseteq \mathcal{K}$. The unique mapping S is denoted by T^* and is called the adjoint of T . It is well known that any adjointable mapping $T : \mathcal{H} \rightarrow \mathcal{K}$ is $\mathcal{A}$ -linear and bounded. $L(\mathcal{H}, \mathcal{K})$, the set of adjointable maps from $\mathcal{H}$ to $\mathcal{K}$ is a C^* -algebra [4]. The C^* -algebra of adjointable maps from $\mathcal{H}$ to $\mathcal{H}$ is denoted by $L(\mathcal{H})$.

In general, bounded $\mathcal{A}$ -linear operator may fail to possess an adjoint (cf., [4]). However, if $\mathcal{H}$ is a Hilbert C^* -module over the C^* -algebra $\mathcal{K} = K(H)$ of all compact operators on a Hilbert space H , then (with another concept of orthonormal basis for Hilbert C^* -modules) D. Bakić and B. Guljaš ([1], Remark 5) showed that each bounded $\mathcal{K}$ -linear operator on $\mathcal{H}$ is necessarily adjointable.

In 2003, Radu [6] employed the following result, due to Diaz and Margolis [2], to prove the stability of a Cauchy functional equation.

Proposition 1.1. *(The fixed point alternative principle). Let (X, d) be a generalized complete metric space and $J : X \rightarrow X$ be a strictly contractive mapping; that is*

$$d(J(x), J(y)) \leq Ld(x, y) \quad (x, y \in X)$$

for some (Lipschitz) constant $0 < L < 1$. Then, for a given element $x \in X$, exactly one of the following assertions is true: either

- (a) $d(J^n x, J^{n+1} x) = \infty$ for all $n \geq 0$, or
- (b) there exists some integer n_0 such that $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$.

Actually, if (b) holds, then

- (b₁) the sequence $\{J^n x\}$ converges to a fixed point x^* of J ,
- (b₂) x^* is the unique fixed point of J in $X_0 := \{y \in X; d(J^{n_0} x, y) < \infty\}$;
- (b₃) $d(y, x^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in X_0$.

Let $\mathcal{H}$ be a Hilbert C^* -module over a C^* -algebra $\mathcal{A}$ and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a mapping. In this paper, we introduce the new functional equation

$$2\langle T(x), T(y) \rangle = \langle T(x), y \rangle + \langle x, T(y) \rangle \quad (x, y \in \mathcal{H}). \quad (\diamond)$$

In the first section, we show that T is a bounded $\mathcal{A}$ -linear mapping and we find all of such bounded $\mathcal{A}$ -linear mappings. Also we show that T is adjointable. We will show that the unique inner product preserving solution of the functional equation $(\diamond)$, is the identity function.

In the next section, we consider the generalized Hyers-Ulam stability for the above functional equation. We use the fixed point alternative theorem to show that if $\mathcal{H}$ is a Hilbert C^* -modules over a C^* -algebra $\mathcal{A}$ and $f : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the inequality

$$\|2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle\| \leq \varphi(x, y) \quad (x, y \in \mathcal{H})$$

then under suitable conditions on $\varphi : \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty)$, there is a unique adjointable $\mathcal{A}$ -linear mapping $T : \mathcal{H} \rightarrow \mathcal{H}$, which suitably approximates f .

2. General Solution

In this section, we show that a function $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfies $(\diamond)$, if and only if, T is an adjointable (bounded) $\mathcal{A}$ -linear operator satisfying the equation $2T^*T = T + T^*$.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert C^* -module over a C^* -algebra $\mathcal{A}$. Any mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfying $(\diamond)$ is a bounded $\mathcal{A}$ -linear operator with $\|T\| \leq 1$.*

Proof. It follows from $(\diamond)$ that

$$\begin{aligned} & 2\langle T(x+y) - T(x) - T(y), T(z) \rangle \\ &= 2\langle T(x+y), T(z) \rangle - 2\langle T(x), T(z) \rangle - 2\langle T(y), T(z) \rangle \\ &= \langle T(x+y), z \rangle + \langle x+y, T(z) \rangle \\ &\quad - \langle T(x), z \rangle - \langle x, T(z) \rangle - \langle T(y), z \rangle - \langle y, T(z) \rangle \\ &= \langle T(x+y) - T(x) - T(y), z \rangle \end{aligned}$$

for all $x, y, z \in \mathcal{H}$, which implies that

$$\langle T(x+y) - T(x) - T(y), T(z) \rangle = \langle T(x+y) - T(x) - T(y), \frac{z}{2} \rangle \quad (2)$$

for all $x, y, z \in \mathcal{H}$. It follows from (2) that

$$\begin{aligned} & \langle T(x+y) - T(x) - T(y), T(x+y) - T(x) - T(y) \rangle \\ &= \langle T(x+y) - T(x) - T(y), T(x+y) \rangle \\ &\quad - \langle T(x+y) - T(x) - T(y), T(x) \rangle \\ &\quad - \langle T(x+y) - T(x) - T(y), T(y) \rangle \\ &= \langle T(x+y) - T(x) - T(y), \frac{x+y}{2} - \frac{x}{2} - \frac{y}{2} \rangle = 0 \end{aligned}$$

for all $x, y \in \mathcal{H}$. From the above equation, we get

$$T(x+y) = T(x) + T(y)$$

for all $x, y \in \mathcal{H}$. Hence T is an additive mapping.

From

$$\begin{aligned}
& 2\langle T(ax) - aT(x), T(ax) - aT(x) \rangle \\
&= 2\langle T(ax), T(ax) \rangle - 2\langle aT(x), T(ax) \rangle \\
&\quad - 2\langle T(ax), aT(x) \rangle + 2\langle aT(x), aT(x) \rangle \\
&= \langle T(ax), x \rangle a^* + a\langle x, T(ax) \rangle - a\langle T(x), x \rangle a^* - a\langle x, T(ax) \rangle \\
&\quad - \langle T(ax), x \rangle a^* - a\langle x, T(x) \rangle a^* + a\langle T(x), x \rangle a^* + a\langle x, T(x) \rangle a^* \\
&= 0
\end{aligned}$$

for all $a \in \mathcal{A}$ and $x \in \mathcal{H}$, we deduce that $T(ax) = aT(x)$. In the same manner, we deduce that $T(\lambda x) = \lambda T(x)$ for all $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$. Thus

$$T(ax + \lambda y) = aT(x) + \lambda T(y)$$

for all $x, y \in \mathcal{H}$, $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, i.e. T is an $\mathcal{A}$ -linear operator.

Putting $x = y$ in $(\diamond)$, we get

$$2\langle T(x), T(x) \rangle = \langle T(x), x \rangle + \langle x, T(x) \rangle \quad (x \in \mathcal{H}). \quad (3)$$

Thus

$$\begin{aligned}
2\|T(x)\|^2 &= 2\| |T(x)|^2 \| = 2\|\langle T(x), T(x) \rangle\| = \|\langle T(x), x \rangle + \langle x, T(x) \rangle\| \\
&\leq 2\|T(x)\|\|x\| \quad (x \in \mathcal{H}).
\end{aligned}$$

and so

$$\|T\|^2 = \sup_{\|x\| \leq 1} \|T(x)\|^2 \leq \sup_{\|x\| \leq 1} \|T(x)\|\|x\| = \|T\|$$

which implies that $\|T\| \leq 1$. This completes the proof. $\square$

Lemma 2.2. *An $\mathcal{A}$ -linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfies $(\diamond)$, if and only if T satisfies*

$$2\langle T(e_i), T(e_j) \rangle = \langle T(e_i), e_j \rangle + \langle e_i, T(e_j) \rangle \quad (i, j \in I). \quad (4)$$

Proof. If T satisfies $(\diamond)$, then trivially T satisfies (4). Assume that T satisfies (4), and let $x, y \in \mathcal{H}$. Then $x = \sum_{i \in I} a_i e_i$ and $y = \sum_{j \in I} b_j e_j$, where $a_i = \langle x, e_i \rangle$ and $b_j = \langle y, e_j \rangle$ for $i, j \in I$. It follows from Theorem 2.1 that

$$2\langle T(x), T(y) \rangle = 2\langle T\left(\sum_{i \in I} a_i e_i\right), T\left(\sum_{j \in I} b_j e_j\right) \rangle$$

$$\begin{aligned}
&= \sum_{i \in I} \sum_{j \in I} a_i (2 \langle T(e_i), T(e_j) \rangle) b_j^* \\
&= \sum_{i \in I} \sum_{j \in I} a_i (\langle T(e_i), e_j \rangle + \langle e_i, T(e_j) \rangle) b_j^* \\
&= \sum_{i \in I} \sum_{j \in I} a_i \langle T(e_i), e_j \rangle b_j^* + \sum_{i \in I} \sum_{j \in I} a_i \langle e_i, T(e_j) \rangle b_j^* \\
&= \langle T(\sum_{i \in I} a_i e_i), \sum_{j \in I} b_j e_j \rangle + \langle \sum_{i \in I} a_i e_i, T(\sum_{j \in I} b_j e_j) \rangle \\
&= \langle T(x), y \rangle + \langle x, T(y) \rangle.
\end{aligned}$$

□

Let $(e_i)_{i \in I}$ be a Schauder basis for Hilbert C^* -module $\mathcal{H}$ over a unital C^* -algebra $\mathcal{A}$ and $T : \mathcal{H} \rightarrow \mathcal{H}$ be an $\mathcal{A}$ -linear operator. Let $a_{ij} = \langle T(e_j), e_i \rangle$ for all $i, j \in I$ and consider the matrix $A = [a_{ij}]$ corresponding to the $\mathcal{A}$ -linear operator T . The next theorem characterizes the bounded $\mathcal{A}$ -linear operators T satisfying $(\diamond)$.

Theorem 2.3. *Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. An $\mathcal{A}$ -linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfies $(\diamond)$, if and only if,*

$$2A^t A^* = A^t + A^*. \quad (5)$$

where A^t is the transpose of A and $A^* = [a_{ij}^*]$.

Proof. Suppose that T satisfies $(\diamond)$, then from Lemma 2.2, T satisfies (4) for all $i, j \in I$. Since for all $i \in I$, $T(e_i) = \sum_{p \in I} \langle T(e_i), e_p \rangle e_p$, then we have

$$\begin{aligned}
&2 \langle \sum_{p \in I} \langle T(e_i), e_p \rangle e_p, \sum_{q \in I} \langle T(e_j), e_q \rangle e_q \rangle \\
&= \langle \sum_{p \in I} \langle T(e_i), e_p \rangle e_p, e_j \rangle + \langle e_i, \sum_{q \in I} \langle T(e_j), e_q \rangle e_q \rangle.
\end{aligned}$$

which implies that

$$\begin{aligned}
&2 \sum_{p \in I} \sum_{q \in I} \langle T(e_i), e_p \rangle \langle e_p, e_q \rangle \langle e_q, T(e_j) \rangle \\
&= \sum_{p \in I} \langle T(e_i), e_p \rangle \langle e_p, e_j \rangle + \sum_{q \in I} \langle e_i, e_q \rangle \langle e_q, T(e_j) \rangle.
\end{aligned}$$

Therefore it follows from the last equation that

$$2 \sum_{p \in I} \langle T(e_i), e_p \rangle \langle e_p, T(e_j) \rangle = \langle T(e_i), e_j \rangle + \langle e_i, T(e_j) \rangle \quad (i, j \in I).$$

Let $a_{ij} = \langle T(e_j), e_i \rangle$ for all $i, j \in I$, then the last equation implies that

$$2 \sum_{p \in I} a_{pi} a_{pj}^* = a_{ji} + a_{ij}^* \quad (i, j \in I).$$

This means that $2A^t A^* = A^t + A^*$. Conversely, if $2A^t A^* = A^t + A^*$, it is easy to see that T satisfies $(\diamond)$. $\square$

Corollary 2.4. *Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. An $\mathcal{A}$ -linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ satisfies $(\diamond)$, if and only if T is adjointable and satisfies the equation $2T^*T = T + T^*$.*

Proof. Suppose that T satisfies $(\diamond)$. From (5) we have

$$2(A^*)^t A = A + (A^*)^t. \quad (6)$$

Define the $\mathcal{A}$ -linear operator $S : \mathcal{H} \rightarrow \mathcal{H}$ by the matrix $(A^*)^t$. Then trivially $\langle S(e_j), e_i \rangle = a_{ji}^*$ for all $i, j \in I$ and it follows that $\langle e_j, S(e_i) \rangle = a_{ij} = \langle T(e_j), e_i \rangle$ for all $i, j \in I$. From $\mathcal{A}$ -linearity of T and S , we get

$$\langle T(x), y \rangle = \langle x, S(y) \rangle \quad (x, y \in \mathcal{H}).$$

So T is adjointable with $T^* = S$. From (6) we get $2T^*T = T + T^*$. Conversely, if T is an adjointable $\mathcal{A}$ -linear operator such that $2T^*T = T + T^*$, then it satisfies $(\diamond)$. $\square$

Remark 2.5. Suppose that the mappings T and S satisfy $(\diamond)$ and $\langle T(x), S(y) \rangle = 0$ for all $x, y \in \mathcal{H}$, then for any complex number $\lambda = (r, \theta)$ with $r = 2 \cos \theta$, the adjointable $\mathcal{A}$ -linear mapping $\lambda T + S$ satisfies $(\diamond)$. Also if the mapping T satisfies $(\diamond)$, then $I - T$ satisfies $(\diamond)$.

Example 2.6. The Hilbert space ℓ^2 is a Hilbert $\mathbb{C}$ -module. The $\mathbb{C}$ -linear operator $T : \ell^2 \rightarrow \ell^2$ defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell^2$, satisfies $(\diamond)$, also $\|T\| < 1$ and $T = T^*$.

Example 2.7. Let $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a $\mathbb{C}$ -linear operator, corresponding with the complex matrix $A = [a_{ij}]$. T satisfies $(\diamond)$, if and only if the complex numbers a_{ij} satisfy the following equations:

$$|a_{11} - \frac{1}{2}|^2 + |a_{21}|^2 = \frac{1}{4}, \quad |a_{22} - \frac{1}{2}|^2 + |a_{12}|^2 = \frac{1}{4},$$

$$|a_{12}| = |a_{21}| \leq \frac{1}{2}, \quad |a_{11} - \frac{1}{2}| = |a_{22} - \frac{1}{2}| \leq \frac{1}{2}.$$

Remark 2.8. Let $\alpha, \beta, \gamma \in \mathbb{C}$ be nonzero complex numbers and $T : \mathcal{H} \rightarrow \mathcal{H}$ be a mapping satisfying

$$\alpha \langle T(x), T(y) \rangle = \beta \langle T(x), y \rangle + \gamma \langle x, T(y) \rangle \quad (x, y \in \mathcal{H}). \quad (\diamond')$$

It is easy to see that T is an adjointable $\mathcal{A}$ -linear mapping with $\|T\| \leq \frac{|\beta|+|\gamma|}{|\alpha|}$. Moreover it follows that $\alpha T^*T = \beta T + \gamma T^*$ or equivalently $(\frac{\alpha}{\beta}T^* - I)(\frac{\alpha}{\gamma}T - I) = I$. Thus if $\frac{\alpha}{\gamma}T - I$ is surjective, then it is invertible in $L(\mathcal{H})$ with $(\frac{\alpha}{\gamma}T - I)^{-1} = \frac{\alpha}{\beta}T^* - I$.

In particular, if the mapping T satisfies $(\diamond')$ with $\alpha = \beta = \gamma = 1$, then $I - T$ is inner product preserving and so is an isometry.

3. Stability

In this section, we prove the generalized Hyers-Ulam stability of the equation $(\diamond)$.

Theorem 3.1. Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $\varphi : \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty)$ be a control function such that

$$\lim_{n \rightarrow \infty} 2^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \quad (7)$$

for all $x, y \in \mathcal{H}$. Assume that there is $0 < L < 1$ such that

$$\psi\left(\frac{x}{2}\right) \leq \frac{L}{2} \psi(x) \quad (8)$$

for all $x \in \mathcal{H}$, where

$$\psi(x) = (4\varphi(x, x) + 2\varphi(2x, x) + 2\varphi(x, 2x) + \varphi(2x, 2x))^{\frac{1}{2}}. \quad (9)$$

If a function $f : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the inequality

$$\|2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle\| \leq \varphi(x, y) \quad (10)$$

for all $x, y \in \mathcal{H}$, then there exists a unique adjointable mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\|f(x) - T(x)\| \leq \frac{\sqrt{2}L}{4(1-L)}\psi(x) \quad (11)$$

for all $x \in \mathcal{H}$.

Proof. Replacing x by $2x$ in (10), we get

$$\|2\langle f(2x), f(y) \rangle - \langle f(2x), y \rangle - 2\langle x, f(y) \rangle\| \leq \varphi(2x, y). \quad (12)$$

From (10) and (12), we have

$$\|2\langle f(2x) - 2f(x), f(y) \rangle - \langle f(2x) - 2f(x), y \rangle\| \leq 2\varphi(x, y) + \varphi(2x, y). \quad (13)$$

Replacing y by $2y$ in (13), we get

$$\|2\langle f(2x) - 2f(x), f(2y) \rangle - 2\langle f(2x) - 2f(x), y \rangle\| \leq 2\varphi(x, 2y) + \varphi(2x, 2y). \quad (14)$$

From (13) and (14), we have

$$\|2\langle f(2x) - 2f(x), f(2y) - 2f(y) \rangle\| \leq \psi(x)^2. \quad (15)$$

Letting $x = y$ in (15), we obtain

$$\|f(2x) - 2f(x)\|^2 = \| |f(2x) - 2f(x)|^2 \| \leq \frac{1}{2}\psi(x)^2.$$

and so

$$\|f(2x) - 2f(x)\| \leq \frac{\sqrt{2}}{2}\psi(x) \quad (16)$$

Let $X = \{g : \mathcal{H} \rightarrow \mathcal{H}, g(0) = 0\}$ and define $d : X \times X \rightarrow [0, \infty]$ by

$$d(g, h) = \inf\{\alpha \geq 0 : \|g(x) - h(x)\| \leq \alpha\psi(x), \forall x \in \mathcal{H}\} \quad (g, h \in X).$$

Define $J : X \rightarrow X$ by $J(g)(x) = 2g(\frac{x}{2})$ for each $x \in \mathcal{H}$. Then (X, d) is a complete generalized metric space and from (8) it follows that J is a strictly contractive mapping on X with the Lipschitz constant $L = \frac{1}{2}$. From (8) and (16) we have

$$\|J(f)(x) - f(x)\| = \|2f(\frac{x}{2}) - f(x)\| = 2\|f(\frac{x}{2}) - \frac{f(x)}{2}\| \leq \frac{\sqrt{2}}{4}L\psi(x)$$

for each $x \in \mathcal{H}$. This means that $d(J(f), f) \leq \frac{\sqrt{2}}{4}L$. Therefore, by Proposition 1.1, J has a unique fixed point in the set $X_0 = \{g \in X : d(f, g) < \infty\}$. Let $T : \mathcal{H} \rightarrow \mathcal{H}$ be the unique fixed point of J . We have $\lim_n(J^n(f), T) = 0$, so T is defined by

$$T(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \quad (x \in \mathcal{H}). \quad (17)$$

On the other hand, we have $d(f, J(f)) \leq \frac{\sqrt{2}}{4}L$ and $J(T) = T$, then

$$d(f, T) \leq d(f, J(f)) + d(J(f), J(T)) \leq \frac{\sqrt{2}}{4}L + Ld(f, T).$$

So

$$d(f, T) \leq \frac{\sqrt{2}L}{4(1-L)},$$

which implies the inequality (11).

It is easy to see that for all $x, y \in \mathcal{H}$, we have

$$\lim_{n \rightarrow \infty} \langle 2^n f\left(\frac{x}{2^n}\right), y \rangle = \langle T(x), y \rangle, \quad (18)$$

$$\lim_{n \rightarrow \infty} \langle x, 2^n f\left(\frac{y}{2^n}\right) \rangle = \langle x, T(y) \rangle. \quad (19)$$

Since for every $x \in \mathcal{H}$ the sequence $\{2^n f(\frac{x}{2^n})\}$ is convergent, so it is bounded. Therefore for every $x \in \mathcal{H}$ there exists $K_x > 0$ such that $\|2^n f(\frac{x}{2^n})\| \leq K_x$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| \langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right) \rangle - \langle T(x), T(y) \rangle \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right) \rangle - \langle 2^n f\left(\frac{x}{2^n}\right), T(y) \rangle \right. \\ & \quad \left. + \langle 2^n f\left(\frac{x}{2^n}\right), T(y) \rangle - \langle T(x), T(y) \rangle \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right) - T(y) \rangle + \langle 2^n f\left(\frac{x}{2^n}\right) - T(x), T(y) \rangle \right\| \\ &\leq \lim_{n \rightarrow \infty} (\|2^n f\left(\frac{x}{2^n}\right)\| \|2^n f\left(\frac{y}{2^n}\right) - T(y)\| + \|2^n f\left(\frac{x}{2^n}\right) - T(x)\| \|T(y)\|) \\ &\leq \lim_{n \rightarrow \infty} (K_x \|2^n f\left(\frac{y}{2^n}\right) - T(y)\| + \|2^n f\left(\frac{x}{2^n}\right) - T(x)\| \|T(y)\|) = 0 \end{aligned}$$

for all $x, y \in \mathcal{H}$. This shows that

$$\lim_{n \rightarrow \infty} \langle 2^n f\left(\frac{x}{2^n}\right), 2^n f\left(\frac{y}{2^n}\right) \rangle = \langle T(x), T(y) \rangle \quad (x, y \in \mathcal{H}). \quad (20)$$

It follows from (7), (18), (19) and (20) that

$$\begin{aligned} & \|2\langle T(x), T(y) \rangle - \langle T(x), y \rangle - \langle x, T(y) \rangle\| \\ &= \lim_{n \rightarrow \infty} 2^{2n} \|2\langle f(\frac{x}{2^n}), f(\frac{y}{2^n}) \rangle - \langle f(\frac{x}{2^n}), \frac{y}{2^n} \rangle - \langle \frac{x}{2^n}, f(\frac{y}{2^n}) \rangle\| \\ &\leq \lim_{n \rightarrow \infty} 2^{2n} \varphi(\frac{x}{2^n}, \frac{y}{2^n}) = 0 \end{aligned}$$

for all $x, y \in \mathcal{H}$. Whence

$$2\langle T(x), T(y) \rangle = \langle T(x), y \rangle + \langle x, T(y) \rangle \quad (21)$$

for all $x, y \in \mathcal{H}$ and so by Corollary (2.4), T is a adjointable $\mathcal{A}$ -linear mapping.

To see the uniqueness of T , let $T' : \mathcal{H} \rightarrow \mathcal{H}$ be another adjointable $\mathcal{A}$ -linear mapping satisfying (11). Then

$$\begin{aligned} \|T(x) - T'(x)\| &= 2^n \|T(\frac{x}{2^n}) - T'(\frac{x}{2^n})\| \\ &\leq 2^n (\|T(\frac{x}{2^n}) - f(\frac{x}{2^n})\| + \|f(\frac{x}{2^n}) - T'(\frac{x}{2^n})\|) \\ &\leq 2^n \frac{\sqrt{2}L}{2(1-L)} \psi(\frac{x}{2^n}) \\ &\leq \frac{\sqrt{2}L}{2(1-L)} L^n \psi(x) \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in \mathcal{H}$. This completes the proof. $\square$

The following Theorem can be proved in a similar way as Theorem 3.1.

Theorem 3.2. *Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and $\varphi : \mathcal{H} \times \mathcal{H} \rightarrow [0, \infty)$ be an control function such that*

$$\lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{2^{2n}} = 0$$

for all $x, y \in \mathcal{H}$. Assume that there is $0 < L < 1$ such that

$$\psi(2x) \leq 2L\psi(x)$$

for all $x \in \mathcal{H}$, where

$$\psi(x) = (4\varphi(x, x) + 2\varphi(2x, x) + 2\varphi(x, 2x) + \varphi(2x, 2x))^{\frac{1}{2}}.$$

If a function $f : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the inequality

$$\|2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{H}$, then there exists a unique adjointable mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\|f(x) - T(x)\| \leq \frac{\sqrt{2}}{4(1-L)}\psi(x) \quad (x \in \mathcal{H}).$$

The next result follows from Theorem 3.1, where $\varphi(x, y) = \|x - y\|^p$ for all $x, y \in \mathcal{H}$.

Corollary 3.3. *Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. If a function $f : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the inequality*

$$\|2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle\| \leq \|x - y\|^p$$

for all $x, y \in \mathcal{H}$ and some $p > 2$, then there exists a unique adjointable mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\|f(x) - T(x)\| \leq \frac{\sqrt{2}}{\sqrt{2^p} - 2} \|x\|^{\frac{p}{2}} \quad (x \in \mathcal{H}).$$

The next result follows from Theorem 3.2, where $\varphi(x, y) = \varepsilon$.

Corollary 3.4. *Let $\mathcal{H}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. If a function $f : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the inequality*

$$\|2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle\| \leq \varepsilon$$

for all $x, y \in \mathcal{H}$, then there exists a unique adjointable mapping $T : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\|f(x) - T(x)\| \leq \frac{3\sqrt{2\varepsilon}}{2} \quad (x \in \mathcal{H}).$$

Example 3.5. Consider the Hilbert $\mathbb{C}$ -module ℓ^2 . Let $\phi : \ell^2 \rightarrow \mathbb{C}$ be an arbitrary bounded function such that $|\phi(x)| \leq \|x\|$ for all $x \in \ell^2$, where $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$. Define the mapping $f : \ell^2 \rightarrow \ell^2$ by

$$f(x_1, x_2, x_3, \dots) = (\phi(x)x_1, x_2, x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell^2$. Then

$$\begin{aligned}
 & |2\langle f(x), f(y) \rangle - \langle f(x), y \rangle - \langle x, f(y) \rangle| \\
 &= |2\phi(x)x_1\overline{\phi(y)y_1} + 2\sum_{n=2}^{\infty} x_n\overline{y_n} \\
 &\quad - \phi(x)x_1\overline{y_1} - \sum_{n=2}^{\infty} x_n\overline{y_n} - x_1\overline{\phi(y)y_1} - \sum_{n=2}^{\infty} x_n\overline{y_n}| \\
 &= |(x_1\overline{y_1})(2\phi(x)\overline{\phi(y)} - \phi(x) - \overline{\phi(y)})| \\
 &\leq |x_1||y_1|(2|\phi(x)||\phi(y)| + |\phi(x)| + |\phi(y)|) \\
 &\leq \|x\|\|y\|(2\|x\|\|y\| + \|x\| + \|y\|) \\
 &= 2\|x\|^2\|y\|^2 + \|x\|^2\|y\| + \|x\|\|y\|^2
 \end{aligned}$$

for all $x, y \in \ell^2$. Thus if the control function is defined by

$$\varphi(x, y) = 2\|x\|^2\|y\|^2 + \|x\|^2\|y\| + \|x\|\|y\|^2 \quad (x, y \in \ell^2),$$

then we have $\lim_{n \rightarrow \infty} 2^{2n}\varphi(2^{-n}x, 2^{-n}y) = 0$ and $\psi(x) = (72\|x\|^4 + 48\|x\|^3)^{\frac{1}{2}}$. Also for constant $0 < L = \frac{\sqrt{2}}{2} < 1$, we have $\psi(\frac{x}{2}) \leq \frac{L}{2}\psi(x)$ for all $x \in \ell^2$. Then the function $T : \ell^2 \rightarrow \ell^2$ defined by

$$T(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell^2$, is the unique adjointable mapping fulfilling the condition (11) in Theorem 3.1.

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